

MULTIDIMENSIONAL WEIGHTED PÓLYA–KNOPP INEQUALITIES WITH SHARP CONSTANTS

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Abstract. Some new Pólya-Knopp inequalities in two and higher dimensions with sharp constants are stated and proved. Furthermore, some new general weighted Pólya-Knopp type inequalities in n -dimension are proved and applied.

1. Introduction

G. H. Hardy stated and proved in his 1925 paper [2] the following inequality:

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x) dx, \quad (1)$$

for $p > 1$ and where f is a positive measurable function on $(0, \infty)$. The inequality is usually called the classical Hardy inequality. Later, in 1928, he proved the following generalization of (1):

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t) dt \right)^p x^\alpha dx \leq \left(\frac{p}{p-1-\alpha} \right)^p \int_0^\infty f^p(x) x^\alpha dx, \quad (2)$$

whenever $p \geq 1$ and $\alpha < p - 1$. Moreover, the constant $\left(\frac{p}{p-1-\alpha} \right)^p$ is sharp (see [3]). After this, a lot of generalizations and complementary results have been published see e.g. the book [7] and the references therein. Next we note that by replacing $f(x)$ with $(f(x))^{1/p}$ and letting $p \rightarrow \infty$ in (2), we obtain the following Pólya-Knopp's weighted inequality

$$\int_0^\infty \exp \left(\frac{1}{x} \int_0^x \log f(t) dt \right) x^\alpha dx \leq e^{(1+\alpha)} \int_0^\infty f(x) x^\alpha dx, \quad (3)$$

for $\alpha > -1$ and f is a positive measurable function on $(0, \infty)$. Moreover, the constant $e^{(1+\alpha)}$ is sharp. Concerning the name Pólya-Knopp's inequality see our Remark 7.

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Also this inequality has been generalized, complemented and discussed in several publications, see e.g. [1], [5], [8], [9], [11] and the references given there. Such remarkable results motivate mathematician to continue the study of this type of inequalities in the higher dimensions.

In particular, the following two dimensional characterization for the general weight functions u and v was proved in [10] (see also [11]):

THEOREM 1. (See [10, Theorem 4.1]) *Let $0 < p \leq q < \infty$, and let u, v and f be positive functions on \mathbb{R}_+^2 . If $0 < b_1, b_2 \leq \infty$, then*

$$\left(\int_0^{b_1} \int_0^{b_2} \left[\exp \left(\frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} \ln f(y_1, y_2) dy_1 dy_2 \right) \right]^q u(x_1, x_2) dx_1 dx_2 \right)^{\frac{1}{q}} \leq C \left(\int_0^{b_1} \int_0^{b_2} f^p(x_1, x_2) v(x_1, x_2) dx_1 dx_2 \right)^{\frac{1}{p}} \tag{4}$$

if and only if

$$D_W(s_1, s_2, p, q) := \sup_{\substack{y_1 \in (0, b_1) \\ y_2 \in (0, b_2)}} y_1^{\frac{s_1-1}{p}} y_2^{\frac{s_2-1}{p}} \left(\int_{y_1}^{b_1} \int_{y_2}^{b_2} x_1^{-\frac{s_1 q}{p}} x_2^{-\frac{s_2 q}{p}} w(x_1, x_2) dx_1 dx_2 \right)^{\frac{1}{q}} < \infty, \tag{5}$$

where $s_1, s_2 > 1$ and

$$w(x_1, x_2) = \left[\exp \left(\frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} \ln v^{-1}(t_1, t_2) dt_1 dt_2 \right) \right]^{\frac{q}{p}} u(x_1, x_2),$$

and the best possible constant C in (4) can be estimated in the following way:

$$\sup_{s_1, s_2 > 1} \left(\frac{e^{s_1}(s_1 - 1)}{e^{s_1}(s_1 - 1) + 1} \right)^{\frac{1}{p}} \left(\frac{e^{s_2}(s_2 - 1)}{e^{s_2}(s_2 - 1) + 1} \right)^{\frac{1}{p}} D_W(s_1, s_2, p, q) \leq C \leq \inf_{s_1, s_2 > 1} e^{\frac{s_1 + s_2 - 2}{p}} D_W(s_1, s_2, p, q). \tag{6}$$

A simple calculation shows that Theorem 1 implies the following:

