

INEQUALITIES FOR MEROMORPHIC UNIVALENT FUNCTIONS WITH NONZERO POLE

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(Communicated by I. Perić)

Abstract. In this article, we obtain the Grunsky inequality and its several consequences for meromorphic univalent functions defined on the unit disk with a nonzero pole $p \in (0, 1)$. As byproducts, we obtain the Goluzin and the Lebedev inequalities for these functions. We also obtain the Grunsky inequality for a subclass of aforesaid functions that have k -quasiconformal extensions onto the extended complex plane.

1. Introduction

Let \mathbb{C} be the finite complex plane, $\widehat{\mathbb{C}}$ be the extended complex plane $\mathbb{C} \cup \{\infty\}$ and \mathcal{S} be the class of analytic univalent functions f defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ satisfying $f(0) = 0 = f'(0) - 1$. Thus each $f \in \mathcal{S}$ has the following Taylor series expansion about the origin

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \quad (1)$$

In 1916, Bieberbach proved that if $f \in \mathcal{S}$ having the Taylor series expansion of the form (1), then $|a_2| \leq 2$, where equality holds only for the Koebe function $k(z) = z/(1 - z)^2$, $z \in \mathbb{D}$ and at the same time he conjectured that $|a_n| \leq n$ for $n \geq 3$. This conjecture remains unsolved for the whole class \mathcal{S} until L. de Branges settled it in 1985. We now introduce another class of functions Σ which is related to the class \mathcal{S} . The class Σ consists of meromorphic univalent functions defined on \mathbb{D} , such that each function $g \in \Sigma$ is analytic in \mathbb{D} except for a simple pole at the origin with residue 1, satisfying $zg(z)|_{z=0} = 1$. The functions in Σ naturally have the following Laurent series expansion

$$g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D} \setminus \{0\}. \quad (2)$$

The Gronwall area theorem states that if $f \in \Sigma$ with expansion (2), then $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$. The generalization of the proof of this theorem leads to a system of inequalities called

Mathematics subject classification (2020): 30C55, 30C62.

Keywords and phrases: Grunsky inequality, Goluzin inequality, meromorphic functions, quasiconformal extension.

The authors would like to thank CSIR, India (Ref. No. 25(0281)/18/EMR-II) for its financial support.

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the *Grunsky inequalities* which are both necessary and sufficient for univalence of the associated function. These inequalities were discovered by H. Grunsky in 1939 and later it was generalized and applied by Milin, FitzGerald, Garabedian, Schiffer and many others to varieties of problems. We urge the readers to go through the Chapters 3 and 4 of [17] and Chapters 4 and 5 of [7] for a detailed study on this area of research. The Grunsky inequalities also played an important role to prove the Bieberbach conjecture for some initial coefficients of functions belonging to \mathcal{S} . Besides that, these inequalities have useful applications to prove many results such as growth and distortion results, estimates for the logarithmic coefficients for the functions belonging to the classes \mathcal{S} and Σ .

For a function $g \in \Sigma$ with an expansion of the form (2), we see that

$$\log \frac{g(z) - g(\xi)}{1/z - 1/\xi} = - \sum_{m,n=1}^{\infty} b_{mn} z^m \xi^n,$$

is analytic in $(z, \xi) \in \mathbb{D} \times \mathbb{D}$. The coefficients b_{mn} are called *Grunsky coefficients* of g . Now, if $g \in \Sigma$ and $\lambda = (\lambda_1, \lambda_2, \dots) \in l^2$, where l^2 is the space of all square summable sequences on \mathbb{C} with finite l^2 -norm denoted by $\|\cdot\|_2$, then the Grunsky coefficients of g satisfy the following inequality

$$\left| \sum_{m,n=1}^{\infty} \sqrt{mn} b_{mn} \lambda_m \lambda_n \right| \leq \|\lambda\|_2^2, \tag{3}$$

which is popularly known as the Grunsky inequality (cf. [9]).

We now move on to describe our work in this article. To this end, we need to discuss about few classes of functions and some new definitions. Let $\mathcal{M}(p)$ be the class of all meromorphic functions defined on \mathbb{D} with nonzero pole at the point $p \in (0, 1)$ of residue 1 and $\Sigma(p)$ be set of all univalent functions in $\mathcal{M}(p)$. We emphasise here that, considering pole at some nonzero point $p \in (0, 1)$ of a meromorphic univalent function does not only change its normalization but provides us with a Taylor series expansion of the same function inside the disk $\{z \in \mathbb{C} : |z| < p\}$ along with its other possible Laurent expansions inside the unit disk \mathbb{D} , namely, in the annuli $0 < |z| < 1$ and $0 < |z - p| < 1 - p$. Another reason to study such functions is that the presence of nonzero poles force non trivial lower bounds for the Taylor coefficients of a subclass of $\Sigma(p)$, namely the class of *concave univalent functions*. This phenomenon is very uncommon for univalent functions. We refer the articles [1, 2, 15, 16] and the references therein for more details. Inspired by these reasons, we intend to further study such functions. Let $g \in \Sigma(p)$ and g has the Laurent expansion of the form

$$g(z) = \frac{1}{z - p} + \sum_{n=0}^{\infty} b_n (z - p)^n, \quad z \in \mathbb{D}_{p,1-p} \setminus \{p\}, \tag{4}$$

where $\mathbb{D}_{p,1-p} := \{z \in \mathbb{C} : |z - p| < 1 - p\}$. One can deduce that for a function $g \in \Sigma(p)$, with the above expansion, the analog of the Gronwall’s area theorem for functions in the class $\Sigma(p)$ can be derived as

$$\sum_{n=1}^{\infty} (1 - p)^{2n} n |b_n|^2 \leq (1 - p)^{-2}. \tag{5}$$

We mention here that, the above inequality yields from the following computation

$$\text{Area}(\widehat{\mathbb{C}} \setminus g(\mathbb{D}_{p,1-p})) = \pi \left((1-p)^{-2} - \sum_{n=1}^{\infty} n|b_n|^2(1-p)^{2n} \right),$$

and noticing that the area of a region in the complex plane is always nonnegative. Likewise the generalizations of the proof of the Gronwall’s Area theorem for the class Σ , we wish to generalize the proof of (5) to establish the Grunsky inequality for functions in the class $\Sigma(p)$. This result subsequently also yields the *Goluzin* and the *Lebedev inequalities* and some of their important consequences.

To describe our next problem, we first need to prepare a little background. To this end, we first start with the definition of a quasiconformal mapping. A sense preserving homeomorphism $f : \mathbb{C} \rightarrow \mathbb{C}$ is called *k-quasiconformal mapping* (or, *K-quasiconformal*, where $K = (1+k)/(1-k)$) if it belongs to the Sobolev space $W_{loc}^{1,2}(\mathbb{C})$ and satisfies the inequality

$$|f_{\bar{z}}| \leq k|f_z|, \text{ a.e. on } \mathbb{C},$$

for some $k \in [0, 1)$ (or, $K \geq 1$), where $f_{\bar{z}} := \partial f / \partial \bar{z}$ and $f_z := \partial f / \partial z$. We can extend the above definition for a quasiconformal map $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, where it satisfies the inequality $|f_{\bar{z}}| \leq k|f_z|$, a.e. on $\mathbb{C} \setminus \{f^{-1}(\infty)\}$. Next, we discuss about the concept of quasiconformal extension of a conformal map. Let f be a conformal map defined on a domain $\Omega \subset \mathbb{C}$. We say that $f : \Omega \rightarrow \mathbb{C}$ has a *k-quasiconformal extension* onto \mathbb{C} or onto $\widehat{\mathbb{C}}$, if there exists a *k-quasiconformal mapping* $F : \mathbb{C} \rightarrow \mathbb{C}$, or $F : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ respectively, such that $F|_{\Omega} = f$. Normalized holomorphic maps defined on \mathbb{D} having quasiconformal extension onto $\widehat{\mathbb{C}}$ play an important role in the Teichmüller space theory as they can be identified with the elements of the universal Teichmüller space (compare [14, chap. III]) and so it is important to study the conformal maps defined on \mathbb{D} or its exterior $\mathbb{D}^* := \{z \in \mathbb{C} : |z| > 1\}$ which have quasiconformal extensions onto $\widehat{\mathbb{C}}$.

