

FRACTIONAL KORN INEQUALITY ON SUBSETS OF THE EUCLIDEAN SPACE

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Abstract. We prove the fractional Korn inequality for sufficiently smooth subsets of \mathbb{R}^d . We also give a framework for obtaining the Korn inequality directly from the appropriate Hardy-type inequality.

1. Introduction

Let D be a bounded open subset of \mathbb{R}^d , $d > 1$, and let $p \in (1, \infty)$. For $x \in \mathbb{R}^d$ we denote $x = (x', x_d)$ with $x' \in \mathbb{R}^{d-1}$, $x_d \in \mathbb{R}$. Whenever we mention a vector field on $D \subseteq \mathbb{R}^d$, we mean a measurable mapping from D into \mathbb{R}^d . The space $L^p(D)$ consists of all the vector fields u for which the norm $\|u\|_{L^p(D)} := (\int_D |u(x)|^p dx)^{1/p}$ is finite.

We define the fractional Sobolev space of vector fields as follows:

$$W^{s,p}(D) = \left\{ u \in L^p(D) : |u|_{W^{s,p}(D)}^p := \int_D \int_D \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dx dy < \infty \right\}.$$

$W^{s,p}(D)$ is endowed with the norm given by the formula $\|u\|_{W^{s,p}(D)} := (\|u\|_{L^p(D)}^p + |u|_{W^{s,p}(D)}^p)^{1/p}$. We also introduce the Sobolev space with projected difference quotient:

$$\mathcal{X}^{s,p}(D) = \left\{ u \in L^p(D) : |u|_{\mathcal{X}^{s,p}(D)}^p := \int_D \int_D \frac{|(u(x) - u(y)) \frac{(x-y)}{|x-y|}|^p}{|x - y|^{d+sp}} dx dy < \infty \right\}.$$

We equip $\mathcal{X}^{s,p}(D)$ with the norm $\|u\|_{\mathcal{X}^{s,p}(D)} := (\|u\|_{L^p(D)}^p + |u|_{\mathcal{X}^{s,p}(D)}^p)^{1/p}$. Furthermore, we define the spaces $W_0^{s,p}(D)$ and $\mathcal{X}_0^{s,p}(D)$ as the closures of $(C_c^1(D))^d$ in the norms $\|\cdot\|_{W^{s,p}(D)}$ and $\|\cdot\|_{\mathcal{X}^{s,p}(D)}$, respectively.

Obviously, $\|u\|_{\mathcal{X}^{s,p}(D)} \leq \|u\|_{W^{s,p}(D)}$. Our main goal here is to establish a reverse inequality with a multiplicative constant on the left-hand side, which is known as the fractional Korn inequality.

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THEOREM 1.1. *Assume that $p > 1$, $s \in (0, 1)$, and $sp \neq 1$. If D is a bounded C^1 open set or a bounded Lipschitz set with sufficiently small Lipschitz constant, then there exists $C \geq 1$ such that for every $u \in \mathcal{X}_0^{s,p}(D)$ we have*

$$C\|u\|_{\mathcal{X}^{s,p}(D)} \geq \|u\|_{W^{s,p}(D)}.$$

In particular, $\mathcal{X}_0^{s,p}(D) = W_0^{s,p}(D)$.

This result was obtained very recently by Mengesha and Scott [3] with the use of a complicated extension operator intrinsic to studying projected seminorms. We present a significantly shorter proof which omits the extension: we first obtain the inequality for the epigraphs in Theorem 3.1 by using the result of Mengesha and Scott [6, Theorem 1.1] for the half-space together with an appropriate change of variables, and then we apply an argument via the partition of unity.

In Section 4 we show that the Korn inequality for vector fields of the class $(C_c^1(D))^d$ can be obtained directly from the appropriate Hardy-type inequality for D with the use of an operator which extends the vector field by 0 to the whole space. This in particular yields a simpler proof of the Korn inequality for the half-space than the original one due to Mengesha [2]. It may also facilitate the proofs for more general sets D in the future.

The usage of the Hardy inequality imposes the conditions of vanishing at the boundary and $sp \neq 1$, which are most likely superfluous for the Korn inequality, but dropping them would require a completely different method of proof.

For applications, open problems, and a wider context concerning the fractional Korn inequality we refer to the aforementioned works of Mengesha and Scott. We remark that the arguments below were obtained independently of the ones in [3].