EXAMPLE 1. Let $\beta_i, \gamma_i > -1$ ($i = 1, 2$). Then, the inequality

$$\int_0^\infty \int_0^\infty \exp \left(\frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} \ln f(y_1, y_2) dy_1 dy_2 \right) x_1^{\beta_1} x_2^{\beta_2} dx_1 dx_2 \leq C \int_0^\infty \int_0^\infty f(x_1, x_2) x_1^{\gamma_1} x_2^{\gamma_2} dx_1 dx_2$$

holds if and only if $\beta_i = \gamma_i$ ($i = 1, 2$). This fact is obvious since in this case (5) is of the form

$$D_W(s_1, s_2, 1, 1) = e^{(\gamma_1 + \gamma_2)} \sup_{y_1, y_2 > 0} \frac{y_1^{\beta_1 - \gamma_1} y_2^{\beta_2 - \gamma_2}}{(s_1 + \gamma_1 - \beta_1 - 1)(s_2 + \gamma_2 - \beta_2 - 1)},$$

provided that $s_i > 1 + \beta_i - \gamma_i$ ($i = 1, 2$).

REMARK 1. A similar characterization of (4) for the case $b_1 = b_2 = \infty$ and $p = q = 1$, but without explicit estimates of the best constant like in (6), was earlier proved in [5].

REMARK 2. The first paper where the sharp constant in a multidimensional Pólya-Knopp inequality was discussed seems to be [8]. In particular, in [8, Theorem 2.2], the authors stated that the inequality

$$\int_0^\infty \int_0^\infty \exp\left(\frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} \ln f(y_1, y_2) dy_1 dy_2\right) x_1^a x_2^a dx_1 dx_2 \leq e^{2(1+a)} \int_0^\infty \int_0^\infty f(x_1, x_2) x_1^a x_2^a dx_1 dx_2$$

holds and that the constant is sharp. Here $a \in (0, \infty)$ and f is a positive and measurable function. This result is correct but the proof contains a minor gap, a gap which is corrected in this paper (see the proof of the more general Theorem 2).

In this paper we prove some new higher dimensional Pólya-Knopp type inequalities with sharp constants. The paper is organized as follows: In Section 2 we state and prove our two dimensional main result (see Theorem 2). In Section 3 we state and prove the corresponding general main results for dimension n , $n = 3, 4, \dots$ (see Theorem 4). An important step for this proof is to first characterize a new multidimensional Pólya-Knopp type inequality with general weights even for the case $0 < p \leq q < \infty$ (see Theorem 3). Finally, Section 4 is reserved for some concluding remarks.

Convention. Throughout this paper we assume that f is a positive and measurable function defined on $\mathbb{R}_+^n := (0, \infty)^n$.

2. The two dimensional main results

Our first main result reads:

THEOREM 2. *Let $a, b > -1$ and f be a positive measurable function defined on \mathbb{R}_+^2 . Then the inequality*

$$\int_0^\infty \int_0^\infty \exp\left(\frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} \ln f(y_1, y_2) dy_1 dy_2\right) x_1^a x_2^b dx_1 dx_2 \leq e^{2+a+b} \int_0^\infty \int_0^\infty f(x_1, x_2) x_1^a x_2^b dx_1 dx_2 \tag{7}$$

holds and the constant is sharp.

Proof. As a result of Example 1, the inequality (7) holds for some finite C . Now, we need to prove that the constant $C = e^{2+a+b}$ is sharp. From Theorem 1, it follows that the best constant C satisfies

$$C \leq e^{a+b} \inf_{s_1, s_2 > 0} \frac{e^{(s_1+s_2)}}{s_1 s_2}.$$

The infimum in the above inequality is attained at $s_i = 1$ ($i = 1, 2$). This implies that

$$C \leq e^{2+a+b}. \tag{8}$$

It only remains to prove that the inequality (8) also holds in the reversed direction. Consider the function

$$f(x_1, x_2) = \begin{cases} x_1^{-\alpha} x_2^{-\beta} : & x_1 > e, x_2 > e, \\ x_1^{-\alpha} : & x_1 > e, x_2 \leq e, \\ x_2^{-\beta} : & x_1 \leq e, x_2 > e, \\ 1 : & x_1 \leq e, x_2 \leq e \end{cases}$$

where $\alpha > a + 1$ and $\beta > b + 1$. By writing the integral $\int_0^\infty \int_0^\infty$ in the form

$$\int_0^\infty \int_0^\infty = \int_0^e \int_0^e + \int_e^\infty \int_0^e + \int_0^e \int_e^\infty + \int_e^\infty \int_e^\infty,$$