Let \mathcal{S}_k be the class of functions in \mathcal{S} which have *k-quasiconformal extension* onto $\widehat{\mathbb{C}}$. Similarly, let Σ_k be the class of functions in Σ which have *k-quasiconformal extensions* onto $\widehat{\mathbb{C}}$. The function classes \mathcal{S}_k and Σ_k are well studied by the authors Kühnau ([11, 12]), Krushkal ([8]), Krzyż ([10]), Lehto ([13]) and others. Getting motivated by the works of these authors, T. Sugawa along with the authors of the present article, have defined (cf. [3]) a class of meromorphic univalent functions defined on the unit disk \mathbb{D} with a nonzero pole at the point $p \in (0, 1)$ having *k-quasiconformal extension* onto the extended complex plane $\widehat{\mathbb{C}}$. We denote this class of functions by $\Sigma_k(p)$. We further have studied area theorem, coefficient estimates, area distortion inequalities, distortion estimate for functions in this class. Interested readers are urged to look into the articles [3, 4, 5] for the said results. We observe that if $g \in \Sigma_k$, then the Grunsky inequality takes the form

$$\left| \sum_{m,n=1}^{\infty} \sqrt{mn} b_{mn} \lambda_m \lambda_n \right| \leq k \|\lambda\|_2^2.$$

We refer [17, Chap. 9.4] for more details about this inequality.

We organize the obtained results in this article as follows. In the next section, we first prove an inequality involving the Grunsky coefficients (content of the Theorem 1) which will essentially lead us in establishing the Grunsky inequality (content of the Theorem 2) for functions in $\Sigma(p)$. In Theorem 3 and Theorem 4, we obtain the Goluzin and the Lebedev inequalities for functions in $\Sigma(p)$ respectively, and derive some consequences of the obtained results as corollaries. Lastly, we find the Grunsky inequality for functions in the class $\Sigma_k(p)$. This is the content of the Theorem 5.

2. Main results

Firstly, we prove an inequality for $\Sigma(p)$ which will lead us to get the Grunsky inequality for functions belonging in this class.

THEOREM 1. *Let $g \in \Sigma(p)$ having an expansion of the form (4). Let $\lambda_1, \lambda_2, \dots, \lambda_l$ are arbitrary complex numbers, then*

$$\sum_{m=1}^{\infty} m \left| \sum_{n=1}^l \lambda_n b_{mn} \right|^2 (1-p)^{2m} \leq \sum_{m=1}^l (|\lambda_m|^2 / m) (1-p)^{-2m}, \tag{6}$$

where b_{mn} 's are the Grunsky coefficients of g that are defined by

$$F(z, \xi) := \log \left(\frac{g(z) - g(\xi)}{1/(z-p) - 1/(\xi-p)} \right) = - \sum_{m,n=1}^{\infty} b_{mn} (z-p)^m (\xi-p)^n, \tag{7}$$

where $z, \xi \in \mathbb{D}_{p,1-p} \setminus \{p\}$.

Proof. Since $g \in \Sigma(p)$, we have $g(p) = \infty$ and $(z-p)g(z)|_{z=p} = 1$. For a fixed $w \in \mathbb{C}$, we define the function

$$F(z) := \log[(z-p)(g(z) - w)], \quad z \in \mathbb{D}. \tag{8}$$

Here, we choose a suitable branch of logarithm for which $F(p) = 0$, so that F becomes analytic on \mathbb{D} for a fixed $w \neq g(z)$ and $|F'(p)| < \infty$. Thus we can consider the Taylor series expansion of F about the point p of the following form

$$F(z) = \log[(z-p)(g(z) - w)] = \sum_{m=1}^{\infty} \frac{\phi_m(w)}{m} (z-p)^m, \quad z \in \mathbb{D}.$$

Differentiating both sides of the above equation w.r.t. z , we get

$$\frac{g'(z)}{g(z) - w} + \frac{1}{z-p} = \sum_{m=1}^{\infty} \phi_m(w) (z-p)^{m-1}, \quad \text{which gives}$$

$$(z-p)g'(z) = (g(z) - w) \left(\sum_{m=0}^{\infty} \phi_m(w) (z-p)^m \right), \quad \text{where } \phi_0(w) = -1.$$

Now, putting the expansion of g from (4) into the above equation, we have

$$\frac{1}{z-p} + \sum_{m=1}^{\infty} mb_m(z-p)^m = \left(\frac{1}{z-p} + b_0 - w + \sum_{m=1}^{\infty} b_m(z-p)^m \right) \left(\sum_{m=0}^{\infty} \phi_m(w)(z-p)^m \right).$$

Equating the coefficients of $(z-p)^m$ in both sides of the above equation, we get

$$\begin{aligned} \phi_{m+1}(w) = & -(b_0 - w)\phi_m(w) - (b_1\phi_{m-1}(w) + b_2\phi_{m-2}(w) + \dots + b_{m-1}\phi_1(w)) \\ & + (m+1)b_m, \quad m = 1, 2, \dots, \text{ which implies} \end{aligned}$$

$$\begin{aligned} \phi_1(w) = b_0 - w, \quad \phi_2(w) = -(b_0 - w)^2 + 2b_1, \quad \phi_3(w) = (b_0 - w)^3 - 3b_1(b_0 - w) + 3b_2, \\ \phi_4(w) = -(b_0 - w)^4 + 4b_1(b_0 - w)^2 - 4b_2(b_0 - w) + 4b_3 - 2b_1^2, \quad \text{and so on.} \end{aligned}$$

Thus ϕ_m is a polynomial in w of degree m , and we call it as the *Faber Polynomial* of a function $g \in \Sigma(p)$. For $g \in \Sigma(p)$ and $z \neq \xi$, we define an analytic function $F(z, \xi)$ of two variables $(z, \xi) \in \mathbb{D} \times \mathbb{D}$ as below

$$F(z, \xi) := \log \left(\frac{g(z) - g(\xi)}{1/(z-p) - 1/(\xi-p)} \right).$$

Here, we note that due to the normalization of g , $F(z, \xi) = 0$ when $z = \xi$. Since g is of the form (4), $F(z, \xi)$ has an expansion as (7). The Grunsky coefficients b_{mn} can be expressed in terms of the coefficients b_n 's of g . In particular, we have $b_{m1} = b_m$, $b_{1n} = b_n$ and $b_{mn} = b_{nm}$ for all $m, n = 1, 2, \dots$. Now putting $w = g(\xi)$, $\xi \neq z$, in (8), we get

$$\log \left(\frac{g(z) - g(\xi)}{1/(z-p) - 1/(\xi-p)} \right) + \log \left(1 - \frac{z-p}{\xi-p} \right) = \sum_{m=1}^{\infty} \frac{\phi_m(g(\xi))}{m} (z-p)^m,$$

which implies

$$\log \left(\frac{g(z) - g(\xi)}{1/(z-p) - 1/(\xi-p)} \right) = \sum_{m=1}^{\infty} (1/m)[\phi_m(g(\xi)) + (\xi-p)^{-m}](z-p)^m.$$

Hence from (7), for $z, \xi \in \mathbb{D}_{p,1-p} \setminus \{p\}$, we have

$$- \sum_{m,n=1}^{\infty} b_{mn}(z-p)^m(\xi-p)^n = \sum_{m=1}^{\infty} (1/m)[\phi_m(g(\xi)) + (\xi-p)^{-m}](z-p)^m.$$

Equating the coefficients of $(z-p)^m$ in the both sides of the above equation, we have

$$- \sum_{n=1}^{\infty} b_{mn}(\xi-p)^n = (1/m)[\phi_m(g(\xi)) + (\xi-p)^{-m}].$$

Thus

$$\phi_m(g(\xi)) = -(\xi-p)^{-m} - m \sum_{n=1}^{\infty} b_{mn}(\xi-p)^n. \tag{9}$$

We now consider a function

$$h(w) := \sum_{m=1}^l (\lambda_m/m)\phi_m(w), \quad w \in \mathbb{C}. \tag{10}$$

