

TAYLOR–TYPE EXPANSIONS IN TERMS OF EXPONENTIAL POLYNOMIALS

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Abstract. The aim of this paper is to derive an extension of the Taylor theorem related to linear differential operators with constant coefficients. For this aim, using divided differences with repeated arguments, the so-called characteristic element from the kernel of the differential operator is described. The extension of the Taylor theorem related to exponential polynomials and its consequences are established with integral remainder terms as well as in the form of mean value type theorems.

1. Introduction

There are two basic variants of the classical Taylor Theorem, which have a huge number of applications and extensions in various settings.

Given a function $f : I \rightarrow \mathbb{R}$, which is n times differentiable at $a \in I$ (where I is a non-degenerate real interval), the polynomial $T_{n,a}(f)$ defined by

$$T_{n,a}(f)(x) := \sum_{j=0}^n f^{(j)}(a) \cdot \frac{(x-a)^j}{j!}, \quad (1)$$

is called the n th-order Taylor polynomial of the function f at the base point a .

The form with integral remainder term can be formulated as follows.

THEOREM 1.1. *Let I be a real interval and let $f : I \rightarrow \mathbb{R}$ be $(n+1)$ times continuously differentiable. Then, for all $a, x \in I$,*

$$f(x) = T_{n,a}(f)(x) + \int_a^x f^{(n+1)}(t) \cdot \frac{(x-t)^n}{n!} dt.$$

The variant as an intermediate value theorem is the following assertion.

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THEOREM 1.2. *Let I be a real interval and let $f : I \rightarrow \mathbb{R}$ be $(n + 1)$ times differentiable. Then, for all $a, x \in I$, there exists a point ξ between a and x such that*

$$f(x) = T_{n;a}(f)(x) + f^{(n+1)}(\xi) \cdot \frac{(x - a)^{n+1}}{(n + 1)!}. \tag{2}$$

These two theorems are contained in most of the textbooks on basic analysis (see, e.g., [2], [3], [10], [11]). There have been several papers where extensions, generalizations and applications of the above fundamental results can be found, cf. [1], [5], [7], [6], [8], [9], [12].

The main content of these results is that they give a high order approximation of the function f near the point $a \in I$ in terms of the polynomial $T_{n;a}(f)$ defined by (1), which is of at most degree n and therefore it is in the kernel of the differential operator D given by $D(f) = f^{(n+1)}$.

It seems to be a natural problem to obtain similar approximations in terms of linear combinations of a given finite set of functions, in particular, in terms of exponential polynomials, which span the kernel of a linear differential operator with constant coefficients. Of course, the remainder term of such an approximation is of interest both in integral form and in terms of a mean value theorem.

The aim of this paper is to accomplish the above goal and derive a general form of the Taylor theorem related to a linear differential operator with constant coefficients. For this aim, in Section 2, we describe the so-called characteristic element from the kernel of the differential operator using divided differences with repeated arguments. The main results, an extension of the Taylor theorem and its consequences with an integral remainder term, are stated in Section 3, while mean value type extensions are established in Section 4.

2. Auxiliary results on linear differential equations

Let \mathbb{K} denote either the field of real or complex numbers. The imaginary unit in \mathbb{C} will be denoted by i .

Given an interval $I \subseteq \mathbb{R}$, let $\mathcal{C}_{\mathbb{K}}(I)$ stand for the space of continuous \mathbb{K} -valued functions defined on I . If additionally $n \in \mathbb{N}$, then let $\mathcal{C}_{\mathbb{K}}^n(I)$ denote the space of n -times continuously differentiable \mathbb{K} -valued functions defined on I . For $c = (c_0, \dots, c_n)$ in \mathbb{K}^{n+1} with $c_n = 1$, let n th-order linear differential operator $D_c : \mathcal{C}_{\mathbb{K}}^n(I) \rightarrow \mathcal{C}_{\mathbb{K}}(I)$ be defined by the formula

$$D_c(f) := c_n f^{(n)} + \dots + c_1 f' + c_0 f \quad (f \in \mathcal{C}_{\mathbb{K}}^n(I)). \tag{3}$$

Let $\omega_c \in \mathcal{C}_{\mathbb{K}}^n(\mathbb{R})$ denote the unique solution of the initial value problem

$$D_c(\omega_c) = 0, \quad \omega_c^{(\ell)}(0) = \delta_{\ell, n-1} \quad (\ell \in \{0, \dots, n-1\}). \tag{4}$$

The function ω_c will be called the *characteristic solution* of the differential equation $D_c(\omega) = 0$. One can see that if $c \in \mathbb{R}^{n+1}$, then ω_c is real-valued and hence it belongs to $\mathcal{C}_{\mathbb{R}}^n(\mathbb{R})$. Let P_c denote the characteristic polynomial of D_c , which is given by

$$P_c(\lambda) := c_n \lambda^n + \dots + c_1 \lambda + c_0 \quad (\lambda \in \mathbb{C}). \tag{5}$$

In order to provide a more or less explicit formula for P_c , we recall the notion of divided differences and their limiting properties.

If $D \subseteq \mathbb{K}$ and $n \in \mathbb{N}$, then let $\sigma_n(D)$ denote the set

$$\sigma_n(D) := \{(\lambda_1, \dots, \lambda_n) \in D^n \mid \lambda_i \neq \lambda_j \text{ for all } i, j \in \{1, \dots, n\} \text{ with } i \neq j\}.$$

For $f : D \rightarrow \mathbb{C}$, and $(\lambda_1, \dots, \lambda_n) \in \sigma_n(D)$, the $(n - 1)$ st order divided difference of f at $(\lambda_1, \dots, \lambda_n)$ is defined by

$$f(\lambda_1, \dots, \lambda_n) := \sum_{i=1}^n \frac{f(\lambda_i)}{\prod_{j \in \{1, \dots, n\} \setminus \{i\}} (\lambda_i - \lambda_j)},$$

see [4] for more details and alternative definitions. To define divided differences with repeated arguments, for $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$, let $(\lambda)^m$ denote the m -tuple $(\lambda, \dots, \lambda) \in \mathbb{C}^m$ and, for $n, k, m_1, \dots, m_k \in \mathbb{N}$ with $m_1 + \dots + m_k = n$ and $(\lambda_1, \dots, \lambda_k) \in \sigma_k(D)$, denote

$$f((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k}) := \lim_{\sigma_n(D) \ni (\mu_1, \dots, \mu_n) \rightarrow ((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k})} f(\mu_1, \dots, \mu_n)$$

provided that the limit exists. In the following lemma, we compute divided differences of f with repeated arguments under natural regularity assumptions.

LEMMA 2.1. *Let $D \subseteq \mathbb{K}$ be open, let $n, k, m_1, \dots, m_k \in \mathbb{N}$ with $m_1 + \dots + m_k = n$, let $(\lambda_1, \dots, \lambda_k) \in \sigma_k(D)$ and define the polynomials $P_1, \dots, P_k, P : \mathbb{C} \rightarrow \mathbb{C}$ by*

$$P_i(\lambda) := \prod_{j \in \{1, \dots, k\} \setminus \{i\}} (\lambda - \lambda_j)^{m_j} \quad (i \in \{1, \dots, k\}) \quad \text{and} \quad P(\lambda) := \prod_{j=1}^k (\lambda - \lambda_j)^{m_j}. \tag{6}$$

If $f : D \rightarrow \mathbb{C}$ is $(m_i - 1)$ times continuously differentiable at λ_i for all $i \in \{1, \dots, k\}$, then

$$f((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k}) = \sum_{i=1}^k \sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i - 1 - \ell)!} \cdot \frac{f^{(\ell)}(\lambda_i)}{\ell!}. \tag{7}$$

Furthermore,

$$f((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k}) = \sum_{i=1}^k \sum_{\ell=0}^{m_i-1} \left(\sum_{j=0}^{m_i-1-\ell} (-1)^j \frac{j! B_{m_i-1-\ell, j}(x_{i,1}, \dots, x_{i, m_i-\ell-j})}{(m_i - 1 - \ell)! x_{i,0}^{j+1}} \right) \frac{f^{(\ell)}(\lambda_i)}{\ell!},$$

where

$$x_{i,\alpha} := \frac{\alpha!}{(m_i + \alpha)!} P^{(m_i+\alpha)}(\lambda_i) \quad (i \in \{1, \dots, k\}, \alpha \in \{0, \dots, m_i\}).$$

Proof. In what follows, the symbol ∂_ℓ will stand for differentiation with respect to the variable λ_ℓ , where $\ell \in \{1, \dots, k\}$.