we find that the left hand side of (7) becomes

$$\frac{e^{2+a+b}}{(a+1)(b+1)} + \frac{e^{2+a+b}}{(\alpha-a-1)(b+1)} + \frac{e^{2+a+b}}{(\beta-b-1)(a+1)} + \frac{e^{2+a+b}}{(\alpha-a-1)(\beta-b-1)}$$

and the right hand side becomes

$$\frac{e^{2+a+b}}{(a+1)(b+1)} + \frac{e^{2+a+b-\alpha}}{(\alpha-a-1)(b+1)} + \frac{e^{2+a+b-\beta}}{(\beta-b-1)(a+1)} + \frac{e^{2+a+b-\alpha-\beta}}{(\alpha-a-1)(\beta-b-1)}.$$

Consequently, the inequality (7) has the form

$$\frac{e^{\alpha+\beta}}{\left[\frac{(a+1)}{\alpha} + \left(1 - \frac{a+1}{\alpha}\right) e^\alpha \right] \left[\frac{(b+1)}{\beta} + \left(1 - \frac{b+1}{\beta}\right) e^\beta \right]} \leq C.$$

By letting $\alpha \rightarrow (a+1)^+$ and $\beta \rightarrow (b+1)^+$, we find that

$$e^{2+a+b} \leq C. \tag{9}$$

The sharpness of the constant in (7) follows from (8) and (9). The proof is complete. \square

As a particular case of Theorem 2 with $a = b$, we have the following result (c.f. [8, Theorem 2.2], but there $a \in (0, \infty)$ instead of $a \in (-1, \infty)$):

COROLLARY 1. *Let $a > -1$ and f be a positive measurable function defined on \mathbb{R}_+^2 . Then the inequality*

$$\begin{aligned} \int_0^\infty \int_0^\infty \exp\left(\frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} \ln f(y_1, y_2) dy_1 dy_2\right) x_1^a x_2^a dx_1 dx_2 \\ \leq e^{2(1+a)} \int_0^\infty \int_0^\infty f(x_1, x_2) x_1^a x_2^a dx_1 dx_2 \end{aligned}$$

holds and the constant is sharp.

3. The n -dimensional main results

Throughout the paper, we use the following notations in the respective variables and parameters: For $n \geq 2$,

$$J_n = \{1, \dots, n\}, \mathbf{x} = (x_1, \dots, x_n), \mathbf{xt} = (x_1t_1, \dots, x_nt_n), I_n = [0, b_1] \times \dots \times [0, b_n],$$

$$d\mathbf{x} = dx_1 \cdots dx_n, |\boldsymbol{\alpha}| = \sum_{i=1}^n \alpha_i, \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbf{y} \leq \mathbf{x} \Leftrightarrow y_i \leq x_i \quad (i = 1, \dots, n),$$

and

$$\int_{\mathbb{R}_+^n} = \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n \text{ times}}, \int_0^{\mathbf{1}} = \underbrace{\int_0^1 \cdots \int_0^1}_{n \text{ times}}, \int_0^{\mathbf{x}} = \int_0^{x_1} \cdots \int_0^{x_n}, \int_{\mathbf{y}}^{\mathbf{b}} = \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n},$$

where $0 \leq y_i < b_i \leq \infty \quad (i = 1, \dots, n)$.

Theorem 1 can be generalized to a n -dimensional setting as follows:

THEOREM 3. *Let $0 < p \leq q < \infty$, and let u, v and f be positive functions on I_n . Then, for $n = 2, 3, \dots$,*

$$\left(\int_0^{\mathbf{b}} \left[\exp \left(\prod_{i=1}^n x_i^{-1} \int_0^{\mathbf{x}} \ln f(\mathbf{y}) d\mathbf{y} \right) \right]^q u(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_0^{\mathbf{b}} f^p(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}} \quad (10)$$

holds for some finite C if and only if for any $\alpha_i > 0 \quad (i = 1, \dots, n)$,

$$A(\boldsymbol{\alpha}) := \sup_{\substack{0 < y_i < b_i \\ i \in J_n}} \prod_{i=1}^n y_i^{\frac{\alpha_i}{p}} \left(\int_{\mathbf{y}}^{\mathbf{b}} \prod_{i=1}^n x_i^{-(1+\alpha_i)\frac{q}{p}} w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} < \infty, \quad (11)$$

where

$$w(\mathbf{x}) = u(\mathbf{x}) \left[\exp \left(\prod_{i=1}^n x_i^{-1} \int_0^{\mathbf{x}} \ln v^{-1}(\mathbf{y}) d\mathbf{y} \right) \right]^{\frac{q}{p}}.$$