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2. Preliminaries and auxiliary results

Let $f: \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be a continuous function. The open set $\{(x', x_d) \in \mathbb{R}^d : f(x') < x_d\}$ will be called the epigraph of f . The epigraph of a Lipschitz (resp. C^1) function will be called a Lipschitz (resp. C^1) epigraph.

DEFINITION 2.1. We say that an open set $D \subseteq \mathbb{R}^d$ is Lipschitz with constant $L > 0$ if there exist balls B_1, \dots, B_n with centers belonging to ∂D , epigraphs U_1, \dots, U_n of functions f_1, \dots, f_n with Lipschitz constant L or better, and rigid motions R_1, \dots, R_n , such that the following conditions are satisfied

- $\partial D \subset \bigcup_{i=1}^n B_i$,
- $R_i(B_i \cap D) \subset U_i$ and $R_i(B_i \cap \partial D) = R_i(B_i) \cap \partial U_i$.

We say that D is a C^1 set if the above conditions are satisfied with the difference that the epigraphs U_1, \dots, U_n and the functions f_1, \dots, f_n are C^1 instead of Lipschitz.

We also consider an additional open set (not necessarily a ball) B_{n+1} , relatively compact in D , such that $D \subset \bigcup_{i=1}^{n+1} B_i$. Note that if D is C^1 , then the balls and the epigraphs may be so chosen that D is Lipschitz with an arbitrarily small constant L .

Throughout the paper we will use certain Lipschitz maps as substitutions in the integration process. Below we establish some basic facts about these transformations.

LEMMA 2.2. *Let U and V be open subsets of \mathbb{R}^d and assume that $T: U \rightarrow V$ is a bijection such that T and T^{-1} are Lipschitz with constant $K \geq 1$. Then for every non-negative measurable function $u: V \rightarrow \mathbb{R}$ we have*

$$(1/K)^d \int_V u(x) dx \leq \int_U u(Tx) dx \leq K^d \int_V u(x) dx.$$

Proof. This fact follows conveniently from the result of Hajlasz [1, Appendix], see also Rado and Reichelderfer [5, V.2.3]. To verify the validity of the constants we first claim that J_T — the Jacobian of T satisfies $|J_T| \leq K^d$ almost everywhere in U . Indeed, let $x_0 \in U$ and $r > 0$ satisfy $B(x_0, r) \subset U$. If we take $f = T$ and $u = \mathbf{1}_{TB(x_0, r)}$ in [1], then we get that

$$\int_{B(x_0, r)} |J_T(x)| dx = \int_{TB(x_0, r)} dy.$$

Thus,

$$\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |J_T(x)| dx = \frac{|TB(x_0, r)|}{|B(x_0, r)|}. \tag{2.1}$$

The limit $r \rightarrow 0^+$ on the left-hand side of (2.1) exists and equals $J_T(x_0)$ for almost every $x_0 \in U$ by the Lebesgue differentiation theorem. Furthermore, we have $TB(x_0, r) \subseteq B(T(x_0), Kr)$, hence the right-hand side of (2.1) is bounded from above by K^d . This proves the claim that $|J_T(x)| \leq K^d$ for almost every $x \in U$. Similarly we show that $|J_{T^{-1}}| \leq K^d$ almost everywhere in V . Thus, the lemma follows from [1] and the formula $|J_{T^{-1}}(Tx)| = |J_T(x)|^{-1}$. \square

We will commonly map the epigraph of a Lipschitz function $f: \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ to the half-space \mathbb{R}_+^d as follows:

$$T(x', x_d) = (x', x_d - f(x')), \quad x' \in \mathbb{R}^{d-1}, x_d > f(x'). \tag{2.2}$$

Clearly this is a bijection with the inverse

$$T^{-1}(x', x_d) = (x', x_d + f(x')), \quad x' \in \mathbb{R}^{d-1}, x_d > 0.$$

LEMMA 2.3. *If f is Lipschitz with constant L and T is defined as in (2.2), then for every x and y in the epigraph of f we have*

$$C(L)^{-1}|x - y| \leq |Tx - Ty| \leq C(L)|x - y|, \tag{2.3}$$

where $C(L) = \sqrt{1 + L(L + 1)}$. In particular, $C(L) \rightarrow 1$ as $L \rightarrow 0^+$.

Proof. Let x and y belong to the epigraph of f . We may reduce the problem to the planar geometry. Let $a = |x' - y'|$, $b = |x_d - y_d|$, $c = |x - y|$, $c' = |Tx - Ty|$, $d = |f(x') - f(y')|$. Note that $d \leq La$.