Using the well-known formula for divided differences with repeated arguments and also the higher-order Leibniz Rule at the very last equality, we get

$$\begin{aligned}
 f((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k}) &= \left(\prod_{\ell=1}^k \frac{\partial_\ell^{m_\ell-1}}{(m_\ell-1)!} \right) f(\lambda_1, \dots, \lambda_k) \\
 &= \left(\prod_{\ell=1}^k \frac{\partial_\ell^{m_\ell-1}}{(m_\ell-1)!} \right) \sum_{i=1}^k \frac{f(\lambda_i)}{\prod_{j \in \{1, \dots, k\} \setminus \{i\}} (\lambda_i - \lambda_j)} \\
 &= \sum_{i=1}^k \left(\prod_{\ell=1}^k \frac{\partial_\ell^{m_\ell-1}}{(m_\ell-1)!} \right) \frac{f(\lambda_i)}{\prod_{j \in \{1, \dots, k\} \setminus \{i\}} (\lambda_i - \lambda_j)} \\
 &= \sum_{i=1}^k \frac{\partial_i^{m_i-1}}{(m_i-1)!} \left(\prod_{\ell \in \{1, \dots, k\} \setminus \{i\}} \frac{\partial_\ell^{m_\ell-1}}{(m_\ell-1)!} \right) \frac{f(\lambda_i)}{\prod_{j \in \{1, \dots, k\} \setminus \{i\}} (\lambda_i - \lambda_j)} \\
 &= \sum_{i=1}^k \frac{\partial_i^{m_i-1}}{(m_i-1)!} f(\lambda_i) \left(\prod_{\ell \in \{1, \dots, k\} \setminus \{i\}} \frac{\partial_\ell^{m_\ell-1} (\lambda_i - \lambda_\ell)^{-1}}{(m_\ell-1)!} \right) \\
 &= \sum_{i=1}^k \frac{\partial_i^{m_i-1}}{(m_i-1)!} f(\lambda_i) \left(\prod_{\ell \in \{1, \dots, k\} \setminus \{i\}} (\lambda_i - \lambda_\ell)^{-m_\ell} \right) \\
 &= \sum_{i=1}^k \frac{(P_i^{-1} \cdot f)^{(m_i-1)}(\lambda_i)}{(m_i-1)!} \\
 &= \sum_{i=1}^k \sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i-1-\ell)!} \cdot \frac{f^{(\ell)}(\lambda_i)}{\ell!}.
 \end{aligned}$$

This proves the first equality of the lemma.

By applying the Faà di Bruno formula (see [3]) for the computation of the $(m_i - 1 - \ell)$ -th-order derivative of $P_i^{-1} = Q \circ P_i$ (with $Q(u) := u^{-1}$), we have

$$(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i) = \sum_{j=0}^{m_i-1-\ell} \frac{(-1)^j j!}{P_i^{j+1}(\lambda_i)} B_{m_i-1-\ell, j}(P_i'(\lambda_i), \dots, P_i^{(m_i-\ell-j)}(\lambda_i)).$$

For $i \in \{1, \dots, k\}$ and $\lambda \in \mathbb{C}$, we have that $P(\lambda) = (\lambda - \lambda_i)^{m_i} P_i(\lambda)$. Thus, using the higher-order Leibniz Rule again, for $\ell \geq 0$ and $i \in \{1, \dots, k\}$, we obtain

$$P^{(m_i+\ell)}(\lambda_i) = \sum_{j=\ell}^{m_i+\ell} \binom{m_i+\ell}{j} \frac{m_i!}{(j-\ell)!} (\lambda_i - \lambda_i)^{j-\ell} P_i^{(j)}(\lambda_i) = \frac{(m_i+\ell)!}{\ell!} P_i^{(\ell)}(\lambda_i).$$

Applying these equalities, for $i \in \{1, \dots, k\}$, we conclude that

$$\begin{aligned}
 &(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i) \\
 &= \sum_{j=0}^{m_i-1-\ell} (-1)^j \frac{j! B_{m_i-1-\ell, j} \left(\frac{1!}{(m_i+1)!} P^{(m_i+1)}(\lambda_i), \dots, \frac{(m_i-\ell-j)!}{(2m_i-\ell-j)!} P^{(2m_i-\ell-j)}(\lambda_i) \right)}{\left(\frac{0!}{m_i!} P^{(m_i)}(\lambda_i) \right)^{j+1}}.
 \end{aligned}$$

Substituting this expression into the first equality of the lemma, we get the second asserted formula. \square

REMARK 2.2. The i th term of the first (equivalently, of the second) formula of the lemma becomes very simple in the particular cases when $1 \leq m_i \leq 3$. Indeed,

$$\sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i-1-\ell)!} \cdot \frac{f^{(\ell)}(\lambda_i)}{\ell!}$$

$$= \begin{cases} \frac{1}{P'(\lambda_i)} f(\lambda_i) & \text{if } m_i = 1, \\ \frac{2}{P''(\lambda_i)} f'(\lambda_i) - \frac{2P'''(\lambda_i)}{3P''(\lambda_i)^2} f(\lambda_i) & \text{if } m_i = 2, \\ \frac{3}{P'''(\lambda_i)} f''(\lambda_i) - \frac{3P^{(4)}(\lambda_i)}{2P'''(\lambda_i)^2} f'(\lambda_i) + \left(\frac{3P^{(4)}(\lambda_i)^2}{8P'''(\lambda_i)^3} - \frac{3P^{(5)}(\lambda_i)}{10P'''(\lambda_i)^2} \right) f(\lambda_i) & \text{if } m_i = 3. \end{cases}$$

LEMMA 2.3. Let $n \in \mathbb{N}$, $c = (c_0, \dots, c_n) \in \mathbb{K}^{n+1}$ with $c_n = 1$, and let $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ be pairwise distinct roots of the characteristic polynomial P_c with multiplicities $m_1, \dots, m_k \in \mathbb{N}$, respectively. Then

$$\omega_c(t) = \sum_{i=1}^k \sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i-1-\ell)!} \cdot \frac{t^\ell \exp(\lambda_i t)}{\ell!},$$

where P_i is defined by (6).

Proof. From the theory of higher-order linear ordinary differential equations, it follows that the functions $t^\ell \exp(\lambda_i t)$ (where $i \in \{1, \dots, k\}$ and $\ell \in \{0, \dots, m_i - 1\}$) form a fundamental system of solutions to the linear differential equation $D_c(\omega) = 0$. Therefore, any linear combination of them is a solution, which proves that $D_c(\omega_c) = 0$ holds. To complete the proof, we need to show that ω_c also satisfies the initial value condition $\omega_c^{(j)}(0) = \delta_{j,n-1}$ for all $j \in \{0, \dots, n - 1\}$.