Since ϕ_m 's are polynomial in w , h is an analytic function of w . Let us also consider the function $\phi(z) := h(g(z))$, $z \in \mathbb{D}_{p,1-p} \setminus \{p\}$, where we change the variable ξ to z . Now, from (9) and using the fact that $b_{mn} = b_{nm}$, we have

$$\begin{aligned} \phi(z) &= \sum_{m=1}^l (\lambda_m/m) \left(-(z-p)^{-m} - m \sum_{n=1}^{\infty} b_{mn}(z-p)^n \right) \\ &= - \sum_{m=1}^l (\lambda_m/m)(z-p)^{-m} - \sum_{m=1}^l \lambda_m \sum_{n=1}^{\infty} b_{mn}(z-p)^n \\ &= - \sum_{m=1}^l (\lambda_m/m)(z-p)^{-m} - \sum_{n=1}^{\infty} \lambda_n \sum_{m=1}^l b_{nm}(z-p)^m \text{ (interchanging } m \text{ and } n) \\ &= - \sum_{m=1}^l (\lambda_m/m)(z-p)^{-m} - \sum_{m=1}^{\infty} d_m(z-p)^m, \quad \text{where } d_m := \sum_{n=1}^l \lambda_n b_{mn}. \end{aligned} \tag{11}$$

Next we find the area of the domain $\widehat{\mathbb{C}} \setminus \phi(\mathbb{D}_{p,1-p})$. To do so, we first derive area of the domain $\widehat{\mathbb{C}} \setminus \phi(\mathbb{D}_{p,r})$, where $\mathbb{D}_{p,r} := \{z : |z-p| < r\}$ ($0 < r < 1$), and then take $r \rightarrow (1-p)^-$. Also let $C(p,r) := g(\{z : |z-p| = r\})$ and $H(p,r)$ be the domain bounded by $C(p,r)$ with finite area. By the symbols ' \curvearrowright ' and ' \curvearrowleft ', we respectively denote orientation of a curve along anticlockwise and clockwise directions. Using the Green's formula for the plane, we have

$$\begin{aligned} \text{Area}(\widehat{\mathbb{C}} \setminus \phi(\mathbb{D}_{p,r})) &= \iint_{H(p,r)} |h'(w)|^2 dudv, \quad \text{where } w = g(z) = u + iv \\ &= (1/2i) \int_{C_{p,r} \curvearrowright} \overline{h(w)} h'(w) dw \\ &= -(1/2i) \int_{|z-p|=r \curvearrowleft} \overline{\phi(z)} \phi'(z) dz \\ &= (1/2i) \int_{|z-p|=r \curvearrowleft} \left(\sum_{m=1}^l (\overline{\lambda_m}/m)(\bar{z}-p)^{-m} + \sum_{m=1}^{\infty} \overline{d_m}(\bar{z}-p)^m \right) \\ &\quad \times \left(\sum_{m=1}^l \lambda_m(z-p)^{-m-1} - \sum_{m=1}^{\infty} m d_m(z-p)^{m-1} \right) dz \\ &= (1/2i) \int_{|t|=r \curvearrowleft} \left(\sum_{m=1}^l (\overline{\lambda_m}/m)(\bar{t})^{-m} + \sum_{m=1}^{\infty} \overline{d_m}(\bar{t})^m \right) \\ &\quad \times \left(\sum_{m=1}^l \lambda_m t^{-m-1} - \sum_{m=1}^{\infty} m d_m t^{m-1} \right) dt, \quad \text{where } t = z-p. \end{aligned}$$

Now using the following computation

$$\int_{|t|=r} (\bar{t})^{-m} t^{-n-1} dt = \begin{cases} 2\pi i r^{-2n}, & m = n \\ 0, & m \neq n, \end{cases} \tag{12}$$

where m, n are integers, we get from above

$$\text{Area}(\widehat{\mathbb{C}} \setminus \phi(\mathbb{D}_{p,r})) = \pi \left[\sum_{m=1}^l (|\lambda_m|^2/m) r^{-2m} - \sum_{m=1}^{\infty} m |d_m|^2 r^{2m} \right].$$

Taking $r \rightarrow (1-p)^-$, we have

$$\text{Area}(\widehat{\mathbb{C}} \setminus \phi(\mathbb{D}_{p,1-p})) = \pi \left[\sum_{m=1}^l (|\lambda_m|^2/m) (1-p)^{-2m} - \sum_{m=1}^{\infty} m |d_m|^2 (1-p)^{2m} \right]. \tag{13}$$

Since area of a region is always nonnegative, we get

$$\sum_{m=1}^{\infty} m |d_m|^2 (1-p)^{2m} \leq \sum_{m=1}^l (|\lambda_m|^2/m) (1-p)^{-2m}.$$

Now putting the value of d_m , we get (6). \square

Next we deduce the Grunsky inequality for meromorphic functions with nonzero pole.

THEOREM 2. *If $g \in \Sigma(p)$ is of the form (4) and $\lambda_m \in \mathbb{C}$, $m = 1, 2, \dots$ be arbitrary complex numbers, then*

$$\sum_{m=1}^{\infty} m \left| \sum_{n=1}^{\infty} \lambda_n b_{mn} \right|^2 (1-p)^{2m} \leq \sum_{m=1}^{\infty} (|\lambda_m|^2/m) (1-p)^{-2m}, \text{ and} \tag{14}$$

$$\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \lambda_m \lambda_n \right| \leq \sum_{m=1}^{\infty} (|\lambda_m|^2/m) (1-p)^{-2m}, \tag{15}$$

where b_{mn} 's are the Grunsky coefficients of g that are defined in (7), provided the series in the right hand side of the above inequalities is convergent.

Proof. Taking $l \rightarrow \infty$ in (6), we get (14). We only need to show that the series $\sum_{n=1}^{\infty} \lambda_n b_{mn}$ is absolutely convergent, whenever the series at the right hand side of (14) is convergent. Taking $\lambda_1 = \lambda_2 = \dots = \lambda_{l-1} = 0$, and $\lambda_l = 1$, we get from (6) that

$$\sum_{m=1}^{\infty} m |b_{ml}|^2 (1-p)^{2m} \leq (1-p)^{-2l}/l.$$

Using the above inequality and the Cauchy-Schwarz inequality, and also noting that $b_{mn} = b_{nm}$ for all $m, n = 1, 2, \dots$, we get

$$\begin{aligned} \sum_{n=1}^{\infty} |\lambda_n b_{mn}| &= \sum_{n=1}^{\infty} |\sqrt{n} b_{mn} (1-p)^n| |(\lambda_n / \sqrt{n})(1-p)^{-n}| \\ &\leq \left(\sum_{n=1}^{\infty} n |b_{mn}|^2 (1-p)^{2n} \right)^{1/2} \left(\sum_{n=1}^{\infty} (|\lambda_n|^2 / n) (1-p)^{-2n} \right)^{1/2} \\ &\leq ((1-p)^{-m} / \sqrt{m}) \left(\sum_{n=1}^{\infty} (|\lambda_n|^2 / n) (1-p)^{-2n} \right)^{1/2} \\ &< \infty. \end{aligned}$$

To prove (15), we again use the Cauchy-Schwarz inequality and note that

$$\begin{aligned} \left| \sum_{m=1}^l \sum_{n=1}^q b_{mn} \lambda_m \lambda_n \right|^2 &= \left| \sum_{m=1}^l (\lambda_m / \sqrt{m})(1-p)^{-m} \mu_m \right|^2, \text{ where } \mu_m = \sum_{n=1}^q \sqrt{m} b_{mn} \lambda_n (1-p)^m \\ &\leq \left(\sum_{m=1}^l (|\lambda_m|^2 / m) (1-p)^{-2m} \right) \left(\sum_{m=1}^l |\mu_m|^2 \right) \\ &= \left(\sum_{m=1}^l (|\lambda_m|^2 / m) (1-p)^{-2m} \right) \left(\sum_{m=1}^l m \left| \sum_{n=1}^q b_{mn} \lambda_n \right|^2 (1-p)^{2m} \right). \end{aligned}$$

Now taking $l, q \rightarrow \infty$ and using (14), we have

$$\begin{aligned} \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \lambda_m \lambda_n \right|^2 &\leq \left(\sum_{m=1}^{\infty} (|\lambda_m|^2 / m) (1-p)^{-2m} \right) \left(\sum_{m=1}^{\infty} m \left| \sum_{n=1}^{\infty} b_{mn} \lambda_n \right|^2 (1-p)^{2m} \right) \\ &\leq \left(\sum_{m=1}^{\infty} (|\lambda_m|^2 / m) (1-p)^{-2m} \right)^2, \end{aligned} \tag{16}$$

which proves (15). \square

REMARK 1. (i) We observe that equality occurs in (15) for $m = 1$ and $n = 1$ (i.e. choosing $\lambda_1 = 1$ and $\lambda_m = 0$ for all $m > 1$) for the function

$$g(z) = \frac{1}{z-p} + \frac{z-p}{(1-p)^2}, \quad z \in \mathbb{D}. \tag{17}$$

It is easy to see that the above function belongs to $\mathcal{M}(p)$ and is univalent on the disk $\mathbb{D}_{p,1-p}$.