Moreover, the best constant C satisfies

$$\sup_{\substack{\alpha_i > 0 \\ i \in J_n}} A(\boldsymbol{\alpha}) \prod_{i=1}^n \left(\frac{\alpha_i e^{1+\alpha_i}}{1 + \alpha_i e^{1+\alpha_i}} \right)^{\frac{1}{p}} \leq C \leq \inf_{\substack{\alpha_i > 0 \\ i \in J_n}} A(\boldsymbol{\alpha}) \exp \left(\frac{|\boldsymbol{\alpha}|}{p} \right). \quad (12)$$

Proof. Sufficiency. Let $g(\mathbf{x}) = f^p(\mathbf{x})v(\mathbf{x})$. Then the inequality (10) is equivalent to

$$\left(\int_0^{\mathbf{b}} \left[\exp \left(\prod_{i=1}^n x_i^{-1} \int_0^{\mathbf{x}} \ln g(\mathbf{y}) d\mathbf{y} \right) \right]^{\frac{q}{p}} w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_0^{\mathbf{b}} g(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}, \quad (13)$$

where

$$w(\mathbf{x}) = u(\mathbf{x}) \left[\exp \left(\prod_{i=1}^n x_i^{-1} \int_0^{\mathbf{x}} \ln v^{-1}(\mathbf{y}) d\mathbf{y} \right) \right]^{\frac{q}{p}}.$$

Let $y_i = x_i t_i$ ($i = 1, \dots, n$). Then the inequality (13) becomes

$$\left(\int_0^{\mathbf{b}} \left[\exp \left(\int_0^{\mathbf{1}} \ln g(\mathbf{x}t) dt \right) \right]^{\frac{q}{p}} w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_0^{\mathbf{b}} g(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}. \tag{14}$$

For $\alpha_i > 0$ ($i = 1, \dots, n$), we trivially have that

$$\exp(|\boldsymbol{\alpha}|) \exp \left(\int_0^{\mathbf{1}} \ln \prod_{i=1}^n t_i^{\alpha_i} dt \right) = 1. \tag{15}$$

Apply the identity (15) and then, by Jensen’s inequality, the left hand side of (14) becomes

$$\begin{aligned} & \exp \left(\frac{|\boldsymbol{\alpha}|}{p} \right) \left(\int_0^{\mathbf{b}} \left[\exp \left(\int_0^{\mathbf{1}} \ln \left(g(\mathbf{x}t) \prod_{i=1}^n t_i^{\alpha_i} \right) dt \right) \right]^{\frac{q}{p}} w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \\ & \leq \exp \left(\frac{|\boldsymbol{\alpha}|}{p} \right) \left(\int_0^{\mathbf{b}} \left(\int_0^{\mathbf{1}} g(\mathbf{x}t) \prod_{i=1}^n t_i^{\alpha_i} dt \right)^{\frac{q}{p}} w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \\ & = \exp \left(\frac{|\boldsymbol{\alpha}|}{p} \right) \left(\int_0^{\mathbf{b}} \left(\int_0^{\mathbf{x}} g(\mathbf{y}) \prod_{i=1}^n y_i^{\alpha_i} d\mathbf{y} \right)^{\frac{q}{p}} \prod_{i=1}^n x_i^{-(1+\alpha_i)\frac{q}{p}} w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}}. \end{aligned}$$

Therefore, by using Minkowski’s integral inequality for n -dimension when $p < q$ and Fubini’s theorem when $p = q$ (see [10, Remark 5.2]), the later expression is less than or equal to

$$\begin{aligned} & \exp \left(\frac{|\boldsymbol{\alpha}|}{p} \right) \left(\int_0^{\mathbf{b}} g(\mathbf{y}) \prod_{i=1}^n y_i^{\alpha_i} \left(\int_{\mathbf{y}} \prod_{i=1}^n x_i^{-(1+\alpha_i)\frac{q}{p}} w(\mathbf{x}) d\mathbf{x} \right)^{\frac{p}{q}} d\mathbf{y} \right)^{\frac{1}{q}} \\ & \leq \exp \left(\frac{|\boldsymbol{\alpha}|}{p} \right) A(\boldsymbol{\alpha}) \left(\int_0^{\mathbf{b}} g(\mathbf{y}) d\mathbf{y} \right)^{\frac{1}{p}}, \end{aligned} \tag{16}$$

so that (13) follows from (11) and (16). Since (10) is equivalent to (13), we conclude that (10) holds and the best constant C satisfies

$$C \leq \inf_{\substack{\alpha_i > 0 \\ i \in J_n}} \exp \left(\frac{|\boldsymbol{\alpha}|}{p} \right) A(\boldsymbol{\alpha}).$$