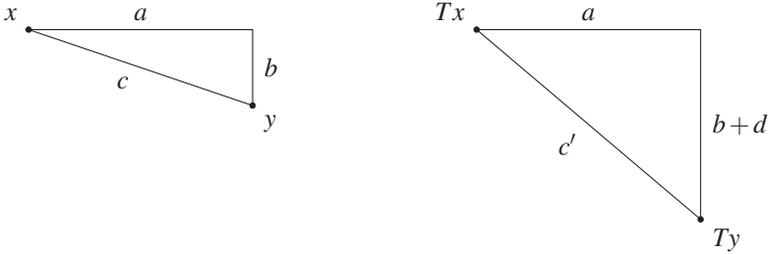


Figure 1: The picture presents the pessimistic variant with $(x_d - y_d)(f(x') - f(y')) > 0$.

We will work with the context given in Figure 1. Assuming that $x \neq y$, we have

$$\frac{(c')^2}{c^2} = 1 + \frac{2bd + d^2}{a^2 + b^2} \leq 1 + \frac{2Lba + L^2a^2}{a^2 + b^2} = \leq 1 + L \frac{(L+1)a^2 + b^2}{a^2 + b^2} \leq 1 + L(L+1).$$

This gives the right-hand side part of (2.3). A similar argument may be used with T^{-1} in place of T , giving the left-hand side of (2.3). \square

LEMMA 2.4. Assume that D is bounded and that $\psi \in C_b^\infty(D)$. Then $|u\psi|_{\mathcal{X}^{s,p}(D)} \lesssim \|u\|_{\mathcal{X}^{s,p}(D)}$. An analogous result holds with $W^{s,p}$ in place of $\mathcal{X}^{s,p}$.

Proof. We have

$$\begin{aligned} \int_D \int_D \frac{|(u\psi(x) - u\psi(y)) \frac{(x-y)}{|x-y|}|^p}{|x-y|^{d+sp}} dx dy &\lesssim \int_D \int_D \frac{|(\psi(x) - \psi(y))u(x) \frac{(x-y)}{|x-y|}|^p}{|x-y|^{d+sp}} dx dy \\ &\quad + \int_D \int_D \frac{|\psi(y)(u(x) - u(y)) \frac{(x-y)}{|x-y|}|^p}{|x-y|^{d+sp}} dx dy. \end{aligned}$$

The latter integral is smaller than $\|\psi\|_{L^\infty(D)}^p \|u\|_{\mathcal{X}^{s,p}(D)}^p$. For the former we use the fact that $|\psi(x) - \psi(y)| \lesssim |x - y|$ to get that it does not exceed $c\|u\|_{L^p(D)}^p$. The proof for $W^{s,p}$ is identical. \square

Let $B_\delta = \{x \in B : d(x, \partial B) > \delta\}$. In the next section we will apply an argument using a partition of unity subordinate to B_1, \dots, B_n, B_{n+1} from Definition 2.1. This in particular will require extending vector fields given on $B_i \cap D$ and supported in $(B_i)_\delta \cap D$ for some fixed $\delta > 0$, to a rotated epigraph ($1 \leq i \leq n$) or to the whole of \mathbb{R}^d ($i = n + 1$). The following result enables us to perform such operations.

LEMMA 2.5. Let the open sets $B, U \subseteq \mathbb{R}^d$ satisfy $U \cap B_\delta \neq \emptyset$ and $U \setminus B \neq \emptyset$. Assume that $u \in \mathcal{X}^{s,p}(U \cap B)$ has support contained in $\overline{U \cap B_\delta}$ for fixed $\delta > 0$. If we

let $\tilde{u} = u$ in $U \cap B$ and $\tilde{u} = 0$ in $U \setminus B$, then $\|\tilde{u}\|_{\mathcal{X}^{s,p}(U)} \lesssim \|u\|_{\mathcal{X}^{s,p}(U \cap B)}$. Analogous result holds with $W^{s,p}$ in place of $\mathcal{X}^{s,p}$.