Denote $E_t(\lambda) := \exp(\lambda t)$. Then, for all $\ell \geq 0$, we have that $E_t^{(\ell)}(\lambda) = t^\ell \exp(\lambda t)$. Therefore, using the definition of ω_c and the previous lemma, we can conclude that

$$\omega_c(t) = \sum_{i=1}^k \sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i-1-\ell)!} \cdot \frac{E_t^{(\ell)}(\lambda_i)}{\ell!}$$

$$= E_t((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k}) = \lim_{\sigma_n(D) \ni (\mu_1, \dots, \mu_n) \rightarrow ((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k})} E_t(\mu_1, \dots, \mu_n).$$

Thus, for $\ell \geq 0$, we obtain

$$\omega_c^{(j)}(t) = \frac{d^j}{dt^j} \left(\lim_{\sigma_n(D) \ni (\mu_1, \dots, \mu_n) \rightarrow ((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k})} E_t(\mu_1, \dots, \mu_n) \right)$$

$$= \lim_{\sigma_n(D) \ni (\mu_1, \dots, \mu_n) \rightarrow ((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k})} \frac{d^j}{dt^j} E_t(\mu_1, \dots, \mu_n)$$

$$\begin{aligned}
 &= \lim_{\sigma_n(D) \ni (\mu_1, \dots, \mu_n) \rightarrow ((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k})} \frac{d^j}{dt^j} \sum_{i=1}^n \frac{\exp(\mu_i t)}{\prod_{\ell \in \{1, \dots, n\} \setminus \{i\}} (\mu_i - \mu_\ell)} \\
 &= \lim_{\sigma_n(D) \ni (\mu_1, \dots, \mu_n) \rightarrow ((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k})} \sum_{i=1}^n \frac{\mu_i^j \exp(\mu_i t)}{\prod_{\ell \in \{1, \dots, n\} \setminus \{i\}} (\mu_i - \mu_\ell)}.
 \end{aligned}$$

Finally, substituting $t = 0$, we arrive at

$$\omega_c^{(j)}(0) = \lim_{\sigma_n(D) \ni (\mu_1, \dots, \mu_n) \rightarrow ((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k})} \sum_{i=1}^n \frac{\mu_i^j}{\prod_{\ell \in \{1, \dots, n\} \setminus \{i\}} (\mu_i - \mu_\ell)}.$$

Observe that the sum in the above expression is equal to the $(n - 1)$ st-order divided difference of the j th monomial function at (μ_1, \dots, μ_n) . Therefore, this sum and hence its limit are equal to $\delta_{j,n-1}$ if $j \in \{0, \dots, n - 1\}$. \square

For the formulation of some consequences of our main results, for $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$ and $\gamma \in \mathbb{C}$, we define the function $\zeta_{n,k,\gamma} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\zeta_{n,k,\gamma}(t) := \sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n}}{(i(n-k) + n)!}. \tag{8}$$

By applying the ratio test, it follows that the series is convergent for all $t \in \mathbb{R}$. For further properties, we have the following statement.

LEMMA 2.4. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$ and $\gamma \in \mathbb{C}$. Then, the function $\zeta_{n,k,\gamma}$ is the (unique) solution of the initial value problem*

$$\zeta^{(n+1)} = \gamma \zeta^{(k+1)}, \quad \zeta^{(i)}(0) = \delta_{i,n} \quad (i \in \{0, \dots, n\}). \tag{9}$$

In addition, if $j \in \{0, \dots, k\}$, then

$$\zeta_{n,k,\gamma}^{(j)} = \zeta_{n-j,k-j,\gamma}. \tag{10}$$

If $\gamma \neq 0$, then, for all $t \in \mathbb{R}$,

$$\zeta_{n,k,\gamma^{n-k}}(t) = \gamma^{-n} \zeta_{n,k,1}(\gamma t). \tag{11}$$

Furthermore, for all $t \in \mathbb{R}$,

$$\zeta_{n,k,0}(t) = \frac{t^n}{n!}, \quad \zeta_{n,0,1}(t) = -1 + \frac{1}{n} \sum_{j=0}^{n-1} \exp\left(\cos\left(\frac{2\pi j}{n}\right)t\right) \cdot \cos\left(\sin\left(\frac{2\pi j}{n}\right)t\right). \tag{12}$$

Proof. The unique solvability of (9) is a consequence of standard results of the theory of linear differential equations. It is easy to see that $\zeta = \zeta_{n,k,\gamma}$ satisfies the

initial value conditions. To see that it fulfills the differential equation in (9), we have the following computation:

$$\begin{aligned} \zeta_{n,k,\gamma}^{(n+1)}(t) &= \left(\sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n}}{(i(n-k)+n)!} \right)^{(n+1)} = \sum_{i=1}^{\infty} \frac{\gamma^i t^{i(n-k)-1}}{(i(n-k)-1)!} \\ &= \gamma \sum_{i=1}^{\infty} \frac{\gamma^{i-1} t^{i(n-k)-1}}{(i(n-k)-1)!} = \gamma \sum_{i=0}^{\infty} \frac{\gamma^i t^{(i+1)(n-k)-1}}{((i+1)(n-k)-1)!} \\ &= \gamma \sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n-k-1}}{(i(n-k)+n-k-1)!} = \gamma \left(\sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n}}{(i(n-k)+n)!} \right)^{(k+1)} = \gamma \zeta_{n,k,\gamma}^{(k+1)}(t). \end{aligned}$$

For the proof of (10) when $j \in \{0, \dots, k\}$, observe that

$$\begin{aligned} \zeta_{n,k,\gamma}^{(j)}(t) &= \left(\sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n}}{(i(n-k)+n)!} \right)^{(j)} = \sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n-j}}{(i(n-k)+n-j)!} \\ &= \sum_{i=0}^{\infty} \frac{\gamma^i t^{i((n-j)-(k-j))+n-j}}{(i((n-j)-(k-j))+n-j)!} = \zeta_{n-j,k-j,\gamma}(t). \end{aligned}$$

We now show that (11) holds for $\gamma \neq 0$. Indeed,

$$\gamma^{-n} \zeta_{n,k,1}(\gamma t) = \gamma^{-n} \sum_{i=0}^{\infty} \frac{(\gamma t)^{i(n-k)+n}}{(i(n-k)+n)!} = \sum_{i=0}^{\infty} \frac{(\gamma^{n-k})^i t^{i(n-k)+n}}{(i(n-k)+n)!} = \zeta_{n,k,\gamma^{n-k}}(t).$$

The formula stated for $\zeta_{n,k,0}$ in (12) is obvious. To compute $\zeta_{n,0,1}$, we use the fact that this function is the unique solution $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ of the initial value problem

$$\zeta^{(n+1)} = \zeta', \quad \zeta^{(\ell)}(0) = \delta_{\ell,n} \quad (\ell \in \{0, \dots, n\}).$$

The characteristic polynomial of this linear differential equation is $P(\lambda) = \lambda^{n+1} - \lambda = (\lambda^n - 1)\lambda$. The roots of this polynomial are the n th roots of unity, $\lambda_j = \exp(i\frac{2\pi j}{n}) = \cos(\frac{2\pi j}{n}) + i \sin(\frac{2\pi j}{n}) =: \alpha_j + i\beta_j$, where $j \in \{0, \dots, n-1\}$ and $\lambda_n = 0$.

Using that $P(\lambda) = \lambda^{n+1} - \lambda$ and that $\lambda_n = 0$ and $\lambda_j^n = 1$ for all $j < n$, Lemma 2.3 implies that

$$\zeta(t) = \sum_{j=0}^n \frac{\exp(\lambda_j t)}{(n+1)\lambda_j^n - 1} = -1 + \sum_{j=0}^{n-1} \frac{\exp(\lambda_j t)}{(n+1)\lambda_j^n - 1} = -1 + \frac{1}{n} \sum_{j=0}^{n-1} \exp(\lambda_j t).$$

Taking into consideration that ζ is real valued, it follows that

$$\zeta(t) = -1 + \frac{1}{n} \sum_{j=0}^{n-1} \Re(\exp(\lambda_j t)) = -1 + \frac{1}{n} \sum_{j=0}^{n-1} \exp\left(\cos\left(\frac{2\pi j}{n}\right)t\right) \cdot \cos\left(\sin\left(\frac{2\pi j}{n}\right)t\right),$$

which proves the second formula in (12). \square

3. A generalization of the Taylor theorem

Our first main result can be stated as follows.

