

VOLTERRA INTEGRAL OPERATOR FROM WEIGHTED BERGMAN SPACES TO GENERAL FUNCTION SPACES

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Abstract. The boundedness, compactness and essential norm of Volterra integral operator V_g from weighted Bergman spaces A_α^p to general function spaces $F(q, qt - 2, s)$ are investigated in this paper.

1. Introduction

Let \mathbb{D} be the unit disk in the complex plane \mathbb{C} , $H(\mathbb{D})$ be the class of functions analytic in \mathbb{D} and H^∞ be the class of bounded analytic functions on \mathbb{D} . The Hardy space $H^p(\mathbb{D})$ ($0 < p < \infty$) is the set of all $f \in H(\mathbb{D})$ with (see [5])

$$\|f\|_{H^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Suppose that $0 < p < \infty$, $\alpha > -1$ and

$$dA_\alpha(z) = (1 - |z|^2)^\alpha dA(z) = \frac{1}{\pi} (1 - |z|^2)^\alpha dx dy.$$

The weighted Bergman space A_α^p consists of all $f \in H(\mathbb{D})$ with (see [39])

$$\|f\|_{A_\alpha^p}^p = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty.$$

Let $0 < \beta < \infty$. The Bloch type space \mathcal{B}^β is the class of all $f \in H(\mathbb{D})$ for which (see [38])

$$\|f\|_{\mathcal{B}^\beta} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |f'(z)| < \infty.$$

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The little Bloch type space \mathcal{B}_0^β consists of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\beta |f'(z)| = 0.$$

For $0 < p < \infty$, $-2 < q < \infty$, $0 \leq s < \infty$, the space $F(p, q, s)$ ([36]) is defined as the space of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q \left(1 - |\varphi_a(z)|^2\right)^s dA(z) < \infty.$$

Here $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ is the Möbius transformation of \mathbb{D} that interchanges 0 and a . When $q = p - 2$ and $s > 1$, it gives the Bloch space \mathcal{B} . If $p = 2$, $q = 0$ and $s = 1$, it is $BMOA$ space, the space of analytic functions in the Hardy space $H^1(\mathbb{D})$ whose boundary functions have bounded mean oscillation.

Let $f, g \in H(\mathbb{D})$. The Volterra integral operator V_g is defined by

$$V_g f(z) = \int_0^z g'(w) f(w) dw, \quad z \in \mathbb{D}.$$

From [2], we see that V_g is bounded on Hardy spaces if and only if $g \in BMOA$. Aleman and Siskakis in [3] showed that V_g is bounded on the Bergman space A^p if and only if $g \in \mathcal{B}$. Siskakis and Zhao in [22] proved that V_g is bounded on $BMOA$ if and only if $g \in BMOA_{\log}$. Xiao in [34] proved that V_g is bounded on Q_s if and only if $g \in Q_{\log}^s$. The boundedness and compactness of a more general operator between $F(p, q, s)$ and Bloch-type spaces were characterized in [27] (for a special case see also [11]). The operator V_g and its generalizations have attracted attention of many authors. See [1, 4, 7, 8, 9, 10, 11, 13, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 37] and the references therein for more results of the operator V_g and their generalizations.

In this paper, we study the boundedness of Volterra integral operator V_g acting from A_α^p to $F(q, qt - 2, s)$. More specifically, we prove that V_g is bounded from A_α^p to $F(q, qt - 2, s)$ if and only if $g \in \mathcal{B}^{\frac{pt-2-\alpha}{p}}$, when $ps > p - q$; V_g is bounded from A_α^p to $F(q, qt - 2, s)$ if and only if $g \in F\left(\frac{pq}{p-q}, (qt - 2 - \frac{q\alpha}{p})\frac{p}{p-q}, \frac{ps}{p-q}\right)$ when $ps \leq p - q$. Moreover, the norm and essential norm of the operator V_g are also investigated. This paper is a continuation of some previous investigations (see [18, 33, 35]), but our results here are more general. Generally speaking, the case of $ps \leq p - q$ can not be deduced by using tent spaces (see [13] and [34]). Thus, the interesting way is deducing with the case $ps \leq p - q$. For higher-dimensional case, we refer to [14, 15].

In this paper, the symbol $f \approx g$ means that $f \lesssim g \lesssim f$. We say that $f \lesssim g$ if there exists a constant C such that $f \leq Cg$.

