

## DIRAC INEQUALITY FOR HIGHEST WEIGHT HARISH–CHANDRA MODULES I

PAVLE PANDŽIĆ\*, ANA PRLIĆ, VLADIMÍR SOUČEK AND VÍT TUČEK

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*Abstract.* Let  $G$  be a connected simply connected noncompact classical simple Lie group of Hermitian type. Then  $G$  has unitary highest weight representations. The proof of the classification of unitary highest weight representations of  $G$  given by Enright, Howe and Wallach is based on the Dirac inequality of Parthasarathy, Jantzen’s formula and Howe’s theory of dual pairs where one group in the pair is compact. In this paper we focus on the Dirac inequality which can be used to prove the classification in a more direct way.

### 1. Introduction

In this introduction, we give an outline of the representation theory context for the results in this paper. However, in each of the cases described in Tables 1 and 2, all the notions become completely explicit and elementary, and the representation theory context may be forgotten. Therefore, the reader who is not familiar with, or interested in, representation theory can mostly ignore the rest of the introduction and only check the concrete definitions given in Tables 1 and 2 before going to Sections 2 and 3 which contain our main results.

Let  $G$  be a connected simply connected noncompact classical simple Lie group of Hermitian type. (Exceptional Lie groups of Hermitian type are treated in [13]). Let  $\Theta$  be a Cartan involution of  $G$  and let  $K$  be the group of fixed points of  $\Theta$ . If  $Z$  denotes the center of  $G$ , then  $K/Z$  is a maximal compact subgroup of  $G/Z$ . Let  $\mathfrak{g}_0$  and  $\mathfrak{k}_0$  be the Lie algebras of  $G$  and  $K$ , respectively, and let  $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$  be the Cartan decomposition. Let  $\mathfrak{t}_0$  be the common Cartan subalgebra of  $\mathfrak{g}_0$  and  $\mathfrak{k}_0$  and let  $\mathfrak{g}$ ,  $\mathfrak{k}$ ,  $\mathfrak{t}$  and  $\mathfrak{p}$  be the complexifications of  $\mathfrak{g}_0$ ,  $\mathfrak{k}_0$ ,  $\mathfrak{t}_0$  and  $\mathfrak{p}_0$ . Let  $\Delta_{\mathfrak{g}}^+ \supset \Delta_{\mathfrak{k}}^+$  denote fixed sets of positive respectively positive compact roots. Since we assume that pair  $(G, K)$  is Hermitian, we have a  $K$ -invariant decomposition  $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$  and  $\mathfrak{p}^{\pm}$  are abelian subalgebras of  $\mathfrak{p}$ . Let  $\rho$  denote the half sum of positive roots for  $\mathfrak{g}$ .

A unitary representation of  $G$  such that the underlying  $(\mathfrak{g}, K)$ -module is an irreducible quotient of a Verma module is called a unitary highest weight module. It is

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\* Corresponding author.

generated by a weight vector that is annihilated by the action of all positive root spaces in  $\mathfrak{g}$ .

For  $\lambda \in \mathfrak{t}^*$  which are  $\Delta_{\mathfrak{k}}^+$ -dominant integral (that means  $\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$  has to be an integer greater or equal to zero), let  $N(\lambda)$  denote the generalized Verma module. By definition  $N(\lambda) = S(\mathfrak{p}^-) \otimes F_\lambda$ , where  $F_\lambda$  is the irreducible  $\mathfrak{k}$ -module with highest weight  $\lambda$ . The generalized Verma module  $N(\lambda)$  is a highest weight module ( $\lambda$  is the highest weight of the  $K$ -type  $F_\lambda$  but also a  $\mathfrak{g}$ -highest weight of  $N(\lambda)$ ) which doesn't have to be irreducible or unitary. Our main goal in this paper is to determine those  $N(\lambda)$  which correspond to unitary irreducible representation of  $G$ . We consider only real highest weights  $\lambda$  since this is a necessary condition for unitarity. In case  $N(\lambda)$  is not irreducible, we will consider the irreducible quotient  $L(\lambda)$  of  $N(\lambda)$  and we will determine those weights  $\lambda$  which correspond to unitarizable  $L(\lambda)$ .

Harish-Chandra has shown that  $G$  admits non-trivial unitary highest weight modules precisely when  $(G, K)$  is a Hermitian symmetric pair and that is precisely when the Lie algebra  $\mathfrak{g}_0$  is one of the Lie algebras listed in tables 1 and 2. To learn more about highest weight modules see [1], [3], [5], [6], [7], [12].

In [5] (and independently in [12]), a complete classification of the unitary highest weight modules was given using the Dirac inequality, Jantzen's formula and Howe's theory of dual pairs. They proved that  $L(\lambda)$  is unitarizable if and only if the strict Dirac inequality holds for all  $K$ -types occurring in  $L(\lambda)$ . This criterion is useful, but it is not easy to use because it is difficult to determine the  $K$ -types of  $L(\lambda)$ . The purpose of this and our future work is to show that the same result can be proved more directly using the Dirac inequality in a more substantial way.