(ii) Inequality (15) can be called as the Grunsky inequality for the class $\Sigma(p)$. Putting $p = 0$ in (15) and considering the arbitrary constants as $\sqrt{m} \lambda_m$, we get the Grunsky inequality for the class Σ , given in (3).

Choosing $\lambda_1 = 1, \lambda_2 = 0, \dots, \lambda_l = 0$, we get from (6) that any function $g \in \Sigma(p)$ satisfies $\sum_{m=1}^{\infty} m|b_m|^2 \leq (1-p)^{-4}$, which gives an estimate of the first Laurent coefficient of g , when expressed in the form of (4). Precisely, we have the following corollary.

COROLLARY 1. *Let $g \in \Sigma(p)$ having an expansion of the form (4), then $|b_1| \leq (1-p)^{-2}$.*

REMARK 2. (i) Equality holds in the above corollary for the function g defined in (17).

(ii) If $g \in \Sigma$ (when $p \rightarrow 0^+$) having an expansion of the form (2), then $|b_1| \leq 1$, where equality holds for the function $1/z+z, z \in \mathbb{D}$ (see [7, p. 134]). This result easily follows from the Corollary 1.

Next we prove the Goluzin inequality for functions in $\Sigma(p)$.

THEOREM 3. *Let $g \in \Sigma(p)$ and $z_1, z_2, \dots, z_r \in \mathbb{D}_{p,1-p}$ and $\gamma_1, \gamma_2, \dots, \gamma_r \in \mathbb{C}$, then*

$$\begin{aligned} & \left| \sum_{i=1}^r \sum_{j=1}^r \gamma_i \gamma_j \log \left(\frac{g(z_i) - g(z_j)}{1/(z_i - p) - 1/(z_j - p)} \right) \right| \\ & \leq - \sum_{i=1}^r \sum_{j=1}^r \gamma_i \overline{\gamma_j} \log (1 - (z_i - p)(\overline{z_j} - p)(1 - p)^{-2}). \end{aligned} \tag{18}$$

Proof. Using (15), we get from (7) that

$$\begin{aligned} \text{l.h.s. of (18)} &= \left| \sum_{i=1}^r \sum_{j=1}^r \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} (z_i - p)^m (z_j - p)^n \right) \gamma_i \gamma_j \right| \\ &= \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \left(\sum_{i=1}^r \gamma_i \left(\sum_{j=1}^r \gamma_j (z_j - p)^n \right) (z_i - p)^m \right) \right| \\ &= \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \left(\sum_{i=1}^r \gamma_i (z_i - p)^m \right) \lambda_n \right|, \text{ where } \lambda_n = \sum_{j=1}^r \gamma_j (z_j - p)^n \\ &= \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \lambda_m \lambda_n \right| \\ &\leq \sum_{m=1}^{\infty} (|\lambda_m|^2 / m) (1 - p)^{-2m} \\ &= \sum_{m=1}^{\infty} (1/m) \left(\sum_{i=1}^r \gamma_i (z_i - p)^m \right) \left(\sum_{j=1}^r \overline{\gamma_j} (\overline{z_j} - p)^m \right) (1 - p)^{-2m} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^r \sum_{j=1}^r \gamma_i \bar{\gamma}_j \left(\sum_{m=1}^{\infty} (z_i - p)^m (\bar{z}_j - p)^m (1 - p)^{-2m} / m \right) \\
 &= - \sum_{i=1}^r \sum_{j=1}^r \gamma_i \bar{\gamma}_j \log \left(1 - (z_i - p)(\bar{z}_j - p)(1 - p)^{-2} \right). \quad \square
 \end{aligned}$$

Next we derive some important corollaries of Theorem 3. First one is the distortion estimate of a function in $\Sigma(p)$.

COROLLARY 2. *If $g \in \Sigma(p)$ and $z \in \mathbb{D}_{p,1-p}$, then*

$$(1 - (1 - p)^{-2}|z - p|^2) \leq |(z - p)^2 g'(z)| \leq (1 - (1 - p)^{-2}|z - p|^2)^{-1}. \quad (19)$$

Proof. Taking $r = 1, \gamma_1 = 1$ in (18), we get for $z, \xi \in \mathbb{D}_{p,1-p}$ that

$$\lim_{\xi \rightarrow z} \left| \log \left(\frac{g(z) - g(\xi)}{1/(z - p) - (\xi - p)} \right) \right| \leq -\log(1 - (1 - p)^{-2}|z - p|^2),$$

which implies

$$|\log(-(z - p)^2 g'(z))| \leq -\log(1 - (1 - p)^{-2}|z - p|^2).$$

Now using the fact that $|\log w| \geq |\log |w||$, we get

$$|\log|(z - p)^2 g'(z)|| \leq |\log(-(z - p)^2 g'(z))| \leq -\log(1 - (1 - p)^{-2}|z - p|^2).$$

Thus

$$\log(1 - (1 - p)^{-2}|z - p|^2) \leq \log|(z - p)^2 g'(z)| \leq -\log(1 - (1 - p)^{-2}|z - p|^2),$$

which yields (19) after exponentiating. \square

REMARK 3. (i) Equality holds for the first inequality in (19) at the point $z = z_0$, for the function

$$g(z) = \left(\frac{1}{z - p} - \frac{1}{z_0 - p} \right) \left(1 - \frac{(\bar{z}_0 - p)(z - p)}{(1 - p)^2} \right),$$

where $z, z_0 \in \mathbb{D}_{p,1-p}$ and z_0 is fixed. We note that $g \in \mathcal{M}(p)$ and it is univalent in $\mathbb{D}_{p,1-p}$. Similarly equality occurs in the second inequality of (19) at the point $z = z_0$, for the function

$$h(z) = \left(\frac{1}{z - p} - \frac{1}{z_0 - p} \right) \left(1 - \frac{(\bar{z}_0 - p)(z - p)}{(1 - p)^2} \right)^{-1}.$$

It is easy to see that h is also univalent in $\mathbb{D}_{p,1-p}$. Here we observe that

$$h'(z) = \frac{-(z - p)^{-2} + 2(1 - p)^{-2}(\bar{z}_0 - p)(z - p)^{-1} - (1 - p)^{-2}(\bar{z}_0 - p)(z_0 - p)^{-1}}{(1 - (1 - p)^{-2}(\bar{z}_0 - p)(z - p))^2}.$$

Hence,

$$h'(z_0) = \frac{-(z_0 - p)^{-2} + (1 - p)^{-2}(\overline{z_0} - p)(z_0 - p)^{-1}}{(1 - (1 - p)^{-2}|z_0 - p|^2)^2}.$$

Therefore,

$$|(z_0 - p)^2 h'(z_0)| = \frac{1 - (1 - p)^{-2}|z_0 - p|^2}{(1 - (1 - p)^{-2}|z_0 - p|^2)^2} = \frac{1}{1 - (1 - p)^{-2}|z_0 - p|^2}.$$

(ii) When $p \rightarrow 0^+$, Corollary 2 yields a well known result for functions in Σ , which states that if $g \in \Sigma$, then

$$1 - |z|^2 \leq |z^2 g'(z)| \leq (1 - |z|^2)^{-1}, \quad z \in \mathbb{D}$$

(see f.i. [7, Corollary 6, p. 127]).

We can also obtain a two point distortion estimate for functions in $\Sigma(p)$.