THEOREM 3.1. *Let $n \in \mathbb{N}$, $c = (c_0, \dots, c_n) \in \mathbb{K}^{n+1}$ with $c_n = 1$, and assume that $f : I \rightarrow \mathbb{K}$ is $(n - 1)$ times differentiable at $a \in I$. Define $T_{a,c}f : \mathbb{R} \rightarrow \mathbb{K}$ by*

$$(T_{a,c}f)(x) := \sum_{j=0}^{n-1} \left(f^{(j)}(a) \sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(x-a) \right),$$

where ω_c is defined by (4). Then, $T_{a,c}f$ belongs to the kernel of D_c and

$$f^{(\ell)}(a) = (T_{a,c}f)^{(\ell)}(a) \quad (\ell \in \{0, \dots, n - 1\}). \tag{13}$$

The function $T_{a,c}f$ is termed the *generalized Taylor polynomial at the point a with respect to the differential operator D_c* .

Proof. The characteristic function ω_c satisfies the differential equation (3), i.e., we have

$$c_n \omega_c^{(n)} + \dots + c_1 \omega_c' + c_0 \omega_c = 0. \tag{14}$$

Differentiating this equality i times (where $i \in \mathbb{N}$), we can see that $\omega_c^{(i)}$ also solves the differential equation (3). It is also obvious that the function $\omega_c^{(i)}$ with the translated argument $(x - a)$ is still a solution to (3). Therefore, $T_{a,c}f$ is a linear combination of solutions of the differential equation (3), which implies that $T_{a,c}f$ belongs to the kernel of D_c .

From the definition of the function $T_{a,c}f$, for $\ell \in \{0, \dots, n - 1\}$, we have that

$$(T_{a,c}f)^{(\ell)}(a) = \sum_{j=0}^{n-1} \left(f^{(j)}(a) \sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i+\ell)}(0) \right).$$

In order to prove that (13) holds, it is sufficient to verify that

$$\sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i+\ell)}(0) = \delta_{j,\ell} \quad (j, \ell \in \{0, \dots, n - 1\}). \tag{15}$$

Using the initial value conditions in (4), we get

$$\sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i+\ell)}(0) = \sum_{i=n-1-\ell}^{n-1-j} c_{i+j+1} \omega_c^{(i+\ell)}(0) = \sum_{\alpha=n+j-\ell}^n c_\alpha \omega_c^{(\alpha+\ell-j-1)}(0). \tag{16}$$

If $j > \ell$, then $n + j - \ell > n$, thus the summation is over the empty set, and therefore (15) is trivially valid.

If $j = \ell$, then

$$\sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i+\ell)}(0) = \sum_{\alpha=n}^n c_\alpha \omega_c^{(\alpha-1)}(0) = c_n \omega_c^{(n-1)}(0) = 1,$$

which proves that (15) holds in this case.

Finally, assume that $j < \ell$. Differentiating the equation (14) $(\ell - j - 1)$ times and then evaluating it at 0, we get

$$0 = \sum_{\alpha=0}^n c_\alpha \omega_c^{(\alpha+\ell-j-1)}(0) = \sum_{\alpha=n+j-\ell}^n c_\alpha \omega_c^{(\alpha+\ell-j-1)}(0),$$

which, combined with (16), shows that (15) is also valid in this case. \square

THEOREM 3.2. *Let $n \in \mathbb{N}$, $c = (c_0, \dots, c_n) \in \mathbb{K}^{n+1}$ with $c_n = 1$. Then, for all $f \in \mathcal{C}_{\mathbb{K}}^n(I)$ and $x, a \in I$, we have*

$$f(x) = (T_{a,c}f)(x) + \int_a^x D_c(f)(t) \cdot \omega_c(x-t) dt. \tag{17}$$

Proof. Let $k \in \{1, \dots, n\}$ be fixed. First we show that, for all $j \in \{0, \dots, k\}$,

$$\int_a^x f^{(k)}(t) \cdot \omega_c(x-t) dt = \sum_{i=0}^{j-1} \left(f^{(k-1-i)}(x) \cdot \omega_c^{(i)}(0) - f^{(k-1-i)}(a) \cdot \omega_c^{(i)}(x-a) \right) + \int_a^x f^{(k-j)}(t) \cdot \omega_c^{(j)}(x-t) dt. \tag{18}$$

We show this equality by induction on j . Clearly, this equality holds if $j = 0$ (because the domain of summation is then empty, and therefore the sum equals 0). Assume that we have proved (18) for some $j \in \{0, \dots, k-1\}$. Then, integrating by parts, we have

$$\int_a^x f^{(k-j)}(t) \cdot \omega_c^{(j)}(x-t) dt = f^{(k-1-j)}(x) \cdot \omega_c^{(j)}(0) - f^{(k-1-j)}(a) \cdot \omega_c^{(j)}(x-a) + \int_a^x f^{(k-1-j)}(t) \cdot \omega_c^{(j+1)}(x-t) dt.$$

Combining this equality with (18), we get

$$\int_a^x f^{(k)}(t) \cdot \omega_c(x-t) dt = \sum_{i=0}^j \left(f^{(k-1-i)}(x) \cdot \omega_c^{(i)}(0) - f^{(k-1-i)}(a) \cdot \omega_c^{(i)}(x-a) \right) + \int_a^x f^{(k-1-j)}(t) \cdot \omega_c^{(j+1)}(x-t) dt,$$

which is exactly the statement (18) for $j+1$ and completes the induction.

Applying (18) for $j = k$, we get that

$$\int_a^x f^{(k)}(t) \cdot \omega_c(x-t) dt = \sum_{i=0}^{k-1} \left(f^{(k-1-i)}(x) \cdot \omega_c^{(i)}(0) - f^{(k-1-i)}(a) \cdot \omega_c^{(i)}(x-a) \right) + \int_a^x f(t) \cdot \omega_c^{(k)}(x-t) dt \tag{19}$$

is valid for $k \in \{1, \dots, n\}$ and, trivially, also for $k = 0$. Multiplying (19) by c_k and adding up the equalities so obtained side by side, and applying (4), we get

$$\begin{aligned}
 & \int_a^x D_c(f)(t) \cdot \omega_c(x-t) dt \\
 &= \sum_{k=0}^n c_k \int_a^x f^{(k)}(t) \cdot \omega_c(x-t) dt \\
 &= \sum_{k=0}^n c_k \left(\sum_{i=0}^{k-1} \left(f^{(k-1-i)}(x) \cdot \omega_c^{(i)}(0) - f^{(k-1-i)}(a) \cdot \omega_c^{(i)}(x-a) \right) \right) \\
 &\quad + \sum_{k=0}^n c_k \left(\int_a^x f(t) \cdot \omega_c^{(k)}(x-t) dt \right) \\
 &= \sum_{k=1}^n c_k \left(\sum_{i=0}^{k-1} \left(f^{(k-1-i)}(x) \cdot \omega_c^{(i)}(0) - f^{(k-1-i)}(a) \cdot \omega_c^{(i)}(x-a) \right) \right) \\
 &\quad + \int_a^x f(t) \cdot D_c(\omega_c)(x-t) dt \\
 &= \sum_{k=1}^n \sum_{i=0}^{k-1} c_k f^{(k-1-i)}(x) \cdot \omega_c^{(i)}(0) - \sum_{k=1}^n \sum_{i=0}^{k-1} c_k f^{(k-1-i)}(a) \cdot \omega_c^{(i)}(x-a) \\
 &= c_n f(x) - \sum_{j=0}^{n-1} \left(f^{(j)}(a) \sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(x-a) \right) = f(x) - (T_{a,c}f)(x).
 \end{aligned}$$

This completes the proof of (17). \square

The following result is a consequence of Theorem 3.2 in which the main part contains the Taylor expansion of order k and the rest is in terms of the function $\zeta_{n,k,\gamma}$.