2. Boundedness of the operator V_g

In this section, we are going to give some auxiliary results and the characterization of the boundedness of $V_g : A_\alpha^p \rightarrow F(q, qt - 2, s)$. Let I be an arc of $\partial\mathbb{D}$ and $|I|$ be the normalized Lebesgue arc length of I . The Carleson square based on I , denoted by $S(I)$, is defined by

$$S(I) = \left\{ z = re^{i\theta} \in \mathbb{D} : 1 - |I| \leq r < 1, e^{i\theta} \in I \right\}.$$

Let μ be a positive Borel measure on \mathbb{D} . For $0 < s < \infty$, μ is called an s -Carleson measure if

$$\sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty.$$

LEMMA 1. [36] *Let $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$ and $q + s > -1$. Then $f \in F(p, q, s)$ if and only if $d\mu(z) = |f'(z)|^p (1 - |z|^2)^{q+s} dA(z)$ is a bounded s -Carleson measure.*

The hyperbolic distance (Bergman metric) of z and w in \mathbb{D} is denoted by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \left| \frac{z-w}{1-\bar{z}w} \right|}{1 - \left| \frac{z-w}{1-\bar{z}w} \right|}.$$

A sequence $\{z_j\}_{j=1}^\infty$ in \mathbb{D} is called an δ -lattice in hyperbolic distance if $\mathbb{D} = \bigcup_{j=1}^\infty \mathbb{D}(z_j, \delta)$, where $\mathbb{D}(z_j, \delta) = \{w \in \mathbb{D} : \beta(w, z_j) < \delta\}$ and $\beta(z_i, z_j) \geq \frac{\delta}{2}$ for $i \neq j$.

LEMMA 2. [39] *Suppose $\alpha > -1$ and $0 < p < \infty$. There exists a positive number δ_0 such that for any $\delta \in (0, \delta_0)$, any δ -lattice $\{z_j\}$ in \mathbb{D} , and any*

$$m > \max \left\{ -1, 2 \left(1 - \frac{1}{p} \right) \right\},$$

the following statements hold.

(1) *If $f \in A_p^\alpha$, then there exists $\{a_j\} \in \ell^p$ with $\|\{a_j\}\|_{\ell^p} \leq C \|f\|_{p,\alpha}$ such that*

$$f(z) = \sum_j a_j \frac{(1 - |z_j|)^m}{(1 - \bar{z}_j z)^{m + \frac{\alpha+2}{p}}}.$$

(2) *If $\{a_j\} \in \ell^p$, then the function f defined by the above series converges in A_p^α and*

$$\|f\|_{p,\alpha} \leq C \|\{a_j\}\|_{\ell^p}.$$

LEMMA 3. (Khinchine’s inequality) [12] *For $x \in [0, 1)$, let $r_j(x) = r_0(2^j x)$, $j = 1, 2, \dots$ with*

$$r_0(y) = \begin{cases} 1, & 0 \leq y - [y] < 1/2, \\ -1, & 1/2 \leq y - [y] < 1. \end{cases}$$

Then for any $0 < p < \infty$ and integer $N > 0$, there exists $c_p > 0$ such that

$$c_p \left(\sum_{j=0}^N |c_j|^2 \right)^{p/2} \leq \int_0^1 \left| \sum_{j=0}^N c_j r_j(x) \right|^p dx \leq \frac{1}{c_p} \left(\sum_{j=0}^N |c_j|^2 \right)^{p/2}.$$

Now we are in a position to state and prove the main result in this section.

THEOREM 1. *Let $1 \leq q \leq p < \infty$, $0 < s \leq 1$, $qt + s > 1$ and $-1 < \alpha < \min\{pt - 2, pt - 1 - \frac{p(1-s)}{q}\}$. Suppose that $g \in H(\mathbb{D})$. Then the following statements hold.*

(1) *When $ps > p - q$, V_g is bounded from A_α^p to $F(q, qt - 2, s)$ if and only if $g \in \mathcal{B}^{\frac{pt-2-\alpha}{p}}$. Moreover,*

$$\|V_g\| \approx \|g\|_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}}.$$

(2) *When $ps \leq p - q$, V_g is bounded from A_α^p to $F(q, qt - 2, s)$ if and only if*

$$g \in F\left(\frac{pq}{p-q}, \left(qt - 2 - \frac{q\alpha}{p}\right) \frac{p}{p-q}, \frac{ps}{p-q}\right),$$

i.e., $|g'(z)|^{\frac{pq}{p-q}} (1 - |z|^2)^{(qt-2+s-\frac{q\alpha}{p})\frac{p}{p-q}} dA(z)$ is a $\frac{ps}{p-q}$ -Carleson measure. Moreover,

$$\|V_g\| \approx \|g\|_{F(\frac{pq}{p-q}, (qt-2-\frac{q\alpha}{p})\frac{p}{p-q}, \frac{ps}{p-q})}.$$

Proof. (1). Suppose $ps > p - q$ and $g \in \mathcal{B}^{\frac{pt-2-\alpha}{p}}$. By Lemma 1, we need to show that the measure

$$d\mu_{V_g(f)}(z) = |V_g(f)'(z)|^q (1 - |z|^2)^{qt-2+s} dA(z) = |f(z)|^q |g'(z)|^q (1 - |z|^2)^{qt-2+s} dA(z)$$

is an s -Carleson measure for any $f \in A_\alpha^p$. If $p = q$, using $g \in \mathcal{B}^{\frac{pt-2-\alpha}{p}}$, we obtain

$$|g'(z)|^p (1 - |z|^2)^{pt-2} \lesssim \|g\|_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}}^p (1 - |z|^2)^\alpha, \quad \text{for } z \in \mathbb{D}.$$