COROLLARY 3. For $g \in \Sigma(p)$ and $z, \xi \in \mathbb{D}_{p,1-p}$, we have

$$\left| \log \left(\frac{g'(z)g'(\xi)(z - \xi)^2}{(g(z) - g(\xi))^2} \right) \right| \leq \log \left(\frac{|1 - (1 - p)^{-2}(z - p)(\overline{\xi} - p)|^2}{(1 - (1 - p)^{-2}|z - p|^2)(1 - (1 - p)^{-2}|\xi - p|^2)} \right). \tag{20}$$

Proof. For $r = 2$, choosing $z_1 = z, z_2 = \xi, \gamma_1 = 1, \gamma_2 = -1$, we get l.h.s. of (18) as

$$\begin{aligned} & \left| \log(-(z - p)^2 g'(z)) + \log(-(\xi - p)^2 g'(\xi)) - 2 \log \left(\frac{g(z) - g(\xi)}{1/(z - p) - 1/(\xi - p)} \right) \right| \\ &= \left| \log \left(\frac{g'(z)g'(\xi)(z - \xi)^2}{(g(z) - g(\xi))^2} \right) \right|. \end{aligned}$$

Again, for the particular values of z_1, z_2, γ_1 and γ_2 chosen above, r.h.s. of (18) is less than or equals to

$$\log \left(\frac{|1 - (z - p)(\overline{\xi} - p)(1 - p)^{-2}|^2}{(1 - |z - p|^2(1 - p)^{-2})(1 - |\xi - p|^2(1 - p)^{-2})} \right).$$

Now from (18), we get (20). \square

REMARK 4. Equality occurs in (20) for the function g defined in (17), since for this function, we have

$$\log \left(\frac{g'(z)g'(\xi)(z - \xi)^2}{(g(z) - g(\xi))^2} \right) = \log \left(\frac{(1 - (1 - p)^{-2}(z - p)^2)(1 - (1 - p)^{-2}(\xi - p)^{-2})}{(1 - (1 - p)^{-2}(z - p)(\xi - p))^2} \right).$$

Now, we choose $z = r_1(1 - p) + p$ and $\xi = r_2(1 - p) + p$, where $r_1, r_2 < 1$ so that z, ξ lies on the real line and belongs to the disk $\mathbb{D}_{p,1-p}$. For this z and ξ , we see that l.h.s. of (20) reduces to $\log[(1 - r_1 r_2)^2 (1 - r_1^2)^{-1} (1 - r_2^2)^{-1}]$, which is the r.h.s. of (20), since $(1 - r_1^2)(1 - r_2^2)(1 - r_1 r_2)^{-2} < 1$.

Using the last result, we get the next corollary.

COROLLARY 4. *Let $g \in \Sigma(p)$ and $z \in \mathbb{D}_{p,1-p}$ such that $g(2p - z) = -g(z)$, then*

$$\frac{1 - |z - p|^2(1 - p)^{-2}}{1 + |z - p|^2(1 - p)^{-2}} \leq \left| \frac{(z - p)g'(z)}{g(z)} \right| \leq \frac{1 + |z - p|^2(1 - p)^{-2}}{1 - |z - p|^2(1 - p)^{-2}}.$$

Proof. Taking $\xi = 2p - z$ in (20) and using the relation $g(2p - z) = -g(z)$, we get

$$\left| \log \left(-\frac{(z - p)g'(z)}{g(z)} \right) \right| \leq \log \left(\frac{1 + |z - p|^2(1 - p)^{-2}}{1 - |z - p|^2(1 - p)^{-2}} \right), \tag{21}$$

which proves the result since $|\log |w|| \leq |\log w|$. \square

REMARK 5. (i) Equality holds in the first and second inequalities of the Corollary 4 at the point $z = z_0$, for the functions g and h respectively, defined as

$$g(z) = \frac{1}{z - p} + \frac{(\overline{z_0} - p)(z - p)}{(1 - p)^2(z_0 - p)},$$

and

$$h(z) = \frac{1}{z - p} - \frac{(\overline{z_0} - p)(z - p)}{(1 - p)^2(z_0 - p)},$$

where $z, z_0 \in \mathbb{D}_{p,1-p}$ and z_0 is fixed. We note that both g and h are univalent in $\mathbb{D}_{p,1-p}$.

(ii) For an odd function $g \in \Sigma$, if we allow $p \rightarrow 0^+$, then (21) gives

$$|\log(-zg'(z)/g(z))| \leq \log((1 + |z|^2)/(1 - |z|^2)), \quad z \in \mathbb{D}$$

(compare [7, Corollary 2, p. 126]).

Considering the real part in (18) and choosing the constants as real numbers, we have the following corollary.

COROLLARY 5. *If $g \in \Sigma(p)$ and $z_i \in \mathbb{D}_{p,1-p}$, $\gamma_i \in \mathbb{R}$ for $i = 1, \dots, r$, then*

$$\begin{aligned} \prod_{i=1}^r \prod_{j=1}^r |1 - (1 - p)^{-2}(z_i - p)(\overline{z_j} - p)|^{\gamma_i \gamma_j} &\leq \prod_{i=1}^r \prod_{j=1}^r \left| \frac{g(z_i) - g(z_j)}{1/(z_i - p) - 1/(z_j - p)} \right|^{\gamma_i \gamma_j} \\ &\leq \prod_{i=1}^r \prod_{j=1}^r |1 - (1 - p)^{-2}(z_i - p)(\overline{z_j} - p)|^{-\gamma_i \gamma_j}. \end{aligned}$$

Next, we derive the Lebedev inequality for functions in $\Sigma(p)$, which is a generalization of the Goluzin inequality.

THEOREM 4. *Let $g \in \Sigma(p)$ and r be a positive integer such that (z_1, z_2, \dots, z_r) and $(\xi_1, \xi_2, \dots, \xi_r)$ be two arbitrary r -tuples of distinct points in $\mathbb{D}_{p,1-p}$, then*

$$\begin{aligned} & \left| \sum_{i=1}^r \sum_{j=1}^r \gamma_i \mu_j \log \left(\frac{g(z_i) - g(\xi_j)}{1/(z_i - p) - 1/(\xi_j - p)} \right) \right|^2 \\ & \leq \left(\sum_{i=1}^r \sum_{j=1}^r \gamma_i \bar{\gamma}_j \log(1 - (z_i - p)(\bar{z}_j - p)(1 - p)^{-2}) \right) \times \\ & \quad \left(\sum_{i=1}^r \sum_{j=1}^r \mu_i \bar{\mu}_j \log(1 - (\xi_i - p)(\bar{\xi}_j - p)(1 - p)^{-2}) \right), \end{aligned} \tag{22}$$

for all complex numbers γ_i and μ_j .

Proof. From (7), we get

l.h.s. of (22) = $\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \lambda_m \lambda_n \right|^2$, where $\lambda_m = \sum_{i=1}^r \gamma_i (z_i - p)^m$ and $\lambda_n = \sum_{j=1}^r \mu_j (\xi_j - p)^n$.

Now using (16), we have

$$\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \lambda_m \lambda_n \right|^2 \leq \left(\sum_{m=1}^{\infty} (|\lambda_m|^2/m)(1 - p)^{-2m} \right) \left(\sum_{n=1}^{\infty} (|\lambda_n|^2/n)(1 - p)^{-2n} \right). \tag{23}$$

Now following the proof of the last part of Theorem 3, we obtain

$$\begin{aligned} \sum_{m=1}^{\infty} (|\lambda_m|^2/m)(1 - p)^{-2m} &= \sum_{m=1}^{\infty} (1/m) \left| \sum_{i=1}^r \gamma_i (z_i - p)^m \right|^2 (1 - p)^{-2m} \\ &= \sum_{i=1}^r \sum_{j=1}^r \gamma_i \bar{\gamma}_j \left(\sum_{m=1}^{\infty} (1/m) (z_i - p)^m (\bar{z}_j - p)^m (1 - p)^{-2m} \right) \\ &= - \sum_{i=1}^r \sum_{j=1}^r \gamma_i \bar{\gamma}_j \log(1 - (z_i - p)(\bar{z}_j - p)(1 - p)^{-2}). \end{aligned}$$

Similarly, we get

$$\sum_{n=1}^{\infty} (|\lambda_n|^2/n)(1 - p)^{-2n} = - \sum_{i=1}^r \sum_{j=1}^r \mu_i \bar{\mu}_j \log(1 - (\xi_i - p)(\bar{\xi}_j - p)(1 - p)^{-2}).$$

Using the last two relations, we get the desired result from (23). \square

Next, we obtain some corollaries of the Lebedev inequality.