Proof. Obviously, it suffices to estimate $|\tilde{u}|_{\mathcal{X}^{s,p}(U)}$. Since $\tilde{u} = 0$ on B_δ^c we have

$$\begin{aligned} \int_U \int_U \frac{|(\tilde{u}(x) - \tilde{u}(y)) \frac{|x-y|}{|x-y|}|^p}{|x-y|^{d+sp}} dx dy &\leq |u|_{\mathcal{X}^{s,p}(U \cap B)}^p + 2 \int_{U \cap B_\delta} |u(x)|^p \int_{U \setminus B} \frac{dy dx}{|x-y|^{d+sp}} \\ &\leq |u|_{\mathcal{X}^{s,p}(U \cap B)}^p + c(\delta) \|u\|_{L^p(U \cap B)}^p \lesssim \|u\|_{\mathcal{X}^{s,p}(U \cap B)}^p. \end{aligned}$$

The proof for $W^{s,p}$ is identical. \square

3. Proof of the Korn inequality

We will show that the Korn inequality holds for the epigraphs with sufficiently small Lipschitz constant and then use this fact to establish Theorem 1.1.

THEOREM 3.1. *Assume that $sp \neq 1$ and that D is the epigraph of a Lipschitz function f with sufficiently small Lipschitz constant L . Then there exists $C \geq 1$ such that for every $u \in \mathcal{X}_0^{s,p}(D)$ we have*

$$C \|u\|_{\mathcal{X}^{s,p}(D)} \geq \|u\|_{W^{s,p}(D)}.$$

Consequently, $\mathcal{X}_0^{s,p}(D) = W_0^{s,p}(D)$.

Proof. Following the approach of Nitsche [4, Remark 3] we will show that there exist $c_1 = c_1(L)$ and $c_2 = c_2(L)$, such that

$$|u|_{W^{s,p}(D)}^p \leq c_1 |u|_{\mathcal{X}^{s,p}(D)}^p + c_2 |u|_{W^{s,p}(D)}^p. \tag{3.1}$$

We will propose an explicit form of c_1 and c_2 so that it will be obvious that for sufficiently small L we have $c_2 < 1$ and the statement will follow by subtracting $c_2 |u|_{W^{s,p}(D)}^p$.

Let $u \in (C_c^1(D))^d \subseteq W_0^{s,p}(D)$. If we substitute $(w', w_d) = (x', x_d - f(x'))$ and $(z', z_d) = (y', y_d - f(y'))$, then by Lemmas 2.2 and 2.3 (see the latter for the definition of $C(L)$) we get

$$\begin{aligned} |u|_{W^{s,p}(D)}^p &\leq C(L)^{2d} \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} \frac{|u(w', w_d + f(w')) - u(z', z_d + f(z'))|^p}{|(w', w_d + f(w')) - (z', z_d + f(z'))|^{d+sp}} dz dw \\ &\leq C(L)^{3d+sp} \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} \frac{|u(w', w_d + f(w')) - u(z', z_d + f(z'))|^p}{|w - z|^{d+sp}} dz dw. \end{aligned} \tag{3.2}$$

Let $v(w', w_d) = u(w', w_d + f(w'))$. Note that the above inequalities are in fact comparisons, in particular the double integral in (3.2) is finite, which means that $v \in W_0^{s,p}(\mathbb{R}_+^d)$. Now, if we let C_K be the constant in the Korn inequality due to Mengesha and Scott

[6, Theorem 1.1], and u^d — the d -th coordinate of u , then we can estimate the double integral in (3.2) as follows:

$$\begin{aligned} & \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} \frac{|v(w) - v(z)|^p}{|w - z|^{d+sp}} dz dw \\ & \leq C_K \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} \frac{|(v(w) - v(z)) \frac{(w-z)}{|w-z|}|^p}{|w - z|^{d+sp}} dz dw \\ & = C_K \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} \frac{|(u(w', w_d + f(w')) - u(z', z_d + f(z'))) \frac{(w-z)}{|w-z|}|^p}{|w - z|^{d+sp}} dz dw \\ & \leq 2^{p-1} C_K \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} |(u(w', w_d + f(w')) - u(z', z_d + f(z'))) \\ & \quad \circ ((w', w_d + f(w')) - (z', z_d + f(z')))|^p |w - z|^{-d-sp-p} dz dw \end{aligned} \tag{3.3}$$

$$+ 2^{p-1} C_K \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} \frac{|(u^d(w', w_d + f(w')) - u^d(z', z_d + f(z')))(f(z') - f(w'))|^p}{|w - z|^{d+sp+p}} dz dw. \tag{3.4}$$