Thus,

$$\begin{aligned} \|V_g f\|_{F(q, qt-2, s)}^p &= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |(V_g f)'(z)|^p (1 - |z|^2)^{pt-2} (1 - |\varphi_a(z)|^2)^s dA(z) \\ &= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^p |g'(z)|^p (1 - |z|^2)^{pt-2} (1 - |\varphi_a(z)|^2)^s dA(z) \\ &\lesssim \|g\|_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}}^p \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha (1 - |\varphi_a(z)|^2)^s dA(z) \\ &\lesssim \|f\|_{A_\alpha^p}^p \|g\|_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}}^p. \end{aligned}$$

If $p > q$, since

$$|g'(z)| (1 - |z|^2)^{\frac{pt-2-\alpha}{p}} \lesssim \|g\|_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}}$$

holds for all $z \in \mathbb{D}$, we have

$$|g'(z)|^q (1 - |z|^2)^{qt-2+s} \lesssim \|g\|_{\mathcal{B}}^q \frac{pt-2-\alpha}{p} (1 - |z|^2)^{\frac{q(2+\alpha)}{p}-2+s}, \quad \text{for } z \in \mathbb{D}.$$

For any arc $I \subset \partial\mathbb{D}$, using Hölder’s inequality, we have

$$\begin{aligned} \mu_{V_g(f)}(S(I)) &= \int_{S(I)} |f(z)|^q |g'(z)|^q (1 - |z|^2)^{qt-2+s} dA(z) \\ &\lesssim \|g\|_{\mathcal{B}}^q \int_{S(I)} |f(z)|^q (1 - |z|^2)^{\frac{q(2+\alpha)}{p}-2+s} dA(z) \\ &\lesssim \|g\|_{\mathcal{B}}^q \frac{pt-2-\alpha}{p} \left(\int_{S(I)} |f(z)|^p (1 - |z|^2)^\alpha dA(z) \right)^{\frac{q}{p}} \left(\int_{S(I)} (1 - |z|^2)^{\frac{ps}{p-q}-2} dA(z) \right)^{\frac{p-q}{p}}. \end{aligned}$$

Noting that $\frac{ps}{p-q} > 1$, we deduce $\frac{ps}{p-q} - 2 > -1$, which implies that (see [33, p. 138])

$$\int_{S(I)} (1 - |z|)^{\frac{ps}{p-q}-2} dA(z) \approx |I|^{\frac{ps}{p-q}}.$$

Hence,

$$\mu_{V_g(f)}(S(I)) \lesssim \|g\|_{\mathcal{B}}^q \frac{pt-2-\alpha}{p} \|f\|_{A_\alpha^p}^q |I|^s,$$

which implies that

$$\|V_g f\|_{F(q,qt-2,s)}^q \approx \sup_{I \subset \partial\mathbb{D}} \frac{\mu_{V_g(f)}(S(I))}{|I|^s} \lesssim \|g\|_{\mathcal{B}}^q \frac{pt-2-\alpha}{p} \|f\|_{A_\alpha^p}^q < \infty.$$

That is, V_g is bounded from A_α^p to $F(q,qt-2,s)$. Moreover,

$$\|V_g\| \lesssim \|g\|_{\mathcal{B}} \frac{pt-2-\alpha}{p}.$$

On the other hand, suppose that V_g is bounded from A_α^p to $F(q,qt-2,s)$. For any $a \in \mathbb{D}$, let

$$f_a(z) = \frac{1 - |a|}{(1 - \bar{a}z)^{1+\frac{\alpha+2}{p}}}.$$

It is not hard to check that $f_a \in A_\alpha^p$ and $\|f_a\|_{A_\alpha^p} \approx 1$. For $b \in \mathbb{D}$ and $r > 0$, let $\mathbb{D}(b,r) = \{z \in \mathbb{D} : \beta(b,z) < r\}$ denote the Bergman metric disk centered at b with radius r . From Proposition 4.3.8 in [39], we see that

$$\frac{(1 - |b|^2)^2}{|1 - \bar{b}z|^4} \approx \frac{1}{(1 - |z|^2)^2} \approx \frac{1}{(1 - |b|^2)^2} \approx \frac{1}{|\mathbb{D}(b,r)|}$$

when $z \in \mathbb{D}(b,r)$, where $|\mathbb{D}(b,r)|$ denotes the area of the Bergman disk $\mathbb{D}(b,r)$. Thus,

$$\begin{aligned} &\int_{\mathbb{D}(a,r)} |f_a(z)|^q |g'(z)|^q (1 - |z|^2)^{qt-2} (1 - |\varphi_a(z)|^2)^s dA(z) \\ &\lesssim \int_{\mathbb{D}} |f_a(z)|^q |g'(z)|^q (1 - |z|^2)^{qt-2} (1 - |\varphi_a(z)|^2)^s dA(z) \lesssim \|V_g f_a\|_{F(q,qt-2,s)}^q. \end{aligned}$$