COROLLARY 6. *If $g \in \Sigma(p)$ and $z, \xi \in \mathbb{D}_{p,1-p}$, then*

$$\begin{aligned} & \left| \log \left(\frac{g(z) - g(\xi)}{1/(z - p) - 1/(\xi - p)} \right) \right|^2 \\ & \leq \log(1 - |z - p|^2(1 - p)^{-2})^{-1} \log(1 - |\xi - p|^2(1 - p)^{-2})^{-1}. \end{aligned} \tag{24}$$

Proof. In (22), we put $r = 1$, $\gamma_1 = \mu_1 = 1$ and $z_1 = z$, $\xi_1 = \xi$ to get (24). If we take $\xi \rightarrow z$, then (24) gives

$$|\log(-(z-p)^2g'(z))| \leq \log(1-|z-p|^2(1-p)^{-2})^{-1},$$

that also implies (19). \square

COROLLARY 7. *For each $g \in \Sigma(p)$, we have*

$$\begin{aligned} & \left| \log \left(\frac{g(z) - g(\xi)}{g(z) + g(\xi)} \right) \left(\frac{z + \xi - 2p}{z - \xi} \right) \right|^2 \\ & \leq \log \left(\frac{1 + |z-p|^2(1-p)^{-2}}{1 - |z-p|^2(1-p)^{-2}} \right) \log \left(\frac{1 + |\xi-p|^2(1-p)^{-2}}{1 - |\xi-p|^2(1-p)^{-2}} \right), \end{aligned}$$

where $z, \xi \in \mathbb{D}_{p,1-p}$ such that $g(2p-z) = -g(z)$ and $g(2p-\xi) = -g(\xi)$.

Proof. In (22), we put $r = 2$, $\gamma_1 = \mu_1 = 1$, $\gamma_2 = \mu_2 = -1$ and $z_1 = z$, $\xi_1 = \xi$, $z_2 = 2p-z$, $\xi_2 = 2p-\xi$. Now, using the relation $g(2p-z) = -g(z)$ and $g(2p-\xi) = -g(\xi)$, we get

$$\begin{aligned} \text{l.h.s. of (22)} &= \left| \log \left(\frac{g(z) - g(\xi)}{1/(z-p) - 1/(\xi-p)} \right)^2 - \log \left(\frac{g(z) + g(\xi)}{1/(z-p) + 1/(\xi-p)} \right)^2 \right|^2 \\ &= \left| \log \left(\frac{g(z) - g(\xi)}{g(z) + g(\xi)} \cdot \frac{z + \xi - 2p}{z - \xi} \right) \right|^2. \end{aligned}$$

Again, with the above values of the constants, r.h.s. of (22) reduces to

$$\log \left(\frac{1 + |z-p|^2(1-p)^{-2}}{1 - |z-p|^2(1-p)^{-2}} \right)^2 \log \left(\frac{1 + |\xi-p|^2(1-p)^{-2}}{1 - |\xi-p|^2(1-p)^{-2}} \right)^2.$$

Comparing the left and right hand side, we get the result. \square

Next, we prove the Grunsky inequality for meromorphic functions with non zero pole having quasiconformal extension onto the extended complex plane. We will apply the *Dirichlet Principle* (compare [17, Chap. 9, p. 289]), as stated below to prove our result.

LEMMA 1. *Let $\mu(z)$ be real harmonic and $v(z)$ be continuously differentiable function in \mathbb{D} and also let μ and v are continuous in $\overline{\mathbb{D}}$. If $\mu(z) = v(z)$ on the boundary $\partial\mathbb{D} := \{z : |z| = 1\}$, then*

$$\iint_{\mathbb{D}} \left| \frac{\partial \mu}{\partial z} \right|^2 dx dy \leq \iint_{\mathbb{D}} \left| \frac{\partial v}{\partial z} \right|^2 dx dy, \quad \text{where } z = x + iy.$$

THEOREM 5. *Let $g \in \Sigma_k(p)$ having expansion of the form (4). Let $\lambda_m \in \mathbb{C}$, $m = 1, 2, \dots$ be arbitrary complex numbers, then*

$$\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \lambda_m \lambda_n \right| \leq k \sum_{m=1}^{\infty} (|\lambda_m|^2 / m) (1 - p)^{-2m}, \tag{25}$$

provided the series in the right hand side of (25) is convergent, where b_{mn} 's are the Grunsky coefficients of g that are defined in (7).

Proof. We follow the technique developed by Pommerenke ([17, Theorem 9.13, p. 289]). Since $g \in \Sigma_k(p)$, g is analytic on $\mathbb{D} \setminus \{p\}$ and $|g_{\bar{z}}| \leq k|g_z|$ on \mathbb{D}^* . Thus

$$\frac{|g_z| + |g_{\bar{z}}|}{|g_z| - |g_{\bar{z}}|} \leq \frac{1+k}{1-k}, \quad z \in \mathbb{D}^*. \tag{26}$$

Let us consider $s(w) := \text{Re}(h(w))$, $w \in \mathbb{C}$, where h is an analytic function of $w = u + iv$, as defined in (10). Let us assume $w = g(z)$ where $z = x + iy \in \mathbb{D}^*$. Now using the fact that $\overline{(g_{\bar{z}})} = (\bar{g})_z$ and $\overline{(g_z)} = (\bar{g})_{\bar{z}}$ and also noting that $\bar{\bar{s}} = s$, we get

$$\begin{aligned} \frac{\partial}{\partial z}(s(g(z))) &= s_w w_z + s_{\bar{w}}(\bar{w})_z = s_w g_z + s_{\bar{w}}(\bar{g})_z \\ &= s_w g_z + \overline{(s_w)} \overline{(g_{\bar{z}})} = s_w g_z + \overline{s_w g_{\bar{z}}}. \end{aligned}$$

Therefore, using (26) we get

$$\begin{aligned} \left| \frac{\partial}{\partial z}(s(g(z))) \right|^2 &= |s_w g_z + \overline{s_w g_{\bar{z}}}|^2 \leq |s_w|^2 (|g_z| + |g_{\bar{z}}|)^2 \\ &\leq \frac{1+k}{1-k} |s_w|^2 J_g(z), \end{aligned} \tag{27}$$

where $J_g(z) := |g_z|^2 - |g_{\bar{z}}|^2$ denotes the Jacobian of the mapping g . Here we note that the last inequality is valid for $z \in \mathbb{D}^*$, but since $g_{\bar{z}} = 0$ for $z \in \mathbb{D} \setminus \{p\}$, we can conclude that it is also valid for larger domain $\mathbb{D}_{p,1-p}^* := \{z : |z - p| > 1 - p\}$. Since h is an analytic function of w and $s(w) = \text{Re}(h(w))$, then $s_w = (1/2)(s_u - is_v)$, which gives $|h'(w)|^2 = s_u^2 + s_v^2 = 4|s_w|^2$. Hence we have

$$|h'(w)| = 2|s_w|. \tag{28}$$

Using this fact and noting that g is univalent on $\widehat{\mathbb{C}}$, we get from (27) that

$$\begin{aligned} \frac{1+k}{1-k} \iint_{\widehat{\mathbb{C}} \setminus g(\mathbb{D}_{p,1-p})} |h'(w)|^2 dudv &= \frac{4(1+k)}{1-k} \iint_{g(\mathbb{D}_{p,1-p}^*)} |s_w|^2 dudv \\ &= \frac{4(1+k)}{1-k} \iint_{\mathbb{D}_{p,1-p}^*} |s_w|^2 J_g(z) dx dy \\ &\geq 4 \iint_{\mathbb{D}_{p,1-p}^*} \left| \frac{\partial}{\partial z}(s(g(z))) \right|^2 dx dy. \end{aligned} \tag{29}$$

Next we see that as $g \in \Sigma_k(p)$, it is continuous from $\widehat{\mathbb{C}}$ onto itself. If we now consider the function $(h \circ g)(z) := h(g(z))$, $z \in \widehat{\mathbb{C}}$, then it is also continuous from $\widehat{\mathbb{C}}$ onto itself, as h is a polynomial (see (10)). From (11), we also see that $h \circ g$ has the following expansion:

$$h(g(z)) = - \sum_{m=1}^l (\lambda_m/m)(z-p)^{-m} - \sum_{m=1}^{\infty} d_m(z-p)^m, \quad z \in \mathbb{D}_{p,1-p} \setminus \{p\}.$$