THEOREM 3.3. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$, and $\gamma \in \mathbb{K}$. Then, for all $f \in \mathcal{C}_{\mathbb{K}}^n(I)$ and $x, a \in I$,*

$$\begin{aligned}
 f(x) &= \sum_{j=0}^{k-1} f^{(j)}(a) \frac{(x-a)^j}{j!} + \sum_{j=k}^{n-1} f^{(j)}(a) \zeta_{n,k,\gamma}^{(n-j)}(x-a) \\
 &\quad + \int_a^x (f^{(n)}(t) - \gamma f^{(k)}(t)) \zeta'_{n,k,\gamma}(x-t) dt.
 \end{aligned} \tag{20}$$

Proof. Consider the particular case of Theorem 3.2 when $c_n = 1$, $c_k = -\gamma$, and $c_i = 0$ for $i \in \{0, \dots, n-1\} \setminus \{k\}$, i.e., when $D_c(f) = f^{(n)} - \gamma f^{(k)}$. We show that $\omega_c = \zeta'_{n,k,\gamma}$. The leading term of this power series is $\frac{t^n}{n!}$, therefore, $\zeta_{n,k,\gamma}^{(i+1)}(0) = \delta_{i,n-1}$ for all $i \in \{0, \dots, n-1\}$. On the other hand,

$$\begin{aligned}
 D_c(\zeta'_{n,k,\gamma})(t) &= \zeta_{n,k,\gamma}^{(n+1)}(t) - \gamma \zeta_{n,k,\gamma}^{(k+1)}(t) \\
 &= \sum_{i=1}^{\infty} \frac{\gamma^i t^{i(n-k)-1}}{(i(n-k)-1)!} - \gamma \sum_{i=0}^{\infty} \frac{\gamma^i t^{(i+1)(n-k)-1}}{((i+1)(n-k)-1)!} = 0.
 \end{aligned}$$

Thus, we have obtained that $\zeta'_{n,k,\gamma}$ is a solution of the initial value problem (4).

For $j \in \{0, \dots, k-1\}$, we get

$$\begin{aligned} \sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(t) &= c_n \zeta_{n,k,\gamma}^{(n-j)}(t) + c_k \zeta_{n,k,\gamma}^{(k-j)}(t) \\ &= \sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+j}}{(i(n-k)+j)!} - \gamma \sum_{i=0}^{\infty} \frac{\gamma^i t^{(i+1)(n-k)+j}}{((i+1)(n-k)+j)!} = \frac{t^j}{j!}. \end{aligned}$$

Similarly, for $j \in \{k, \dots, n-1\}$, we get

$$\sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(t) = c_n \zeta_{n,k,\gamma}^{(n-j)}(t).$$

Putting these formulas together, we can see that (17) simplifies to the equality (20), which was to be proved. \square

The subsequent results will be corollaries of Theorem 3.3. First we note that the classical Taylor theorem with an integral remainder term follows from Theorem 3.3 by taking $k = 0$ and $\gamma = 0$.

COROLLARY 3.4. For all $f \in \mathcal{C}_{\mathbb{K}}^2(I)$ and $a, x \in I$, we have

$$f(x) = f(a) \cos(x-a) + f'(a) \sin(x-a) + \int_a^x (f''(t) + f(t)) \sin(x-t) dt. \quad (21)$$

Proof. Let $n = 2$, $k = 0$ and $\gamma = -1$ in Theorem 3.3. Then, $\zeta_{2,0,-1}(t) = 1 - \cos(t)$. Therefore, the equality (20) reduces to (21). \square

COROLLARY 3.5. For all $f \in \mathcal{C}_{\mathbb{K}}^2(I)$ and $a, x \in I$, we have

$$f(x) = f(a) \cosh(x-a) + f'(a) \sinh(x-a) + \int_a^x (f''(t) - f(t)) \sinh(x-t) dt. \quad (22)$$

Proof. Let $n = 2$, $k = 0$ and $\gamma = 1$ in Theorem 3.3. Then, $\zeta_{2,0,1}(t) = \cosh(t) - 1$. Hence, the equality (20) simplifies to (22). \square

COROLLARY 3.6. For all $f \in \mathcal{C}_{\mathbb{K}}^4(I)$ and $a, x \in I$, we have

$$\begin{aligned} f(x) &= f(a) \frac{\cosh(x-a) + \cos(x-a)}{2} + f'(a) \frac{\sinh(x-a) + \sin(x-a)}{2} \\ &\quad + f''(a) \frac{\cosh(x-a) - \cos(x-a)}{2} + f''(a) \frac{\sinh(x-a) - \sin(x-a)}{2} \\ &\quad + \int_a^x (f'''(t) - f(t)) \frac{\sinh(x-t) - \sin(x-t)}{2} dt. \end{aligned} \quad (23)$$

Proof. Let $n = 4$, $k = 0$ and $\gamma = 1$ in Theorem 3.3. Then, it is easy to see that

$$\zeta_{4,0,1}(t) = \sum_{i=0}^{\infty} \frac{t^{4i+j}}{(4i+4)!} = \frac{\cosh(t) + \cos(t)}{2} - 1.$$

Thus, the equality (20) of Theorem 3.3 can be rewritten as (23). \square

COROLLARY 3.7. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. Then, for all $f \in \mathcal{C}_{\mathbb{K}}^4(I)$ and $a, x \in I$, we have*

$$\begin{aligned} f(x) &= f(a) \frac{\beta^2 \cos(\alpha(x-a)) - \alpha^2 \cos(\beta(x-a))}{\beta^2 - \alpha^2} \\ &+ f'(a) \frac{\beta^3 \sin(\alpha(x-a)) - \alpha^3 \sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\ &+ f''(a) \frac{\cos(\alpha(x-a)) - \cos(\beta(x-a))}{\beta^2 - \alpha^2} \\ &+ f'''(a) \frac{\beta \sin(\alpha(x-a)) - \alpha \sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\ &+ \int_a^x (f''''(t) + (\alpha^2 + \beta^2)f''(t) + \alpha^2\beta^2 f(t)) \frac{\beta \sin(\alpha(x-t)) - \alpha \sin(\beta(x-t))}{\alpha\beta(\beta^2 - \alpha^2)} dt. \end{aligned} \tag{24}$$

Proof. Let $n = 4$ and apply Theorem 3.2 in the setting $c_4 = 1$, $c_3 = c_1 = 0$, $c_2 = \alpha^2 + \beta^2$, $c_0 = \alpha^2\beta^2$, that is, when $D_c(f) = f'''' + (\alpha^2 + \beta^2)f'' + \alpha^2\beta^2 f$. Then the corresponding characteristic polynomial is $P_c(\lambda) = \lambda^4 + (\alpha^2 + \beta^2)\lambda^2 + \alpha^2\beta^2$ whose roots are $\pm\alpha i$ and $\pm\beta i$. Therefore, $\sin(\alpha x)$, $\cos(\alpha x)$, $\sin(\beta x)$, and $\cos(\beta x)$ form a fundamental system of solutions for the differential equation $D_c(f) = 0$. Then, the characteristic solution ω_c is given by

$$\omega_c(t) = \frac{\beta \sin(\alpha t) - \alpha \sin(\beta t)}{\alpha\beta(\beta^2 - \alpha^2)}.$$

We can easily obtain

$$\sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(t) = \begin{cases} \frac{\beta^2 \cos(\alpha t) - \alpha^2 \cos(\beta t)}{\beta^2 - \alpha^2} & \text{if } j = 0, \\ \frac{\beta^3 \sin(\alpha t) - \alpha^3 \sin(\beta t)}{\alpha\beta(\beta^2 - \alpha^2)} & \text{if } j = 1, \\ \frac{\cos(\alpha t) - \cos(\beta t)}{\beta^2 - \alpha^2} & \text{if } j = 2, \\ \frac{\beta \sin(\alpha t) - \alpha \sin(\beta t)}{\alpha\beta(\beta^2 - \alpha^2)} & \text{if } j = 3. \end{cases}$$

Therefore, the equalities (17) and (24) turn out to be equivalent. \square

The limiting case of the above corollary (i.e., when $\alpha^2 = \beta^2 \neq 0$) is formulated as follows.