Noting that

$$|f_a(z)| \approx \frac{1}{(1 - |a|^2)^{\frac{2+\alpha}{p}}} \approx \frac{1}{(1 - |z|^2)^{\frac{2+\alpha}{p}}}, \quad z \in \mathbb{D}(a, r),$$

we get

$$\begin{aligned} (1 - |a|^2)^{-\frac{q(2+\alpha)}{p} + qt} |g'(a)|^q &\lesssim \int_{\mathbb{D}(a,r)} |f_a(z)|^q |g'(z)|^q (1 - |z|^2)^{qt-2} (1 - |\varphi_a(z)|^2)^s dA(z) \\ &\lesssim \|V_g f_a\|_{F(q,qt-2,s)}^q \lesssim \|V_g\|^q, \end{aligned}$$

which implies that $g \in \mathcal{B}^{\frac{pt-2-\alpha}{p}}$ and

$$\|g\|_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}} \lesssim \|V_g\|.$$

From the above proof, we see that

$$\|V_g\| \approx \|g\|_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}}.$$

(2) Suppose that $|g'(z)|^{\frac{pq}{p-q}} (1 - |z|^2)^{(qt-2+s-\frac{q\alpha}{p})\frac{p}{p-q}} dA(z)$ is a $\frac{ps}{p-q}$ -Carleson measure. By Lemma 1, we need to show that the measure

$$d\mu_{V_g(f)}(z) = |V_g(f)'(z)|^q (1 - |z|)^{qt-2+s} dA(z)$$

is an s -Carleson measure for any $f \in A_\alpha^p$. For any arc $I \subset \partial\mathbb{D}$, we have

$$\begin{aligned} \mu_{V_g(f)}(S(I)) &= \int_{S(I)} |f(z)|^q |g'(z)|^q (1 - |z|)^{qt-2+s} dA(z) \\ &\lesssim \left(\int_{S(I)} |f(z)|^p (1 - |z|)^\alpha dA(z) \right)^{\frac{q}{p}} \left(\int_{S(I)} |g'(z)|^{\frac{qp}{p-q}} (1 - |z|)^{(qt-2+s-\frac{q\alpha}{p})\frac{p}{p-q}} dA(z) \right)^{\frac{p-q}{p}} \\ &\lesssim \|f\|_{A_\alpha^p}^q \left(|I|^{\frac{ps}{p-q}} \right)^{\frac{p-q}{p}} \|g\|_{F(\frac{pq}{p-q}, (qt-2-\frac{q\alpha}{p})\frac{p}{p-q}, \frac{ps}{p-q})}^q \\ &\lesssim \|f\|_{A_\alpha^p}^q \|g\|_{F(\frac{pq}{p-q}, (qt-2-\frac{q\alpha}{p})\frac{p}{p-q}, \frac{ps}{p-q})}^q |I|^s, \end{aligned}$$

which implies that

$$\|V_g f\|_{F(q,qt-2,s)}^q \approx \sup_{I \subset \partial\mathbb{D}} \frac{\mu_{V_g(f)}(S(I))}{|I|^s} \lesssim \|g\|_{F(\frac{pq}{p-q}, (qt-2-\frac{q\alpha}{p})\frac{p}{p-q}, \frac{ps}{p-q})}^q \|f\|_{A_\alpha^p}^q < \infty.$$

That is, V_g is bounded from A_α^p to $F(q,qt-2,s)$. Moreover,

$$\|V_g\| \lesssim \|g\|_{F(\frac{pq}{p-q}, (qt-2-\frac{q\alpha}{p})\frac{p}{p-q}, \frac{ps}{p-q})}.$$

On the other hand, suppose that V_g is bounded from A_α^p to $F(q,qt-2,s)$. Let $\{z_j\}$ be a δ -lattice in \mathbb{D} , $m > \max\{-1, 2(1 - 1/p)\}$ and

$$f_j(z) = \frac{(1 - |z_j|)^m}{(1 - \bar{z}_j z)^{m + \frac{2+\alpha}{p}}}, \quad j = 1, 2, \dots.$$

By Lemma 2, we know that for sufficient small $\delta > 0$, and any $\Delta = \{\lambda_j\} \in \ell^p$, the function

$$f_\Delta(z) = \sum_j \lambda_j f_j(z)$$

is a function in A_α^p and

$$\|f_\Delta\|_{A_\alpha^p} \leq C\|\Delta\|_{\ell^p}.$$

For every $x \in [0, 1)$, let $\Delta(x) = \{\lambda_j r_j(x)\}$. It is clear that

$$\|\Delta(x)\|_{\ell^p} = \|\Delta\|_{\ell^p}, \quad \text{for } x \in [0, 1).$$

Using Lemma 1 and the fact that V_g is bounded from A_α^p to $F(q, qt - 2, s)$, we have

$$\begin{aligned} & \frac{1}{|I|^s} \int_{S(I)} |f_{\Delta(x)}(z)|^q |g'(z)|^q (1 - |z|)^{qt-2+s} dA(z) \\ & \lesssim \|V_g\|^q \|f_{\Delta(x)}\|_{A_\alpha^p}^q \lesssim \|V_g\|^q \|\Delta(x)\|_{\ell^p}^q \lesssim \|V_g\|^q \|\Delta\|_{\ell^p}^q. \end{aligned}$$