Necessity. To prove that (10), or equivalently (13), implies (11), we define the test function g by

$$g(\mathbf{x}) = \prod_{i=1}^n \left(y_i^{-1} \chi_{[0,y_i]}(x_i) + \frac{y_i^{\alpha_i} e^{-(\alpha_i+1)}}{x_i^{\alpha_i+1}} \chi_{(y_i,b_i)}(x_i) \right),$$

for fixed $y_i, 0 < y_i < b_i$ ($i = 1, \dots, n$). Then, the right hand side of (13) becomes

$$\begin{aligned} C \left(\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}} &= C \prod_{i=1}^n \left[\int_0^{b_i} \left(y_i^{-1} \chi_{[0,y_i]}(x_i) + \frac{y_i^{\alpha_i} e^{-(\alpha_i+1)}}{x_i^{\alpha_i+1}} \chi_{(y_i,b_i)}(x_i) \right) dx_i \right]^{\frac{1}{p}} \\ &= C \prod_{i=1}^n \left[1 + \frac{e^{-(1+\alpha_i)}}{\alpha_i} \left(1 - \left(\frac{y_i}{b_i} \right)^{\alpha_i} \right) \right]^{\frac{1}{p}} \\ &\leq C \prod_{i=1}^n \left(1 + \frac{e^{-(1+\alpha_i)}}{\alpha_i} \right)^{\frac{1}{p}} = C \prod_{i=1}^n \left(\frac{1 + \alpha_i e^{1+\alpha_i}}{\alpha_i e^{1+\alpha_i}} \right)^{\frac{1}{p}}. \end{aligned} \tag{17}$$

On the other hand, for $\mathbf{0} \leq \mathbf{y} < \mathbf{b}$, we have that

$$\begin{aligned} &\left(\int_{\mathbf{y}}^{\mathbf{b}} \left[\exp \left(\prod_{i=1}^n x_i^{-1} \int_0^{\mathbf{x}} \ln g(\mathbf{t}) d\mathbf{t} \right) \right]^{\frac{q}{p}} w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \\ &\leq \left(\int_{\mathbf{0}}^{\mathbf{b}} \left[\exp \left(\prod_{i=1}^n x_i^{-1} \int_0^{\mathbf{x}} \ln g(\mathbf{t}) d\mathbf{t} \right) \right]^{\frac{q}{p}} w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \end{aligned} \tag{18}$$

Moreover, for $\mathbf{y} \leq \mathbf{x} < \mathbf{b}$,

$$\begin{aligned} \int_0^{\mathbf{x}} \ln g(\mathbf{t}) d\mathbf{t} &= \sum_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n x_j \int_0^{x_i} \ln \left(y_i^{-1} \chi_{[0,y_i]}(t_i) + \frac{y_i^{\alpha_i} e^{-(\alpha_i+1)}}{t_i^{\alpha_i+1}} \chi_{(y_i,b_i)}(t_i) \right) dt_i \\ &= \left(\prod_{j=1}^n x_j \right) \sum_{i=1}^n \ln \frac{y_i^{\alpha_i}}{x_i^{(\alpha_i+1)}}. \end{aligned} \tag{19}$$

It follows from (13) and (17)–(19) that

$$\prod_{i=1}^n y_i^{\frac{\alpha_i}{p}} \left(\int_{\mathbf{y}}^{\mathbf{b}} \prod_{i=1}^n x_i^{-(1+\alpha_i)\frac{q}{p}} w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \prod_{i=1}^n \left(\frac{1 + \alpha_i e^{1+\alpha_i}}{\alpha_i e^{1+\alpha_i}} \right)^{\frac{1}{p}}.$$

Therefore,

$$A(\boldsymbol{\alpha}) = \sup_{\substack{0 < y_i < b_i \\ i \in J_n}} \prod_{i=1}^n y_i^{\frac{\alpha_i}{p}} \left(\int_{\mathbf{y}}^{\mathbf{b}} \prod_{i=1}^n x_i^{-(1+\alpha_i)\frac{q}{p}} w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \prod_{i=1}^n \left(\frac{1 + \alpha_i e^{1+\alpha_i}}{\alpha_i e^{1+\alpha_i}} \right)^{\frac{1}{p}} < \infty.$$