By going back to the old variables and by using Lemma 2.3 once more, we get that (3.3) is estimated from above by

$$\begin{aligned} & 2^{p-1} C_K C(L)^{3d+sp+p} \int_D \int_D \frac{|(u(x) - u(y))(x - y)|^p}{|x - y|^{d+sp+p}} dx dy \\ & = 2^{p-1} C_K C(L)^{3d+sp+p} |u|_{\mathcal{X}^{s,p}(D)}^p. \end{aligned}$$

In (3.4) we also substitute the old variables so that it is estimated from above by

$$\begin{aligned} & 2^{p-1} C_K C(L)^{3d+sp+p} \int_D \int_D \frac{|(u^d(x) - u^d(y))(f(y') - f(x'))|^p}{|x - y|^{d+sp+p}} dx dy \\ & \leq 2^{p-1} C_K C(L)^{3d+sp+p} L^p \int_D \int_D \frac{|u^d(x) - u^d(y)|^p}{|x - y|^{d+sp}} dx dy \\ & \leq 2^{p-1} C_K C(L)^{3d+sp+p} L^p |u|_{W^{s,p}(D)}^p. \end{aligned}$$

Overall, we get (3.1):

$$|u|_{W^{s,p}(D)}^p \leq 2^{p-1} C_K C(L)^{6d+2sp+p} |u|_{\mathcal{X}^{s,p}(D)}^p + 2^{p-1} C_K C(L)^{6d+2sp+p} L^p |u|_{W^{s,p}(D)}^p.$$

Since $C(L) \approx 1$ for small L , the second constant can be made arbitrarily small, which yields the Korn inequality for $u \in (C_c^1(D))^d$.

Now let $u \in \mathcal{X}_0^{s,p}(D)$. There exists a sequence $u_n \in (C_c^1(D))^d$ such that $\|u_n - u\|_{\mathcal{X}^{s,p}(D)} \rightarrow 0$ as $n \rightarrow \infty$. We may assume without loss of generality that $u_n \rightarrow u$ almost everywhere. By the Korn inequality for the smooth functions we find that for every $m, n \in \mathbb{N}$,

$$|u_n|_{W^{s,p}(D)} \leq |u_n - u_m|_{W^{s,p}(D)} + |u_m|_{W^{s,p}(D)} \leq C|u_n - u_m|_{\mathcal{X}^{s,p}(D)} + C|u_n|_{\mathcal{X}^{s,p}(D)}.$$

By letting m go to infinity, and then computing \liminf with respect to n and using Fatou’s lemma, we conclude that $\|u\|_{W^{s,p}(D)} \leq C \|u\|_{\mathcal{X}^{s,p}(D)}$, which ends the proof. \square

Proof of Theorem 1.1. First, suppose that $u \in W_0^{s,p}(D)$. We may assume that D is a Lipschitz set with the constant L satisfying the assumptions of Theorem 3.1. Let $B_1, \dots, B_n, B_{n+1}, R_1, \dots, R_n, f_1, \dots, f_n$, and U_1, \dots, U_n be as in Definition 2.1, and let $U_{n+1} = \mathbb{R}^d$ and $R_{n+1} = I$. We consider a smooth partition of unity $\psi_1, \dots, \psi_{n+1}$ subordinate to B_1, \dots, B_{n+1} , i.e., $0 \leq \psi_i \leq 1$, $\text{supp}(\psi_i) \subset B_i$, and $\sum \psi_i = 1$ on D .

We define $u_i = u\psi_i$, $i = 1, \dots, n + 1$. By the triangle inequality we have

$$\|u\|_{W^{s,p}(D)} \lesssim \sum_{i=1}^{n+1} \|u_i\|_{W^{s,p}(D)}.$$

Furthermore, $\|u_i\|_{W^{s,p}(D)} \lesssim \|u_i\|_{W^{s,p}(B_i \cap D)}$ by Lemma 2.5.

Now, for $i = 1, \dots, n + 1$ we extend $R_i(u_i)(R_i^{-1}(\cdot))$ from $R_i(B_i \cap D)$ to U_i by 0 and we call the resulting vector fields \tilde{u}_i . Unlike $\|u\|_{\mathcal{X}^{s,p}(D)}$, the norm $\|u\|_{W^{s,p}(D)}$ is invariant under the rotations of u , so it is crucial that we rotate u_1, \dots, u_n at this point, so that they agree with the new coordinate system. Obviously, we have $\|u_i\|_{W^{s,p}(B_i \cap D)} \leq \|\tilde{u}_i\|_{W^{s,p}(U_i)}$ and by Lemma 2.5 the right-hand sides are finite for all i , hence $\tilde{u}_i \in W_0^{s,p}(U_i)$. By using Theorem 3.1 for $\tilde{u}_1, \dots, \tilde{u}_n$ and the Korn inequality for the whole space [6, Theorem 1.1] for \tilde{u}_{n+1} , we obtain

$$\|u\|_{W^{s,p}(D)} \lesssim \sum_{i=1}^{n+1} \|\tilde{u}_i\|_{\mathcal{X}^{s,p}(U_i)}.$$