Integrating both sides of the above inequality over $[0, 1)$ with respect to x , and then using Lemma 3, we obtain

$$\int_{S(I)} \left(\sum_j |\lambda_j|^2 |f_j(z)|^2 \right)^{\frac{q}{2}} |g'(z)|^q (1 - |z|)^{qt-2+s} dA(z) \lesssim |I|^s \|V_g\|^q \|\Delta\|_{\ell^p}^q.$$

For convenience, write $D_j = \mathbb{D}(z_j, \delta)$. We have

$$\begin{aligned} & \int_{D_j} |f_j(z)|^q |g'(z)|^q (1 - |z|)^{qt-2+s} dA(z) \\ & \approx (1 - |z_j|)^{-\frac{q(2+\alpha)}{p}} \int_{D_j} |g'(z)|^q (1 - |z|)^{qt-2+s} dA(z), \end{aligned}$$

which implies

$$\begin{aligned} & \sum_{j:D_j \subset S(I)} \frac{|\lambda_j|^q}{(1 - |z_j|)^{\frac{q(2+\alpha)}{p}}} \int_{D_j} |g'(z)|^q (1 - |z|)^{qt-2+s} dA(z) \\ & \approx \sum_{j:D_j \subset S(I)} \int_{D_j} \left(|\lambda_j|^2 |f_j(z)|^2 \right)^{q/2} |g'(z)|^q (1 - |z|)^{qt-2+s} dA(z) \\ & \lesssim \sum_{j:D_j \subset S(I)} \int_{D_j} \left(\sum_k |\lambda_k|^2 |f_k(z)|^2 \right)^{q/2} |g'(z)|^q (1 - |z|)^{qt-2+s} dA(z) \\ & \lesssim \int_{S(I)} \left(\sum_k |\lambda_k|^2 |f_k(z)|^2 \right)^{q/2} |g'(z)|^q (1 - |z|)^{qt-2+s} dA(z) \\ & \lesssim |I|^s \|V_g\|^q \|\Delta\|_{\ell^p}^q. \end{aligned}$$

This further implies, by the duality relation of $\ell^{\frac{p}{q}}$ and $\ell^{\frac{p}{p-q}}$, that

$$\left(\sum_{j:D_j \subset S(I)} \left(\frac{1}{(1-|z_j|)^{\frac{q(2+\alpha)}{p}}} \int_{D_j} |g'(z)|^q (1-|z|)^{qt-2+s} dA(z) \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}} \lesssim |I|^s \|V_g\|^q.$$

Let $z \in D_j$ and $\mathbb{D}(z, r) \subseteq D_j$. Using the sub-mean property of $|g'|^q$, that is,

$$|g'(z)|^q \lesssim \frac{1}{(1-|z|^2)^2} \int_{\mathbb{D}(z,r)} |g'(w)|^q dA(w),$$

we easily deduce that

$$\begin{aligned} & |g'(z)|^q (1-|z|^2)^{qt-2+s-\frac{q\alpha}{p}} \\ & \lesssim (1-|z|^2)^{-\frac{q\alpha}{p}-2} \int_{\mathbb{D}(z,r)} |g'(w)|^q (1-|w|^2)^{qt-2+s} dA(w) \\ & \lesssim (1-|z|^2)^{-\frac{q\alpha}{p}-2} \int_{D_j} |g'(w)|^q (1-|w|^2)^{qt-2+s} dA(w). \end{aligned}$$

Therefore,

$$\begin{aligned} & \int_{D_j} |g'(z)|^{\frac{pq}{p-q}} (1-|z|^2)^{(qt-2+s-\frac{q\alpha}{p})\frac{p}{p-q}} dA(z) \\ & \lesssim (1-|z_j|)^{-\frac{q\alpha-2p}{p-q}+2} \left(\int_{D_j} |g'(z)|^q (1-|z|^2)^{qt-2+s} dA(z) \right)^{\frac{p}{p-q}} \\ & \approx \left(\frac{1}{(1-|z_j|)^{\frac{q(2+\alpha)}{p}}} \int_{D_j} |g'(z)|^q (1-|z|^2)^{qt-2+s} dA(z) \right)^{\frac{p}{p-q}}. \end{aligned}$$

That is,

$$\sum_{j:D_j \subset S(I)} \int_{D_j} |g'(z)|^{\frac{pq}{p-q}} (1-|z|^2)^{(qt-2+s-\frac{q\alpha}{p})\frac{p}{p-q}} dA(z) \lesssim |I|^{\frac{ps}{p-q}} \|V_g\|^{\frac{pq}{p-q}}.$$