We conclude that (10) and (11) hold and that the sharp constant C satisfies (12). The proof is complete. \square

Theorem 3 implies the following inequality for homogeneous power weights:

EXAMPLE 2. Let $0 < p \leq q < \infty$, let $\beta_i, \gamma_i > -1$ ($i = 1, \dots, n$). Then, the inequality

$$\left(\int_0^{\mathbf{b}} \left[\exp \left(\prod_{i=1}^n x_i^{-1} \int_0^{\mathbf{x}} \ln f(\mathbf{y}) \, d\mathbf{y} \right) \right]^q \prod_{i=1}^n x_i^{\beta_i} \, d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_0^{\mathbf{b}} f^p(\mathbf{x}) \prod_{i=1}^n x_i^{\gamma_i} \, d\mathbf{x} \right)^{\frac{1}{p}} \tag{20}$$

holds if and only if

$$\frac{1 + \beta_i}{q} = \frac{1 + \gamma_i}{p}, \tag{21}$$

for all $i = 1, \dots, n$. This fact is obvious since in this case (11) is of the form

$$\begin{aligned} A(\boldsymbol{\alpha}) &= \exp \left(\frac{|\boldsymbol{\gamma}|}{p} \right) \sup_{\substack{y_i \in (0, b_i) \\ i \in J_n}} \prod_{i=1}^n \frac{y_i^{\frac{1+\beta_i}{q} - \frac{1+\gamma_i}{p}}}{\left[(1 + \alpha_i + \gamma_i) \frac{q}{p} - 1 - \beta_i \right]^{\frac{1}{q}}} \left[1 - \left(\frac{y_i}{b_i} \right)^{(1+\alpha_i+\gamma_i)\frac{q}{p} - 1 - \beta_i} \right]^{\frac{1}{q}} \\ &\leq \exp \left(\frac{|\boldsymbol{\gamma}|}{p} \right) \sup_{\substack{y_i \in (0, b_i) \\ i \in J_n}} \prod_{i=1}^n \frac{y_i^{\frac{1+\beta_i}{q} - \frac{1+\gamma_i}{p}}}{\left((1 + \alpha_i + \gamma_i) \frac{q}{p} - (1 + \beta_i) \right)^{\frac{1}{q}}}, \end{aligned}$$

provided that $\alpha_i > \frac{p}{q}(1 + \beta_i) - (1 + \gamma_i)$ ($i = 1, \dots, n$).

Moreover, we state and prove the following n -dimensional version of Theorem 2:

THEOREM 4. Let $n = 2, 3, \dots$, let $\gamma_i > -1$ ($i = 1, \dots, n$), and let f be a positive measurable function defined on \mathbb{R}_+^n . Then the inequality

$$\int_{\mathbb{R}_+^n} \exp \left(\prod_{i=1}^n x_i^{-1} \int_0^{\mathbf{x}} \ln f(\mathbf{y}) \, d\mathbf{y} \right) \prod_{i=1}^n x_i^{\gamma_i} \, d\mathbf{x} \leq \exp(n + |\boldsymbol{\gamma}|) \int_{\mathbb{R}_+^n} f(\mathbf{x}) \prod_{i=1}^n x_i^{\gamma_i} \, d\mathbf{x} \tag{22}$$

holds and the constant is sharp.

Proof. In view of Example 2, the inequality (22) holds. Now, we prove that the best constant $C = \exp(n + |\boldsymbol{\gamma}|)$. From Theorem 3, it follows that the best constant C satisfies

$$C \leq \exp(|\boldsymbol{\gamma}|) \inf_{\substack{\alpha_i > 0 \\ i \in J_n}} \left(\prod_{i=1}^n \frac{e^{\alpha_i}}{\alpha_i} \right).$$