COROLLARY 3.8. Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. Then, for all $f \in \mathcal{C}_{\mathbb{K}}^4(I)$ and $a, x \in I$, we have

$$\begin{aligned} f(x) = & f(a) \frac{2 \cos(\alpha(x-a)) + \alpha(x-a) \sin(\alpha(x-a))}{2} \\ & + f'(a) \frac{3 \sin(\alpha(x-a)) - \alpha(x-a) \cos(\alpha(x-a))}{2\alpha} \\ & + f''(a) \frac{(x-a) \sin(\alpha(x-a))}{2\alpha} + f'''(a) \frac{\sin(\alpha(x-a)) - \alpha(x-a) \cos(\alpha(x-a))}{2\alpha^3} \\ & + \int_a^x (f''''(t) + 2\alpha^2 f''(t) + \alpha^4 f(t)) \frac{\sin(\alpha(x-t)) - \alpha(x-t) \cos(\alpha(x-t))}{2\alpha^3} dt. \end{aligned}$$

The proofs of this and of the next two corollaries are completely similar to that of Corollary 3.7, and hence they are omitted. The results in terms of hyperbolic functions in the next corollaries are analogous to Corollary 3.7 and Corollary 3.8 which are in terms of trigonometric functions.

COROLLARY 3.9. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. Then, for all $f \in \mathcal{C}_{\mathbb{K}}^4(I)$ and $a, x \in I$, we have

$$\begin{aligned} f(x) = & f(a) \frac{\beta^2 \cosh(\alpha(x-a)) - \alpha^2 \cosh(\beta(x-a))}{\beta^2 - \alpha^2} \\ & + f'(a) \frac{\beta^3 \sinh(\alpha(x-a)) - \alpha^3 \sinh(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\ & + f''(a) \frac{\cosh(\beta(x-a)) - \cosh(\alpha(x-a))}{\beta^2 - \alpha^2} \\ & + f'''(a) \frac{\alpha \sinh(\beta(x-a)) - \beta \sinh(\alpha(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\ & + \int_a^x (f''''(t) - (\alpha^2 + \beta^2) f''(t) + \alpha^2 \beta^2 f(t)) \\ & \quad \times \frac{\alpha \sinh(\beta(x-t)) - \beta \sinh(\alpha(x-t))}{\alpha\beta(\beta^2 - \alpha^2)} dt. \end{aligned} \tag{25}$$

COROLLARY 3.10. Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. Then, for all $f \in \mathcal{C}_{\mathbb{K}}^4(I)$ and $a, x \in I$, we have

$$\begin{aligned} f(x) = & f(a) \frac{2 \cosh(\alpha(x-a)) - \alpha(x-a) \sinh(\alpha(x-a))}{2} \\ & + f'(a) \frac{3 \sinh(\alpha(x-a)) - \alpha(x-a) \cosh(\alpha(x-a))}{2\alpha} \\ & + f''(a) \frac{(x-a) \sinh(\alpha(x-a))}{2\alpha} \\ & + f'''(a) \frac{\alpha(x-a) \cosh(\alpha(x-a)) - \sinh(\alpha(x-a))}{2\alpha^3} \\ & + \int_a^x (f''''(t) - 2\alpha^2 f''(t) + \alpha^4 f(t)) \frac{\alpha(x-t) \cosh(\alpha(x-t)) - \sinh(\alpha(x-t))}{2\alpha^3} dt. \end{aligned}$$

4. A generalization of the Taylor mean value theorem

Before describing the mean value form of Theorem 3.2, we recall the extended mean value theorem for integrals.

LEMMA 4.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function and $g : [a, b] \rightarrow \mathbb{R}$ a nonnegative (or nonpositive) integrable function. Then there exists $\xi \in [a, b]$ such that*

$$\int_a^b fg = f(\xi) \int_a^b g.$$

In the rest of this paper, for any continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$, let $\rho^+(h) \in [0, +\infty]$ (resp. $\rho^-(h) \in [-\infty, 0]$) denote the infimum of the positive roots (resp. the supremum of the negative roots) of h .

LEMMA 4.2. *Let $n \in \mathbb{N}$ and $c = (c_0, \dots, c_n) \in \mathbb{R}^{n+1}$ with $c_n = 1$. Then, for $k \in \{1, \dots, n - 1\}$,*

$$[\rho^-(\omega_c^{(k)}), \rho^+(\omega_c^{(k)})] \subseteq [\rho^-(\omega_c^{(k-1)}), \rho^+(\omega_c^{(k-1)})]. \tag{26}$$

Furthermore, $[\rho^-(\omega_c^{(n-1)}), \rho^+(\omega_c^{(n-1)})]$ is a neighborhood of 0.

Proof. To prove (26), let $k \in \{1, \dots, n - 1\}$ and $t \in [\rho^-(\omega_c^{(k)}), \rho^+(\omega_c^{(k)})] \setminus \{0\}$. Assume first that $t > 0$. Then $\omega_c^{(k)}$ does not vanish in the open interval $]0, t[$. Therefore, $\omega_c^{(k-1)}$ is strictly monotone on $[0, t]$. Hence, for all $s \in]0, t[$, we have $0 = \omega_c^{(k-1)}(0) \neq \omega_c^{(k-1)}(s)$, which shows that $t \in [\rho^-(\omega_c^{(k-1)}), \rho^+(\omega_c^{(k-1)})]$. This completes the proof of the inclusion in (26) for positive elements. For negative elements, the proof is completely analogous.

We show that 0 is an interior point to the interval $[\rho^-(\omega_c^{(n-1)}), \rho^+(\omega_c^{(n-1)})]$. In view of $\omega_c^{(n-1)}(0) = 1$, it follows that $\omega_c^{(n-1)}$ is positive on $[-r, r]$ for some $r > 0$. Clearly, $[-r, r] \subseteq [\rho^-(\omega_c^{(n-1)}), \rho^+(\omega_c^{(n-1)})]$. \square

THEOREM 4.3. *Let $n \in \mathbb{N}$, $c = (c_0, \dots, c_n) \in \mathbb{R}^{n+1}$ with $c_n = 1$. Then, for all $f \in \mathcal{C}_{\mathbb{R}}^n(I)$ and $a, x \in I$ with $\rho^-(\omega_c) \leq x - a \leq \rho^+(\omega_c)$, there exists a point ξ between a and x such that*

$$f(x) = (T_{a,c}f)(x) + D_c(f)(\xi) \cdot \int_0^{x-a} \omega_c(t) dt. \tag{27}$$

Proof. In view of Theorem 3.2, we have that (17) holds. The statement is trivial if $x = a$. Assume first that $a < x$. Then, by our assumption, $x - a \leq \rho^+(\omega_c)$. If $t \in]a, x[$, then $0 < x - t < x - a$ and hence $\omega_c(x - t)$ has the same sign for all $t \in]a, x[$. Using this, by Lemma 4.1, we conclude that there exists a point $\xi \in [a, x]$ such that

$$\int_a^x D_c(f)(t) \cdot \omega_c(x - t) dt = D_c(f)(\xi) \cdot \int_a^x \omega_c(x - t) dt = D_c(f)(\xi) \cdot \int_0^{x-a} \omega_c(t) dt.$$

Using this equality, formula (17) implies the assertion.