For an arc I , let $L_I = \{j : D_j \cap S(I) \neq \emptyset\}$. Clearly

$$S(I) \subset \bigcup_{j \in L_I} D_j.$$

Let J be the smallest arc on $\partial\mathbb{D}$ such that

$$\bigcup_{j \in L_I} D_j \subset S(J).$$

It is not hard to see that $I \subset J$ and $|J| \approx |I|$ (see [33]). Thus,

$$\begin{aligned} & \int_{S(I)} |g'(z)|^{\frac{pq}{p-q}} (1 - |z|^2)^{\left(qt - 2 + s - \frac{q\alpha}{p}\right) \frac{p}{p-q}} dA(z) \\ & \lesssim \sum_{j \in L_I} \int_{D_j} |g'(z)|^{\frac{pq}{p-q}} (1 - |z|^2)^{\left(qt - 2 + s - \frac{q\alpha}{p}\right) \frac{p}{p-q}} dA(z) \\ & \lesssim \sum_{j: D_j \subset S(J)} \int_{D_j} |g'(z)|^{\frac{pq}{p-q}} (1 - |z|^2)^{\left(qt - 2 + s - \frac{q\alpha}{p}\right) \frac{p}{p-q}} dA(z) \\ & \lesssim |J|^{\frac{ps}{p-q}} \|V_g\|^{\frac{pq}{p-q}} \lesssim |I|^{\frac{ps}{p-q}} \|V_g\|^{\frac{pq}{p-q}}, \end{aligned}$$

which implies that $g \in F\left(\frac{pq}{p-q}, \left(qt - 2 - \frac{q\alpha}{p}\right) \frac{p}{p-q}, \frac{ps}{p-q}\right)$ and

$$\|g\|_{F\left(\frac{pq}{p-q}, \left(qt - 2 - \frac{q\alpha}{p}\right) \frac{p}{p-q}, \frac{ps}{p-q}\right)} \lesssim \|V_g\|.$$

From the above proof, we see that

$$\|V_g\| \approx \|g\|_{F\left(\frac{pq}{p-q}, \left(qt - 2 - \frac{q\alpha}{p}\right) \frac{p}{p-q}, \frac{ps}{p-q}\right)}.$$

The proof is complete. \square

3. Essential norm of the operator V_g

Let us recall the definition of essential norm. Suppose that X be a Banach space and T is a bounded linear operator on X . The essential norm of T is the distance of T to the closed ideals of compact operators, that is

$$\|T\|_e = \inf\{\|T - S\| : S \text{ is a compact operator on } X\}.$$

For some results on essential norm of integral-type operators see, e.g., [6, 19, 23, 24, 26, 28, 29, 30, 35]. Note that T is compact if and only if $\|T\|_e = 0$. Let X and Y be two Banach spaces with $X \subset Y$. If $f \in Y$, then the distance from f to X is defined as

$$\text{dist}_Y(f, X) = \inf_{g \in X} \|f - g\|_Y.$$

LEMMA 4. [37] *If $\alpha > 0$ and $f \in \mathcal{B}^\alpha$, then*

$$\limsup_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha |f'(z)| \approx \text{dist}_{\mathcal{B}^\alpha}(f, \mathcal{B}_0^\alpha) \approx \limsup_{r \rightarrow 1^-} \|f - f_r\|_{\mathcal{B}^\alpha}.$$

Here $f_r(z) = f(rz)$, $0 < r < 1$, $z \in \mathbb{D}$.

LEMMA 5. Let $1 \leq q \leq p < \infty$, $0 < s < \infty$, $qt + s > 1$ and $-1 < \alpha < \min\{pt - 2, pt - 1 - \frac{p(1-s)}{q}\}$. If $g \in F(\frac{pq}{p-q}, (qt - 2 - \frac{q\alpha}{p})\frac{p}{p-q}, \frac{ps}{p-q})$, then

$$\begin{aligned} & \text{dist}_{F(\frac{pq}{p-q}, (qt-2-\frac{q\alpha}{p})\frac{p}{p-q}, \frac{ps}{p-q})} \left(g, F_0\left(\frac{pq}{p-q}, (qt-2-\frac{q\alpha}{p})\frac{p}{p-q}, \frac{ps}{p-q}\right) \right) \\ & \approx \left(\limsup_{|a| \rightarrow 1} \int_{\mathbb{D}} |g'(z)|^{\frac{pq}{p-q}} (1 - |z|^2)^{(qt-2-\frac{q\alpha}{p})\frac{p}{p-q}} (1 - |\varphi_a(z)|^2)^{\frac{ps}{p-q}} dA(z) \right)^{\frac{p-q}{pq}} \\ & \approx \limsup_{|r| \rightarrow 1} \|g - g_r\|_{F(\frac{pq}{p-q}, (qt-2-\frac{q\alpha}{p})\frac{p}{p-q}, \frac{ps}{p-q})}. \end{aligned}$$

Proof. The proof of the lemma is similar to Lemma 2.5 of [37], thus, we omit it. \square

LEMMA 6. Let $0 < r < 1$, $1 \leq q \leq p < \infty$, $0 < s < \infty$, $qt + s > 1$, $-1 < \alpha < \min\{pt - 2, pt - 1 - \frac{p(1-s)}{q}\}$ and $ps > p - q$. Let $g \in \mathcal{B}^{\frac{pt-2-\alpha}{p}}$. Then $V_{g_r} : A_{\alpha}^p \rightarrow F(q, qt - 2, s)$ is compact.