The infimum in the above inequality is attained at $\alpha_i = 1$ ($i = 1, \dots, n$). Hence,

$$C \leq \exp(n + |\boldsymbol{\gamma}|). \tag{23}$$

It only remains to prove that the inequality (23) also holds in the reversed direction. Consider the function

$$f(\mathbf{x}) = \prod_{i=1}^n \left(\chi_{[0,e]}(x_i) + x_i^{-\beta_i} \chi_{(e,\infty)}(x_i) \right),$$

where $\beta_i > \gamma_i + 1$ ($i = 1, \dots, n$). Then, the integral part of the right hand side of (22) becomes

$$\begin{aligned} \int_{\mathbb{R}_+^n} f(\mathbf{x}) \prod_{i=1}^n x_i^{\gamma_i} \, d\mathbf{x} &= \int_{\mathbb{R}_+^n} \prod_{i=1}^n x_i^{\gamma_i} \left(\chi_{[0,e]}(x_i) + x_i^{-\beta_i} \chi_{(e,\infty)}(x_i) \right) \, d\mathbf{x} \\ &= \prod_{i=1}^n \int_{\mathbb{R}_+} x_i^{\gamma_i} \left(\chi_{[0,e]}(x_i) + x_i^{-\beta_i} \chi_{(e,\infty)}(x_i) \right) \, dx_i \\ &= \prod_{i=1}^n e^{(\gamma_i+1)} \left(\frac{1}{\gamma_i + 1} + \frac{e^{-\beta_i}}{\beta_i - \gamma_i - 1} \right) \end{aligned} \tag{24}$$

and the left hand side of (22) becomes

$$\begin{aligned} \int_{\mathbb{R}_+^n} \exp \left(\prod_{i=1}^n x_i^{-1} \int_0^{\mathbf{x}} \ln f(\mathbf{y}) \, d\mathbf{y} \right) \prod_{i=1}^n x_i^{\gamma_i} \, d\mathbf{x} \\ &= \int_{\mathbb{R}_+^n} \exp \left(\prod_{j=1}^n x_j^{-1} \int_0^{\mathbf{x}} \ln \prod_{i=1}^n \left(\chi_{[0,e]}(y_i) + y_i^{-\beta_i} \chi_{(e,\infty)}(y_i) \right) \, d\mathbf{y} \right) \prod_{i=1}^n x_i^{\gamma_i} \, d\mathbf{x} \\ &= \int_{\mathbb{R}_+^n} \prod_{i=1}^n x_i^{\gamma_i} \exp \left(x_i^{-1} \int_0^{x_i} \ln \left(\chi_{[0,e]}(y_i) + y_i^{-\beta_i} \chi_{(e,\infty)}(y_i) \right) \, dy_i \right) \, d\mathbf{x} \\ &= \prod_{i=1}^n \int_{\mathbb{R}_+} x_i^{\gamma_i} \exp \left(x_i^{-1} \int_0^{x_i} \ln \left(\chi_{[0,e]}(y_i) + y_i^{-\beta_i} \chi_{(e,\infty)}(y_i) \right) \, dy_i \right) \, dx_i \\ &= \prod_{i=1}^n e^{(\gamma_i+1)} \left(\frac{1}{\gamma_i + 1} + \frac{1}{\beta_i - \gamma_i - 1} \right). \end{aligned} \tag{25}$$

It follows from (22), (24) and (25) that

$$\prod_{i=1}^n \frac{e^{\beta_i}}{\left(\frac{\gamma_i+1}{\beta_i} + \left(1 - \frac{\gamma_i+1}{\beta_i} \right) e^{\beta_i} \right)} \leq C.$$

By letting $\beta_i \rightarrow (\gamma_i + 1)^+$ ($i = 1, \dots, n$), we find that

$$\exp \left(n + \sum_{i=1}^n \gamma_i \right) \leq C. \tag{26}$$

Therefore, the sharpness of the constant in (22) follows by combining (23) and (26). The proof is complete. \square

REMARK 3. Note that for $n = 2$ Theorem 4 coincides with Theorem 2. In fact, just put $n = 2$, $\gamma_1 = a$ and $\gamma_2 = b$ in Theorem 4 to confirm this fact.

As a particular case of Theorem 4 with $\gamma_i = a$ ($i = 1, \dots, n$), we have the following result:

COROLLARY 2. Let $n = 2, 3, \dots$, let $a > -1$ and let f be a positive measurable function defined on \mathbb{R}_+^n . Then the inequality

$$\int_{\mathbb{R}_+^n} \exp\left(\prod_{i=1}^n x_i^{-1} \int_0^{\mathbf{x}} \ln f(\mathbf{y}) d\mathbf{y}\right) \prod_{i=1}^n x_i^a d\mathbf{x} \leq e^{n(1+a)} \int_{\mathbb{R}_+^n} f(\mathbf{x}) \prod_{i=1}^n x_i^a d\mathbf{x}$$

holds and the constant is sharp.