We now consider the function

$$\begin{aligned} \psi(z) := & h \left(g \left(p + \frac{(1-p)^2}{z-p} \right) \right) + \sum_{m=1}^l (\lambda_m/m)(1-p)^{-2m}(z-p)^m \\ & - \sum_{m=1}^l (\overline{\lambda}_m/m)(z-p)^{-m}, \quad z \in \widehat{\mathbb{C}}. \end{aligned}$$

It now follows that $\psi : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is continuous and has the following form:

$$\psi(z) = - \sum_{m=1}^l (\overline{\lambda}_m/m)(z-p)^{-m} - \sum_{m=1}^{\infty} d_m(1-p)^{2m}(z-p)^{-m}, \quad z \in \mathbb{D}_{p,1-p}^*. \tag{30}$$

From (30), it also follows that ψ is analytic on $\mathbb{D}_{p,1-p}^*$, which is symmetric w.r.t. the real axis. Thus the function $\rho(z) := \overline{\psi(\bar{z})}$ is also analytic on that domain and $\rho'(z) = \overline{\psi'(\bar{z})}$. Therefore, $\mu(z) := \text{Re}(\rho(z)) = \text{Re}(\psi(\bar{z}))$ is harmonic on $\mathbb{D}_{p,1-p}^*$. Let us also denote $v(z) := s(g(z)) = \text{Re}(h(g(z)))$, $z \in \widehat{\mathbb{C}}$. Let us now make a change of variable $\zeta = (1-p)/(z-p)$, so that $z = p + (1-p)/\zeta$. Hence the functions μ and v transform to

$$\tilde{\mu}(\zeta) := \mu(p + (1-p)/\zeta) \quad \text{and} \quad \tilde{v}(\zeta) := v(p + (1-p)/\zeta), \quad \zeta \in \widehat{\mathbb{C}}.$$

From the constructions of $\tilde{\mu}$ and \tilde{v} , it readily follows that they are continuous from $\widehat{\mathbb{C}}$ onto itself. Moreover, $\tilde{\mu}$ is harmonic on \mathbb{D} , \tilde{v} has continuous partial derivatives on \mathbb{D} that are square-integrable. Also from the expressions of ψ (in (30)) and $h \circ g$ (in (11)), we see that $\tilde{\mu}$ and \tilde{v} respectively have the following forms:

$$\begin{aligned} \tilde{\mu}(\zeta) = & -\text{Re} \left(\sum_{m=1}^l (\overline{\lambda}_m/m)(1-p)^{-m}\bar{\zeta}^m + \sum_{m=1}^{\infty} d_m(1-p)^m\bar{\zeta}^m \right), \quad \zeta \in \mathbb{D}, \\ \tilde{v}(\zeta) = & -\text{Re} \left(\sum_{m=1}^l (\lambda_m/m)(1-p)^{-m}\zeta^m + \sum_{m=1}^{\infty} d_m(1-p)^m\zeta^{-m} \right), \quad \zeta \in \mathbb{D}^*. \end{aligned}$$

Since $\tilde{\mu}$ and \tilde{v} are continuous on $\widehat{\mathbb{C}}$, it now follows that $\tilde{\mu}(\zeta) = \tilde{v}(\zeta)$ on $\partial\mathbb{D}$. Thus $\tilde{\mu}$ and \tilde{v} satisfy conditions of the Lemma 1. Hence from the Lemma 1, we get

$$\iint_{\mathbb{D}} \left| \frac{\partial}{\partial \zeta} (\tilde{\mu}(\zeta)) \right|^2 d\tau d\eta \leq \iint_{\mathbb{D}} \left| \frac{\partial}{\partial \zeta} (\tilde{v}(\zeta)) \right|^2 d\tau d\eta, \quad \text{where } \zeta = \tau + i\eta. \tag{31}$$

By the chain rule of differentiation, we have

$$-\frac{(z-p)^2}{1-p} \frac{\partial}{\partial z}(\mu(z)) = \frac{\partial}{\partial \zeta}(\tilde{\mu}(\zeta)), \quad \text{and} \quad -\frac{(z-p)^2}{1-p} \frac{\partial}{\partial z}(v(z)) = \frac{\partial}{\partial \zeta}(\tilde{v}(\zeta)).$$

Again the Jacobian of the transformation from z -variable to ζ -variable is given by $J_\zeta(z) = (1-p)^2|z-p|^{-4}$. Hence returning back to the original variable z , we get from (31) that

$$\iint_{\mathbb{D}_{p,1-p}^*} \left| \frac{\partial}{\partial z}(\mu(z)) \right|^2 dx dy \leq \iint_{\mathbb{D}_{p,1-p}^*} \left| \frac{\partial}{\partial z}(v(z)) \right|^2 dx dy. \tag{32}$$

Since ρ is analytic on $\mathbb{D}_{p,1-p}^*$ and $\mu(z) = \text{Re}(\psi(\bar{z})) = \text{Re}(\rho(z))$, therefore using a similar method of computation as to derive (28), we have

$$|\rho'(z)| = 2 \left| \frac{\partial}{\partial z}(\text{Re}(\rho(z))) \right|.$$

Using this fact and the inequality (32), we get from (29) that

$$\begin{aligned} \frac{1+k}{1-k} \iint_{\hat{\mathbb{C}} \setminus g(\mathbb{D}_{p,1-p})} |h'(w)|^2 dudv &\geq 4 \iint_{\mathbb{D}_{p,1-p}^*} \left| \frac{\partial}{\partial z}(v(z)) \right|^2 dx dy \\ &\geq 4 \iint_{\mathbb{D}_{p,1-p}^*} \left| \frac{\partial}{\partial z}(\mu(z)) \right|^2 dx dy \\ &= 4 \iint_{\mathbb{D}_{p,1-p}^*} \left| \frac{\partial}{\partial z}(\text{Re}(\rho(z))) \right|^2 dx dy \\ &= \iint_{\mathbb{D}_{p,1-p}^*} |\rho'(z)|^2 dx dy. \end{aligned} \tag{33}$$

From (30), we have

$$\rho(z) = -\sum_{m=1}^l \left((\lambda_m/m) + \overline{d_m}(1-p)^{2m} \right) (z-p)^{-m} - \sum_{m=l+1}^{\infty} \overline{d_m}(1-p)^{2m} (z-p)^{-m},$$

where $z \in \mathbb{D}_{p,1-p}^*$. Therefore, using (12) and the Green's formula (see also proof of the Theorem 1), we get

$$\begin{aligned} &\iint_{\mathbb{D}_{p,1-p}^*} |\rho'(z)|^2 dx dy \\ &= -\frac{1}{2i} \int_{|z-p|=1-p} \overline{\rho(\bar{z})} \rho'(z) dz \\ &= -\frac{1}{2i} \int_{|z-p|=1-p} \left(-\sum_{m=1}^l \left((\overline{\lambda_m}/m) + d_m(1-p)^{2m} \right) (\bar{z}-p)^{-m} \right. \\ &\quad \left. - \sum_{m=l+1}^{\infty} d_m(1-p)^{2m} (\bar{z}-p)^{-m} \right) \left(\sum_{m=1}^l m \left((\lambda_m/m) + \overline{d_m}(1-p)^{2m} \right) (z-p)^{-m-1} \right. \\ &\quad \left. + \sum_{m=l+1}^{\infty} m \overline{d_m}(1-p)^{2m} (z-p)^{-m-1} \right) dz \end{aligned}$$

$$\begin{aligned}
 &= \pi \sum_{m=1}^l m |(\overline{\lambda_m}/m)(1-p)^{-m} + d_m(1-p)^m|^2 + \pi \sum_{m=l+1}^{\infty} m |d_m|^2 (1-p)^{2m} \\
 &= \pi \sum_{m=1}^l m \left((\overline{\lambda_m}/m)(1-p)^{-m} + d_m(1-p)^m \right) \left((\lambda_m/m)(1-p)^{-m} + \overline{d_m}(1-p)^m \right) \\
 &\quad + \pi \sum_{m=l+1}^{\infty} m |d_m|^2 (1-p)^{2m}.
 \end{aligned}$$