By the definition of \tilde{u}_i and Lemmas 2.4 and 2.5, we get that for every $i = 1, \dots, n + 1$,

$$\|\tilde{u}_i\|_{\mathcal{X}^{s,p}(U_i)} \lesssim \|R_i(u_i)(R_i^{-1}(\cdot))\|_{\mathcal{X}^{s,p}(R_i(B_i \cap D))} = \|u_i\|_{\mathcal{X}^{s,p}(B_i \cap D)} \lesssim \|u\|_{\mathcal{X}^{s,p}(D)}.$$

This concludes the proof of the Korn inequality for $u \in W_0^{s,p}(D)$. The result for $\mathcal{X}_0^{s,p}(D)$ is obtained as in the last part of the proof of Theorem 3.1. \square

4. Application of the Hardy inequality

In [2, Theorem 2.3] Mengesha gives a Hardy-type inequality for the half-space \mathbb{R}_+^d , $p \geq 1$, $s \in (0, 1)$, $sp \neq 1$, and $u \in C_c^1(\mathbb{R}_+^d)$:

$$\int_{\mathbb{R}_+^d} \frac{|u(x)|^p}{x_d^{sp}} dx \lesssim |u|_{\mathcal{X}^{s,p}(\mathbb{R}_+^d)}^p.$$

In this section we give a simple framework which allows to obtain the Korn inequality for open sets D directly from its counterpart for the whole space and the Hardy inequality for D .

PROPOSITION 4.1. Let $p > 1$, $s \in (0, 1)$. Assume that the open set $D \subset \mathbb{R}^d$ admits the following Hardy inequality for $u \in (C_c^1(D))^d$:

$$\int_D \frac{|u(x)|^p}{d(x, D^c)^{sp}} dx \lesssim |u|_{\mathcal{H}^{s,p}(D)}^p.$$

Then the Korn inequality holds for D , that is, there exists $C \geq 1$ such that for $u \in (C_c^1(D))^d$,

$$C|u|_{\mathcal{H}^{s,p}(D)} \geq |u|_{W^{s,p}(D)}.$$

Proof. Let $u \in (C_c^1(D))^d$ and let \tilde{u} be the vector field u extended to the whole of \mathbb{R}^d by 0. First, by the Korn inequality for the whole space [6, Theorem 1.1] we obtain

$$|u|_{W^{s,p}(D)} \leq |\tilde{u}|_{W^{s,p}(\mathbb{R}^d)} \lesssim |\tilde{u}|_{\mathcal{H}^{s,p}(\mathbb{R}^d)}.$$

We estimate the right-hand side as follows:

$$|\tilde{u}|_{\mathcal{H}^{s,p}(\mathbb{R}^d)}^p \leq |u|_{\mathcal{H}^{s,p}(D)}^p + 2 \int_D |u(x)|^p \int_{D^c} \frac{dy}{|x-y|^{d+sp}} dx.$$

By using the polar coordinates we see that for every $x \in D$,

$$\int_{D^c} |x-y|^{-d-sp} dy \leq \int_{B(0,d(x,D^c))^c} |y|^{-d-sp} dy \lesssim d(x, D^c)^{-sp}.$$

Therefore, by the Hardy inequality we get

$$\int_D |u(x)|^p \int_{D^c} \frac{dy}{|x-y|^{d+sp}} dx \lesssim \int_D \frac{|u(x)|^p}{d(x, D^c)^{sp}} dx \lesssim |u|_{\mathcal{H}^{s,p}(D)}^p,$$

which ends the proof. \square

REMARK 4.2. Thanks to the above result, we can significantly simplify the proof of the Korn inequality for the half-space by Mengesha by omitting the discussion of the extension operator [2, Section 4.1]. If we had at our disposal the Hardy inequality for the sets discussed in Theorem 1.1, we would obtain a slightly stronger statement: $|u|_{W^{s,p}(D)} \lesssim |u|_{\mathcal{H}^{s,p}(D)}$, that is, the estimate without the L^p norm of u on the right-hand side.

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