The structure of  $S(\mathfrak{p}^-)$  is very well known (see [15]). The  $K$ -types of  $S(\mathfrak{p}^-)$  are called the Schmid modules. For each of the Lie algebras in Table 2, the general Schmid module  $s$  is a nonnegative integer combination of the so called basic Schmid modules. The basic Schmid modules for each classical Lie algebra  $\mathfrak{g}_0$  for which  $(G, K)$  is a Hermitian symmetric pair are given in Table 2.

Let  $U(\mathfrak{g})$  be the universal enveloping algebra of  $\mathfrak{g}$  and let  $C(\mathfrak{p})$  be the Clifford algebra of  $\mathfrak{p}$ . The Dirac operator is an element of  $U(\mathfrak{g}) \otimes C(\mathfrak{p})$  defined as  $D = \sum_i b_i \otimes d_i$  where  $b_i$  is a basis of  $\mathfrak{p}$  and  $d_i$  is the dual basis of  $\mathfrak{p}$  with respect to the Killing form  $B$ . It is easy to show that  $D$  is independent of the choice of  $b_i$  and that it is  $K$ -invariant for the adjoint action on both factors. The Dirac operator acts on the tensor product  $X \otimes S$  where  $X$  is a  $(\mathfrak{g}, K)$ -module, and  $S$  is the spin module for  $C(\mathfrak{p})$ . The square of the Dirac operator is very simple:

$$D^2 = -(\text{Cas}_{\mathfrak{g}} \otimes 1 + \|\rho\|^2) + (\text{Cas}_{\mathfrak{k}_\Delta} + \|\rho_{\mathfrak{k}}^2\|),$$

where  $\rho_{\mathfrak{k}}$  is a half sum of the compact positive roots and  $\mathfrak{k}_\Delta$  is the diagonal embedding of  $\mathfrak{k}$  into  $U(\mathfrak{g}) \otimes C(\mathfrak{p})$  defined on  $X \in \mathfrak{k}$  by  $\Delta(X) = X \otimes 1 + 1 \otimes \alpha(X)$ , where  $\alpha$  is the action map  $\mathfrak{k} \rightarrow \mathfrak{so}(\mathfrak{p})$  followed by the usual identifications  $\mathfrak{so}(\mathfrak{p}) \cong \wedge^2 \mathfrak{p} \hookrightarrow C(\mathfrak{p})$ . There are many applications of the Dirac operators in representation theory (see [2], [4], [8], [10], [11], [9]).

The Dirac inequality is a very useful necessary condition for unitarity. More precisely, if a  $(\mathfrak{g}, K)$ -module is unitary, then  $D$  is a self adjoint with respect to an inner

product, so  $D^2 \geq 0$ . By the formula for  $D^2$  the Dirac inequality becomes explicit on any  $K$ -type  $F_\tau$  of  $L(\lambda) \otimes S$

$$\|\tau + \rho_\mathfrak{k}\|^2 \geq \|\lambda + \rho\|^2.$$

In [5] it was proved that  $L(\lambda)$  is unitary if and only if  $D^2 > 0$  on  $F_\mu \otimes \bigwedge^{\text{top}} \mathfrak{p}^+$  for any  $K$ -type  $F_\mu$  of  $L(\lambda)$  other than  $F_\lambda$ , that is if and only if

$$\|\mu + \rho\|^2 > \|\lambda + \rho\|^2.$$

As we already said, it is difficult to determine the  $K$ -types of  $L(\lambda)$ .

Table 1:  $\rho$  and  $W_\mathfrak{k}$

Lie algebra	$\rho$	generators of $W_\mathfrak{k}$
$\mathfrak{sp}(2n, \mathbb{R})$	$(n, n-1, \dots, 2, 1)$	$s_{\varepsilon_i - \varepsilon_j}, 1 \leq i < j \leq n$
$\mathfrak{so}^*(2n)$	$(n-1, n-2, \dots, 1, 0)$	$s_{\varepsilon_i - \varepsilon_j}, 1 \leq i < j \leq n$
$\mathfrak{su}(p, q) \ p \leq q$ ,	$(\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{-n+3}{2}, \frac{-n+1}{2})$	$s_{\varepsilon_i - \varepsilon_j}, 1 \leq i < j \leq p$ or $p+1 \leq i < j \leq n$
$\mathfrak{so}(2, 2n-2)$	$(n-1, n-2, \dots, 1, 0)$	$s_{\varepsilon_i \pm \varepsilon_j}, 2 \leq i < j \leq n$
$\mathfrak{so}(2, 2n-1)$	$(n-\frac{1}{2}, n-\frac{3}{2}, \dots, \frac{1}{2})$	$s_{\varepsilon_i \pm \varepsilon_j}, 2 \leq i < j \leq n,$ $s_{\varepsilon_i}, 2 \leq i \leq n$

Table 2: The weights of basic Schmid modules and the condition for the  $\mathfrak{k}$ -highest weights  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$