In the case when $x < a$, the proof is analogous. \square

THEOREM 4.4. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$ and $\gamma \in \mathbb{R}$ and define $\zeta_{n,k,\gamma}: \mathbb{R} \rightarrow \mathbb{R}$ by (8). Then, for all $f \in \mathcal{C}_{\mathbb{R}}^n(I)$ and $x, a \in I$ with $\rho^-(\zeta'_{n,k,\gamma}) \leq x - a \leq \rho^+(\zeta'_{n,k,\gamma})$, there exists a point ξ between a and x such that*

$$f(x) = \sum_{j=0}^{k-1} f^{(j)}(a) \frac{(x-a)^j}{j!} + \sum_{j=k}^{n-1} f^{(j)}(a) \zeta_{n,k,\gamma}^{(n-j)}(x-a) + (f^{(n)}(\xi) - \gamma f^{(k)}(\xi)) \zeta_{n,k,\gamma}(x-a). \tag{28}$$

Proof. Let the vector $c \in \mathbb{R}^{n+1}$ be given by $c_n = 1$, $c_k := -\gamma$ and $c_j = 0$ otherwise. Then $D_c(f) = f^{(n)} - \gamma f^{(k)}$ and, as we have seen it in the proof of Theorem 3.3, the characteristic solution ω_c of the differential equation $D_c(\omega) = 0$ is equal to $\zeta'_{n,k,\gamma}$.

Assume first that $a < x$. Then, by our assumption, $x - a \leq \rho^+(\zeta'_{n,k,\gamma})$. If $t \in]a, x[$, then $0 < x - t < x - a$ and hence $\zeta'_{n,k,\gamma}(x - t)$ has the same sign for all $t \in]a, x[$. Using this, by Lemma 4.1, we conclude that there exists a point $\xi \in [a, x]$ such that

$$\int_a^x (f^{(n)}(t) - \gamma f^{(k)}(t)) \zeta'_{n,k,\gamma}(x-t) dt = (f^{(n)}(\xi) - \gamma f^{(k)}(\xi)) \int_a^x \zeta'_{n,k,\gamma}(x-t) dt = (f^{(n)}(\xi) - \gamma f^{(k)}(\xi)) \zeta_{n,k,\gamma}(x-a).$$

By Theorem 3.3, we have formula (20), combining it with the above equality, we get (28). \square

The classical Taylor Mean Value Theorem (stated as Theorem 1.2 in the introduction) is the particular case of Theorem 4.4 when $k = 0$ and $\gamma = 0$. In this setting, we have that $\zeta_{n,0,0}(t) = \frac{t^n}{n!}$ and hence $\rho^\pm(\zeta'_{n,0,0}) = \pm\infty$ and (28) simplifies to (2).

COROLLARY 4.5. *For all $f \in \mathcal{C}_{\mathbb{R}}^2(I)$ and $a, x \in I$ with $|a - x| \leq \pi$, there exists a point ξ between a and x such that*

$$f(x) = f(a) \cos(x-a) + f'(a) \sin(x-a) + (f''(\xi) + f(\xi))(1 - \cos(x-a)). \tag{29}$$

Proof. Let $n = 2$, $k = 0$ and $\gamma = -1$ in Theorem 4.4. Then, $\zeta_{2,0,-1}(t) = 1 - \cos(t)$ and hence $\rho^\pm(\zeta'_{2,0,-1}) = \pm\pi$. Therefore, we can apply the statement of Theorem 4.4 and the equality (28) reduces to (29). \square

To see that the condition $|a - x| \leq \pi$ of the above corollary cannot be omitted, consider the function $f(x) := x$. Then, for $x = 2\pi$ and $a = 0$, the equality (29) simplifies to $2\pi = \xi \cdot 0$, which cannot be valid for any $\xi \in \mathbb{R}$.

COROLLARY 4.6. *For all $f \in \mathcal{C}_{\mathbb{R}}^2(I)$ and $a, x \in I$, there exists a point ξ between a and x such that*

$$f(x) = f(a) \cosh(x-a) + f'(a) \sinh(x-a) + (f''(\xi) - f(\xi))(\cosh(x-a) - 1). \tag{30}$$

Proof. Let $n = 2$, $k = 0$ and $\gamma = 1$ in Theorem 4.4. Then, $\zeta_{2,0,1}(t) = \cosh(t) - 1$ and hence $\rho^\pm(\zeta'_{2,0,-1}) = \pm\infty$. Therefore, we can apply the statement of Theorem 4.4 and the equality (28) simplifies to (30). \square

COROLLARY 4.7. *For all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ and $a, x \in I$, there exists a point ξ between a and x such that*

$$\begin{aligned}
 f(x) = & f(a) \frac{\cosh(x-a) + \cos(x-a)}{2} + f'(a) \frac{\sinh(x-a) + \sin(x-a)}{2} \\
 & + f''(a) \frac{\cosh(x-a) - \cos(x-a)}{2} + f'''(a) \frac{\sinh(x-a) - \sin(x-a)}{2} \quad (31) \\
 & + (f''''(t) - f(t)) \frac{\cosh(x-a) + \cos(x-a) - 2}{2}.
 \end{aligned}$$

Proof. Let $n = 4$, $k = 0$ and $\gamma = 1$ in Theorem 4.4. Then, one can easily see that $\zeta_{4,0,1}(t) = \frac{\cosh(t) + \cos(t) - 2}{2}$ and hence $\rho^\pm(\zeta'_{4,0,1}) = \pm\infty$. Thus, by applying the statement of Theorem 4.4, the equality (28) can be rewritten as (31). \square

COROLLARY 4.8. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$ and let t_0 be the smallest positive root of the equation*

$$\beta \sin(\alpha t) = \alpha \sin(\beta t). \quad (32)$$

Then, for all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ and $a, x \in I$ with $|x - a| \leq t_0$, there exists a point ξ between a and x such that

$$\begin{aligned}
 f(x) = & f(a) \frac{\beta^2 \cos(\alpha(x-a)) - \alpha^2 \cos(\beta(x-a))}{\beta^2 - \alpha^2} \\
 & + f'(a) \frac{\beta^3 \sin(\alpha(x-a)) - \alpha^3 \sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\
 & + f''(a) \frac{\cos(\alpha(x-a)) - \cos(\beta(x-a))}{\beta^2 - \alpha^2} \quad (33) \\
 & + f'''(a) \frac{\beta \sin(\alpha(x-a)) - \alpha \sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\
 & + (f''''(\xi) + (\alpha^2 + \beta^2)f''(\xi) + \alpha^2\beta^2 f(\xi)) \\
 & \times \frac{\alpha^2(\cos(\beta(x-a)) - 1) - \beta^2(\cos(\alpha(x-a)) - 1)}{\alpha^2\beta^2(\beta^2 - \alpha^2)}.
 \end{aligned}$$

Proof. Let $n = 4$ and apply Theorem 4.3 in the setting $c_4 = 1$, $c_3 = c_1 = 0$, $c_2 = \alpha^2 + \beta^2$, $c_0 = \alpha^2\beta^2$, that is, when $D_c(f) = f'''' + (\alpha^2 + \beta^2)f'' + \alpha^2\beta^2 f$. Then $\sin(\alpha x)$, $\cos(\alpha x)$, $\sin(\beta x)$, and $\cos(\beta x)$ form a fundamental system of solutions for the differential equation $D_c(f) = 0$ and

$$\omega_c(t) = \frac{\beta \sin(\alpha t) - \alpha \sin(\beta t)}{\alpha\beta(\beta^2 - \alpha^2)}$$

is a solution to the initial value problem (4). Observe that $\rho^\pm(\omega_c) = \pm t_0$. As we have seen it in the proof of Corollary 3.7, the equalities in (24) hold. On the other hand,

$$\int_0^{x-a} \omega_c(t) dt = \frac{\alpha^2(\cos(\beta(x-a)) - 1) - \beta^2(\cos(\alpha(x-a)) - 1)}{\alpha^2\beta^2(\beta^2 - \alpha^2)},$$