Proof. Let $\{f_n\}$ be any function sequence such that $\|f_n\|_{A_{\alpha}^p} \lesssim 1$ and $f_n \rightarrow 0$ uniformly on compact subsets of \mathbb{D} as $n \rightarrow \infty$. We only need to show that

$$\lim_{n \rightarrow \infty} \|V_{g_r} f_n\|_{F(q, qt-2, s)} = 0.$$

From the growth of A_{α}^p , we have $|f(z)| \lesssim \frac{\|f\|_{A_{\alpha}^p}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}$. Noting that

$$|g'_r(z)| \lesssim \frac{\|g\|_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}}}{(1-r^2)^{\frac{pt-2-\alpha}{p}}}, \quad z \in \mathbb{D},$$

we get

$$\begin{aligned} & \int_{\mathbb{D}} |f_n(z)|^q |g'_r(z)|^q (1 - |z|^2)^{qt-2} (1 - |\varphi_a(z)|^2)^s dA(z) \\ & \lesssim \frac{\|g\|_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}}^q}{(1-r^2)^{\frac{q(pt-2-\alpha)}{p}}} \int_{\mathbb{D}} |f_n(z)|^q (1 - |z|^2)^{qt-2} dA(z) \\ & \lesssim \frac{\|f_n\|_{A_{\alpha}^p}^q \|g\|_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}}^q}{(1-r^2)^{\frac{q(pt-2-\alpha)}{p}}} \int_{\mathbb{D}} (1 - |z|^2)^{qt-2-\frac{q(2+\alpha)}{p}} dA(z) \\ & \lesssim \|f_n\|_{A_{\alpha}^p}^q \|g\|_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}}^q. \end{aligned}$$

where we used the fact that $\int_{\mathbb{D}} (1 - |z|^2)^{\alpha} dA(z) < \infty$, $\alpha > -1$. That is

$$\|V_{g_r} f_n\|_{F(q, qt-2, s)}^q \lesssim \|f_n\|_{A_{\alpha}^p}^q \|g\|_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}}^q.$$

In addition,

$$|f_n(z)|^q(1 - |z|^2)^{qt-2} \lesssim (1 - |z|^2)^{qt-2-\frac{q(2+\alpha)}{p}}.$$

Then by the Dominated Convergence Theorem we get the desired result. \square

Noting that

$$g \in F\left(\frac{pq}{p-q}, \left(qt - 2 - \frac{q\alpha}{p}\right) \frac{p}{p-q}, \frac{ps}{p-q}\right) \subseteq \mathcal{B}^{\frac{(pt-\alpha)(p-q)}{p^2}},$$

we get the following lemma by the similar argument as Lemma 6.

LEMMA 7. *Let $0 < r < 1$, $1 \leq q \leq p < \infty$, $0 < s < \infty$, $qt + s > 1$, $-1 < \alpha < \min\{pt - 2, pt - 1 - \frac{p(1-s)}{q}\}$ and $ps \leq p - q$. If $g \in F(\frac{pq}{p-q}, (qt - 2 - \frac{q\alpha}{p}) \frac{p}{p-q}, \frac{ps}{p-q})$, then $V_{gr} : A_\alpha^p \rightarrow F(q, qt - 2, s)$ is compact.*

We also need the following lemma.

LEMMA 8. [32, Lemma 3.7] *Let X, Y be two Banach spaces of analytic functions on \mathbb{D} . Suppose that*

- (1) *The point evaluation functionals on Y are continuous.*
- (2) *The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.*
- (3) *$T : X \rightarrow Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if for any bounded sequence $\{f_j\}$ in X such that $\{f_j\}$ converges to zero uniformly on every compact set of \mathbb{D} , then the sequence $\{Tf_j\}$ converges to zero in the norm of Y .

THEOREM 2. *Let $1 \leq q \leq p < \infty$, $0 < s < \infty$, $qt + s > 1$ and $-1 < \alpha < \min\{pt - 2, pt - 1 - \frac{p(1-s)}{q}\}$. Suppose that $g \in H(\mathbb{D})$ such that $V_g : A_\alpha^p \rightarrow F(q, qt - 2, s)$ is bounded. Then the following statements hold.*

(1)

$$\|V_g\|_e \approx \limsup_{|z| \rightarrow 1^-} (1 - |z|^2)^{\frac{pt-2-\alpha}{p}} |g'(z)| \approx \text{dist}_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}} (g, \mathcal{B}_0^{\frac{pt-2-\alpha}{p}}),$$

when $ps > p - q$;

(2)

$$\begin{aligned} \|V_g\|_e &\approx \text{dist}_{F(\frac{pq}{p-q}, (qt-2-\frac{q\alpha}{p})\frac{p}{p-q}, \frac{ps}{p-q})} \left(g, F_0\left(\frac{pq}{p-q}, \left(qt - 2 - \frac{q\alpha}{p}\right) \frac{p}{p-q}, \frac{ps}{p-q}\right) \right) \\ &\approx \left(\limsup_{|a| \rightarrow 1} \int_{\mathbb{D}} |g'(z)|^{\frac{pq}{p-q}} (1 - |z|^2)^{(qt-2-\frac{q\alpha}{p})\frac{p}{p-q}} (1 - |\varphi_a(z)|^2)^{\frac{ps}{p-q}} dA(z) \right)^{\frac{p-q}{pq}}, \end{aligned}$$

when $ps \leq p - q$.