Finally, we state the following consequence of Theorem 4 for another type of homogeneous weight.

COROLLARY 3. Let $k \in \mathbb{N}$ and $a \in \mathbb{R}$ such that $ak > -1$, and let f be a positive function defined on \mathbb{R}_+^n ($n = 2, 3, \dots$). Then the inequality

$$\int_{\mathbb{R}_+^n} \exp\left(\prod_{i=1}^n x_i^{-1} \int_0^{\mathbf{x}} \ln f(\mathbf{y}) d\mathbf{y}\right) \left(\sum_{i=1}^n x_i^a\right)^k d\mathbf{x} \leq e^{(n+ak)} \int_{\mathbb{R}_+^n} f(\mathbf{x}) \left(\sum_{i=1}^n x_i^a\right)^k d\mathbf{x} \tag{27}$$

holds and the constant is sharp.

Proof. Let $k \in \mathbb{N}$ and $a \in \mathbb{R}$ such that $ak > -1$. Then, by applying the multinomial theorem and Theorem 4, we obtain that

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \exp\left(\prod_{i=1}^n x_i^{-1} \int_0^{\mathbf{x}} \ln f(\mathbf{y}) d\mathbf{y}\right) \left(\sum_{i=1}^n x_i^a\right)^k d\mathbf{x} \\ &= \sum_{\substack{m_1 + \dots + m_n = k \\ m_i \in \mathbb{N}_0}} \binom{k}{m_1, \dots, m_n} \int_{\mathbb{R}_+^n} \exp\left(\prod_{i=1}^n x_i^{-1} \int_0^{\mathbf{x}} \ln f(\mathbf{y}) d\mathbf{y}\right) \prod_{i=1}^n x_i^{am_i} d\mathbf{x}, \\ &\leq \sum_{\substack{m_1 + \dots + m_n = k \\ m_i \in \mathbb{N}_0}} \binom{k}{m_1, \dots, m_n} \exp\left(n + a \sum_{i=1}^n m_i\right) \int_{\mathbb{R}_+^n} f(\mathbf{x}) \prod_{i=1}^n x_i^{am_i} d\mathbf{x} \\ &= e^{(n+ak)} \int_{\mathbb{R}_+^n} f(\mathbf{x}) \left(\sum_{i=1}^n x_i^a\right)^k d\mathbf{x}, \end{aligned}$$

where

$$\binom{k}{m_1, \dots, m_n} = \frac{k!}{m_1! \dots m_n!}$$

is a multinomial coefficient. Since the constant in Theorem 4 is sharp, then the sharpness of the constant $e^{(n+ak)}$ in (27) is guaranteed so the proof is complete. \square

4. Concluding remarks

REMARK 4. The application pointed out in Corollary 3 is not unique. In fact, Theorem 4 can be used to derive sharp inequalities of Pólya-Knopp type also for other homogeneous weights.

REMARK 5. By using a similar approach as that in Theorems 2 and 4, we can prove the sharpness of the constant $e^{\alpha+1}$ in the inequality (3).

REMARK 6. Let $0 < p \leq q < \infty$. If the condition (21) in Example 2 holds, then the best constant C satisfies

$$C \leq \exp\left(\frac{n}{q} + \frac{|\gamma|}{p}\right).$$

In particular, if $p = q$ and $b_i = \infty$ ($i = 1, \dots, n$), then by replacing $f(x)$ by $f^p(x)$ in Theorem 4, we obtain that the best constant

$$C = \exp\left(\frac{n + |\gamma|}{p}\right).$$

OPEN QUESTION 1. To find the best constant for the case $0 < p < q < \infty$ is an open question, which seems to be open even for $n = 1$.

REMARK 7. Concerning the name Pólya-Knopp's inequality for inequality (3) with $\alpha = 0$: some author's referred to this inequality as 'Knopp's inequality' with reference to his 1928 paper [6]. It was later on discovered that Hardy in his famous 1925 paper [2] pointed out the inequality as a limit inequality of his inequality but that it was his friend Pólya which informed him about this fact. In this connection we also refer to the first really important book [4] with main focus on inequalities.

OPEN QUESTION 2. In view of Remark 7 and the limit argument pointed out in our introduction it seems of interest to try to characterize (10) as a limit inequality of a corresponding n -dimensional Hardy inequality (which exists). For the case $n = 1$ this problem was solved in [9].

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