Therefore, we can apply Theorem 4.3 and hence there exists a point ξ between a and x such that the equality (27) holds, which reduces to (33). \square

REMARK 4.9. For the applicability of the previous corollary, it is essential to find the zeroes of the equation (32). In general, beyond the trivial solution $t = 0$, the other solutions cannot be established algebraically. On the other hand, if $\frac{\alpha}{\beta}$ is rational, say $|\frac{\alpha}{\beta}| = \frac{n}{m}$, where n, m are coprime natural numbers. Let $s := \frac{|\alpha|}{n} = \frac{|\beta|}{m} \neq 0$. Then $\alpha = \pm ns$ and $\beta = \pm ms$ and (32) is now equivalent to

$$m \sin(nst) = n \sin(mst).$$

In the case when $t = \frac{k}{s}\pi$ for some $k \in \mathbb{N}$, then both sides are equal to zero. If t is not of this form, then $\sin(st) \neq 0$, thus this equation can be rewritten as

$$mU_{n-1}(\cos(st)) = m \frac{\sin(nst)}{\sin(st)} = n \frac{\sin(mst)}{\sin(st)} = nU_{m-1}(\cos(st)),$$

where U_k denotes the k th degree Chebyshev polynomial of the second kind. Therefore, the last equation is an algebraic equation for $\cos(st)$. Solving this equation for $\cos(st)$, the smallest positive solution t_0 can easily be computed.

The limiting case of Corollary 4.8 (i.e., when $\alpha^2 = \beta^2 \neq 0$) is formulated as follows.

COROLLARY 4.10. *Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$ and let t_0 be the smallest positive root of the equation*

$$\sin(\alpha t) = \alpha t \cos(\alpha t). \tag{34}$$

Then, for all $f \in \mathcal{C}_{\mathbb{R}}^4(I)$ and $a, x \in I$ with $|x - a| \leq t_0$, there exists a point ξ between a and x such that

$$\begin{aligned} f(x) = & f(a) \frac{2 \cos(\alpha(x-a)) + \alpha(x-a) \sin(\alpha(x-a))}{2} \\ & + f'(a) \frac{3 \sin(\alpha(x-a)) - \alpha(x-a) \cos(\alpha(x-a))}{2\alpha} \\ & + f''(a) \frac{(x-a) \sin(\alpha(x-a))}{2\alpha} \\ & + f'''(a) \frac{\sin(\alpha(x-a)) - \alpha(x-a) \cos(\alpha(x-a))}{2\alpha^3} \\ & + (f''''(\xi) + 2\alpha^2 f''(\xi) + \alpha^4 f(\xi)) \\ & \times \frac{2 - 2 \cos(\alpha(x-a)) - \alpha(x-a) \sin(\alpha(x-a))}{2\alpha^4}. \end{aligned} \tag{35}$$

The proof of this result is completely analogous to that of Corollary 4.8, therefore it is omitted.

COROLLARY 4.11. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. Then, for all $f \in \mathcal{C}_{\mathbb{R}}^4(I)$ and $a, x \in I$, there exists a point ξ between a and x such that*

$$\begin{aligned}
 f(x) = & f(a) \frac{\beta^2 \cosh(\alpha(x-a)) - \alpha^2 \cosh(\beta(x-a))}{\beta^2 - \alpha^2} \\
 & + f'(a) \frac{\beta^3 \sinh(\alpha(x-a)) - \alpha^3 \sinh(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\
 & + f''(a) \frac{\cosh(\beta(x-a)) - \cosh(\alpha(x-a))}{\beta^2 - \alpha^2} \\
 & + f'''(a) \frac{\alpha \sinh(\beta(x-a)) - \beta \sinh(\alpha(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\
 & + (f''''(\xi) - (\alpha^2 + \beta^2)f''(\xi) + \alpha^2\beta^2 f(\xi)) \\
 & \times \frac{\alpha^2(\cosh(\beta(x-a)) - 1) - \beta^2(\cosh(\alpha(x-a)) - 1)}{\alpha^2\beta^2(\beta^2 - \alpha^2)}.
 \end{aligned} \tag{36}$$

Proof. Let $n = 4$ and apply Theorem 4.3 in the setting $c_4 = 1$, $c_3 = c_1 = 0$, $c_2 = -(\alpha^2 + \beta^2)$, $c_0 = \alpha^2\beta^2$, that is, when $D_c(f) = f'''' - (\alpha^2 + \beta^2)f'' + \alpha^2\beta^2 f$. Then $\sinh(\alpha x)$, $\cosh(\alpha x)$, $\sinh(\beta x)$, and $\cosh(\beta x)$ form a fundamental system of solutions for the differential equation $D_c(f) = 0$ and

$$\omega_c(t) = \frac{\alpha \sinh(\beta t) - \beta \sinh(\alpha t)}{\alpha\beta(\beta^2 - \alpha^2)}$$

is a solution to the initial value problem (4).

Now, we prove that $\rho^\pm(\omega_c) = \pm\infty$. First observe that

$$\left(\frac{\sinh(x)}{x}\right)' = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}\right)' = \frac{1}{x} \sum_{n=1}^{\infty} \frac{2nx^{2n}}{(2n+1)!},$$

which shows that the mapping $x \mapsto \sinh(x)/x$ is strictly monotone on \mathbb{R}_+ . Assume that $|\alpha| < |\beta|$. Then, for all nonzero t ,

$$\frac{\sinh(\alpha t)}{\alpha t} = \frac{\sinh(|\alpha t|)}{|\alpha t|} < \frac{\sinh(|\beta t|)}{|\beta t|} = \frac{\sinh(\beta t)}{\beta t}.$$

Therefore, any nonzero number t cannot be a root of ω_c , which implies $\rho^\pm(\omega_c) = \pm\infty$.

As we have seen it in the proof of Corollary 3.9, the equalities in (25) hold. On the other hand,

$$\int_0^{x-a} \omega_c(t) dt = \frac{\alpha^2(\cosh(\beta(x-a)) - 1) - \beta^2(\cosh(\alpha(x-a)) - 1)}{\alpha^2\beta^2(\beta^2 - \alpha^2)}.$$

Therefore, we can apply Theorem 4.3 and hence there exists a point ξ between a and x such that the equality (27) holds, which reduces to (36). \square

COROLLARY 4.12. *Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. Then, for all $f \in \mathcal{C}_{\mathbb{R}}^4(I)$ and $a, x \in I$, there exists a point ξ between a and x such that*

$$\begin{aligned} f(x) = & f(a) \frac{2 \cosh(\alpha(x-a)) - \alpha(x-a) \sinh(\alpha(x-a))}{2} \\ & + f'(a) \frac{3 \sinh(\alpha(x-a)) - \alpha(x-a) \cosh(\alpha(x-a))}{2\alpha} + f''(a) \frac{(x-a) \sinh(\alpha(x-a))}{2\alpha} \\ & + f'''(a) \frac{\alpha(x-a) \cosh(\alpha(x-a)) - \sinh(\alpha(x-a))}{2\alpha^3} \\ & + (f'''(\xi) - 2\alpha^2 f''(\xi) + \alpha^4 f(\xi)) \\ & \times \frac{2 - 2 \cosh(\alpha(x-a)) + \alpha(x-a) \sinh(\alpha(x-a))}{2\alpha^4}. \end{aligned}$$

The proof of the above statement is omitted.

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