Hence from above, we have

$$\begin{aligned}
 \iint_{\mathbb{D}_{p,1-p}^*} |\rho'(z)|^2 dx dy &= \pi \left(\sum_{m=1}^l (|\lambda_m|^2/m)(1-p)^{-2m} + \sum_{m=1}^l m |d_m|^2 (1-p)^{2m} \right. \\
 &\quad \left. + 2\operatorname{Re} \left(\sum_{m=1}^l \lambda_m d_m \right) + \sum_{m=l+1}^{\infty} m |d_m|^2 (1-p)^{2m} \right). \tag{34}
 \end{aligned}$$

Combining (33) and (34), we get from (13) that

$$\begin{aligned}
 &\frac{1+k}{1-k} \left(\sum_{m=1}^l (|\lambda_m|^2/m)(1-p)^{-2m} - \sum_{m=1}^{\infty} m |d_m|^2 (1-p)^{2m} \right) \\
 &= \frac{1+k}{\pi(1-k)} \iint_{\mathbb{C} \setminus g(\mathbb{D}_{p,1-p})} |h'(w)|^2 dudv \\
 &\geq \frac{1}{\pi} \iint_{\mathbb{D}_{p,1-p}^*} |\rho'(z)|^2 dx dy \\
 &= \sum_{m=1}^l (|\lambda_m|^2/m)(1-p)^{-2m} + \sum_{m=1}^l m |d_m|^2 (1-p)^{2m} + 2\operatorname{Re} \left(\sum_{m=1}^l \lambda_m d_m \right) \\
 &\quad + \sum_{m=l+1}^{\infty} m |d_m|^2 (1-p)^{2m},
 \end{aligned}$$

which implies

$$2k \sum_{m=1}^l (|\lambda_m|^2/m)(1-p)^{-2m} - 2 \sum_{m=1}^{\infty} m |d_m|^2 (1-p)^{2m} \geq 2(1-k) \operatorname{Re} \left(\sum_{m=1}^l \lambda_m d_m \right).$$

Therefore,

$$\begin{aligned}
 \frac{1-k}{1+k} \operatorname{Re} \left(\sum_{m=1}^l \lambda_m d_m \right) &\leq \frac{k}{1+k} \sum_{m=1}^l (|\lambda_m|^2/m)(1-p)^{-2m} \\
 &\quad - \frac{1}{1+k} \sum_{m=1}^{\infty} m |d_m|^2 (1-p)^{2m},
 \end{aligned}$$

which again implies

$$\begin{aligned}
 \operatorname{Re}\left(\sum_{m=1}^l \lambda_m d_m\right) &\leq \frac{k}{1+k} \sum_{m=1}^l (|\lambda_m|^2/m)(1-p)^{-2m} - \frac{1}{1+k} \sum_{m=1}^{\infty} m|d_m|^2(1-p)^{2m} \\
 &\quad + \frac{2k}{1+k} \operatorname{Re}\left(\sum_{m=1}^l \lambda_m d_m\right) \\
 &= k \sum_{m=1}^l (|\lambda_m|^2/m)(1-p)^{-2m} - \frac{k^2}{1+k} \sum_{m=1}^l (|\lambda_m|^2/m)(1-p)^{-2m} \\
 &\quad - \frac{1}{1+k} \sum_{m=1}^{\infty} m|d_m|^2(1-p)^{2m} + \frac{k}{1+k} \sum_{m=1}^l (\lambda_m d_m + \overline{\lambda_m d_m}) \\
 &= k \sum_{m=1}^l (|\lambda_m|^2/m)(1-p)^{-2m} - \frac{1}{1+k} \sum_{m=1}^l \left[(k^2|\lambda_m|^2/m)(1-p)^{-2m} \right. \\
 &\quad \left. + m|d_m|^2(1-p)^{2m} - k(\lambda_m d_m + \overline{\lambda_m d_m}) \right] \\
 &\quad - \frac{1}{1+k} \sum_{m=l+1}^{\infty} m|d_m|^2(1-p)^{2m} \\
 &= k \sum_{m=1}^l (|\lambda_m|^2/m)(1-p)^{-2m} - \frac{1}{1+k} \sum_{m=l+1}^{\infty} m|d_m|^2(1-p)^{2m} \\
 &\quad - \frac{1}{1+k} \sum_{m=1}^l m \left| \frac{k\overline{\lambda_m}(1-p)^{-m}}{m} - d_m(1-p)^m \right|^2.
 \end{aligned}$$

Thus we get from the above inequality that

$$\operatorname{Re}\left(\sum_{m=1}^l \lambda_m d_m\right) \leq k \sum_{m=1}^l (|\lambda_m|^2/m)(1-p)^{-2m}.$$

Since the constants λ_m 's are chosen arbitrarily, it follows that

$$\left| \sum_{m=1}^l \lambda_m d_m \right| \leq k \sum_{m=1}^l (|\lambda_m|^2/m)(1-p)^{-2m}.$$

Now putting $d_m = \sum_{n=1}^l \lambda_n b_{mn}$ and taking $l \rightarrow \infty$, we get (25). \square

Next we state some important corollaries which follows from Theorem 5. Choosing $m = n = 1$ and $\lambda_1 = 1$ in (25), we easily get the following corollary.

COROLLARY 8. *Let $g \in \Sigma_k(p)$ having an expansion of the form (4), then $|b_1| \leq k(1-p)^{-2}$.*

If $g \in \Sigma_k(p)$ and $z_1, z_2, \dots, z_r \in \mathbb{D}_{p,1-p}$ and $\gamma_1, \gamma_2, \dots, \gamma_r \in \mathbb{C}$, then the Goluzin inequality takes the following form:

$$\left| \sum_{i=1}^r \sum_{j=1}^r \gamma_i \gamma_j \log \left(\frac{g(z_i) - g(z_j)}{1/(z_i - p) - 1/(z_j - p)} \right) \right| \leq -k \sum_{i=1}^r \sum_{j=1}^r \gamma_i \overline{\gamma_j} \log (1 - (z_i - p)(\overline{z_j} - p)(1 - p)^{-2}).$$

As a consequence, we get the following results:

COROLLARY 9. *If $g \in \Sigma_k(p)$ and $z \in \mathbb{D}_{p,1-p}$, then*

$$(1 - (1 - p)^{-2}|z - p|^2)^k \leq |(z - p)^2 g'(z)| \leq (1 - (1 - p)^{-2}|z - p|^2)^{-k}.$$

COROLLARY 10. *Let $g \in \Sigma_k(p)$ and $z \in \mathbb{D}_{p,1-p}$ such that $g(2p - z) = -g(z)$, then*

$$\left| \log \left(-\frac{(z - p)g'(z)}{g(z)} \right) \right| \leq k \log \left(\frac{1 + |z - p|^2(1 - p)^{-2}}{1 - |z - p|^2(1 - p)^{-2}} \right).$$

We can prove the Corollary 9 in a similar way like we proved the Corollary 2. Corollary 10 follows in a similar way like we proved the Corollary 4.

REMARK 6. Equality holds in (25) (for $m = 1$) and the last three corollaries for the following k -quasiconformal mapping on $\widehat{\mathbb{C}}$ (or, its modification):

$$G(z) = \begin{cases} \frac{1}{z-p} + \frac{k(z-p)}{(1-p)^2}, & |z - p| < 1 - p, \\ \frac{1}{z-p} + \frac{k}{\overline{z-p}}, & |z - p| \geq 1 - p. \end{cases}$$

But we see that this function does not belong to $\Sigma_k(p)$.

3. Concluding remark

For a function $g \in \Sigma(p)$, we have considered the Laurent expansion about the point p of the form (4). Instead of this expansion, we can also assume the following Laurent expansion of g about the origin:

$$g(z) = \frac{1}{z - p} + \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D} \setminus \{p\}. \tag{35}$$

In this case, area theorem for $\Sigma(p)$ was proved by P. N. Chichra (compare [6]), which says that if $g \in \Sigma(p)$ having an expansion of the form (35), then $\sum_{n=1}^{\infty} n|b_n|^2 \leq (1 - p^2)^{-2}$. In the previous section, we have generalized the proof of an area theorem for $\Sigma(p)$ given in (5) to get the Grunsky inequality. So, we may expect another form of the Grunsky inequality by generalizing the proof of the Chichra’s area theorem. It will be

interesting to establish the Grunsky inequality for the classes $\Sigma(p)$ and $\Sigma_k(p)$ where the functions have expansion of the form (35). But we see that, this form of g does not work for the Theorem 1, as we cannot determine the Grunsky coefficients from equation (7).

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(Received November 24, 2020)

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