Proof. Let $\{I_n\}$ be the subarc sequence of $\partial\mathbb{D}$ such that $|I_n| \rightarrow 0$ as $n \rightarrow \infty$. Set $w_n = (1 - |I_n|)\zeta_n \in \mathbb{D}$, where ζ_n is the center of I_n . $n = 1, 2, \dots$. Then

$$1 - |w_n| \approx |1 - \overline{w_n}z| \approx |I_n|, \quad z \in S(I_n).$$

Take

$$f_n(z) = \frac{(1 - |w_n|)}{(1 - \overline{w_n}z)^{1 + \frac{2+\alpha}{p}}}.$$

Then $f_n \rightarrow 0$ uniformly on the compact subsets of \mathbb{D} as $n \rightarrow \infty$ and $\|f_n\|_{A_\alpha^p} \lesssim 1$. Thus, for any compact operator S from A_α^p to $F(q, qt - 2, s)$, by Lemma 8, we have

$$\lim_{n \rightarrow \infty} \|Sf_n\|_{F(q, qt-2, s)} = 0.$$

(1) From the above facts, by using Lemma 1 we deduce

$$\begin{aligned} \|V_g - S\| &\gtrsim \limsup_{n \rightarrow \infty} (\|V_g f_n\|_{F(q, qt-2, s)} - \|Sf_n\|_{F(q, qt-2, s)}) \\ &= \limsup_{n \rightarrow \infty} \|V_g f_n\|_{F(q, qt-2, s)} \\ &\gtrsim \limsup_{n \rightarrow \infty} \left(\frac{1}{|I_n|} \int_{S(I_n)} |(V_g f_n)'(z)|^q (1 - |z|^2)^{qt-2+s} dA(z) \right)^{\frac{1}{q}} \\ &= \limsup_{n \rightarrow \infty} \left(\frac{1}{|I_n|^s} \int_{S(I_n)} |f_n(z)|^q |g'(z)|^q (1 - |z|^2)^{qt-2+s} dA(z) \right)^{\frac{1}{q}} \\ &\approx \limsup_{n \rightarrow \infty} \left(\frac{1}{|I_n|^{s + \frac{q(2+\alpha)}{p}}} \int_{S(I_n)} |g'(z)|^q (1 - |z|^2)^{qt-2+s} dA(z) \right)^{\frac{1}{q}} \\ &\gtrsim \limsup_{n \rightarrow \infty} (1 - |w_n|^2)^{\frac{pt-2-\alpha}{p}} |g'(w_n)|, \end{aligned}$$

which implies that

$$\|V_g\|_e \gtrsim \limsup_{n \rightarrow \infty} (1 - |w_n|^2)^{\frac{pt-2-\alpha}{p}} |g'(w_n)|$$

and hence

$$\|V_g\|_e \gtrsim \limsup_{|z| \rightarrow 1^-} (1 - |z|^2)^{\frac{pt-2-\alpha}{p}} |g'(z)| \approx \text{dist}(g, \mathcal{B}_0^{\frac{pt-2-\alpha}{p}}).$$

On the other hand, by Lemma 6, $V_{g_r} : A_\alpha^p \rightarrow F(q, qt - 2, s)$ is a compact operator. Combining this with Theorem 1 and the linearity of V_g respect to g implies

$$\|V_g\|_e \leq \|V_g - V_{g_r}\| = \|V_{g-g_r}\| \approx \|g - g_r\|_{\mathcal{B}^{\frac{pt-2-\alpha}{p}}},$$

where the last asymptotic is deduced by the boundedness of V_{g-r} . By Lemma 4, we have

$$\|V_g\|_e \lesssim \limsup_{r \rightarrow 1^-} \|f - f_r\|_{\mathcal{B}^\alpha} \lesssim \limsup_{|z| \rightarrow 1^-} (1 - |z|^2)^{\frac{pt-2-\alpha}{p}} |g'(z)| \approx \text{dist}(g, \mathcal{B}_0^{\frac{pt-2-\alpha}{p}}).$$

(2) By using Theorem 1, Lemmas 5 and 7, similarly to the proof of (1) we can get the desired result. The proof is complete. \square

COROLLARY 1. *Let $1 \leq q \leq p < \infty$, $0 < s < \infty$, $qt + s > 1$ and $-1 < \alpha < \min\{pt - 2, pt - 1 - \frac{p(1-s)}{q}\}$. Suppose $g \in H(\mathbb{D})$ such that $V_g : A_\alpha^p \rightarrow F(q, qt - 2, s)$ is bounded. Then the following statements hold.*

(1) V_g is compact from A_α^p to $F(q, qt - 2, s)$ if and only if $g \in \mathcal{B}_0^{\frac{pt-2-\alpha}{p}}$ when $ps > p - q$;

(2) V_g is compact from A_α^p to $F(q, qt - 2, s)$ if and only if $g \in F_0(\frac{pq}{p-q}, (qt - 2 - \frac{q\alpha}{p})\frac{p}{p-q}, \frac{ps}{p-q})$ when $ps \leq p - q$.

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