Lie algebra	basic Schmid modules	highest weights
$\mathfrak{sp}(2n, \mathbb{R})$	$s_i = (\underbrace{2, \dots, 2}_i, 0, \dots, 0),$ $i = 1, \dots, n$	$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n,$ $\lambda_i - \lambda_j \in \mathbb{Z},$ $1 \leq i, j \leq n$
$\mathfrak{so}^*(2n)$	$s_i = (\underbrace{1, \dots, 1}_{2i}, 0, \dots, 0),$ $i = 1, \dots, [n/2]$	$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n,$ $\lambda_i - \lambda_j \in \mathbb{Z},$ $1 \leq i, j \leq n$
$\mathfrak{su}(p, q) \ p \leq q$ ,	$s_i = (\underbrace{1, \dots, 1}_i, 0, \dots, 0   0, \dots, 0, \underbrace{-1, \dots, -1}_i),$ $i = 1, \dots, p$	$\lambda_1 \geq \dots \geq \lambda_p;$ $\lambda_{p+1} \geq \dots \geq \lambda_n,$ $\lambda_i - \lambda_j \in \mathbb{Z},$ $1 \leq i < j \leq p$ or $p+1 \leq i < j \leq n$
$\mathfrak{so}(2, 2n-2)$	$s_1 = (1, 1, 0, \dots, 0),$ $s_2 = (2, 0, 0, \dots, 0)$	$\lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_{n-1} \geq  \lambda_n ,$ $\lambda_i - \lambda_j \in \mathbb{Z}, 2 \leq i, j \leq n$
$\mathfrak{so}(2, 2n-1)$	$s_1 = (1, 1, 0, \dots, 0),$ $s_2 = (2, 0, 0, \dots, 0)$	$\lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n \geq 0,$ $\lambda_i - \lambda_j \in \mathbb{Z}$ and $2\lambda_i \in \mathbb{Z},$ $2 \leq i, j \leq n$

The results of this paper provide examples for the following theorem:

**THEOREM 1.1.** *Let us assume that  $\mathfrak{g}, \rho, \lambda, s$  are as in tables 1 and 2.*

(1) *Let  $s_0$  be a Schmid module such that the strict Dirac inequality*

$$\|(\lambda - s)^+ + \rho\|^2 > \|\lambda + \rho\|^2 \tag{1.1}$$

*holds for any Schmid module  $s$  of strictly lower level than  $s_0$ , and such that*

$$\|(\lambda - s_0)^+ + \rho\|^2 < \|\lambda + \rho\|^2.$$

*Then  $L(\lambda)$  is not unitary.*

(2) *If*

$$\|(\lambda - s)^+ + \rho\|^2 > \|\lambda + \rho\|^2 \tag{1.2}$$

*holds for all Schmid modules  $s$ , then  $N(\lambda)$  is irreducible and unitary.*

In Theorem 1.1,  $(\lambda - s)^+$  is the unique  $\mathfrak{k}$ -dominant  $W_{\mathfrak{k}}$ -conjugate of  $\lambda - s$ , which means  $(\lambda - s)^+$  is as in the third column of Table 2.

The proof of the above theorem requires some tools from representation theory, so we will omit it in this paper and prove it in [14].

Another possible reason to provide this detailed study of the Dirac inequality is that [5] prove a relationship between the norms of certain  $K$ -types in which the eigenvalue of the Dirac operator appears. Together with results of the current paper this could be potentially used to study convergence of  $K$ -type decompositions / series in the Hilbert spaces involved.

In Table 1,  $s_{\alpha}(\lambda) = \lambda - \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$  is the reflection of  $\lambda$  with respect to the hyperplane orthogonal to a root  $\alpha$ ,  $W_{\mathfrak{k}}$  is the Weyl group of  $\mathfrak{k}$  generated by the  $s_{\alpha}$ .

Here  $\lambda$  and  $\rho$  are elements of  $\mathfrak{t}^*$  which is identified with  $\mathbb{C}^n$ , and  $\varepsilon_i$  denotes the projection to the  $i$ -th coordinate. The roots are certain functionals on  $\mathfrak{t}^*$  and the relevant ones are those in the subscripts of the reflections  $s$  in Table 1, like  $\varepsilon_i - \varepsilon_j$  or  $\varepsilon_i + \varepsilon_j$ .

The rest of this paper is devoted to analyzing the Dirac inequality (1.2) for various choices of  $s$ . Our analysis will be case by case for the Lie algebras as in the above tables but first we need two auxiliary results that are going to help us in each of the cases.

### 2. Some technical lemmas

**LEMMA 2.1.** *Let  $\mathfrak{g}$  be one of the Lie algebras listed in the above tables. Let  $\mu$  and  $\nu$  be weights as in the last column of Table 2. Let  $w_1, w_2 \in W_{\mathfrak{k}}$ . Then*

$$\|(w_1\mu - w_2\nu)^+ + \rho\|^2 \geq \|(\mu - \nu)^+ + \rho\|^2.$$

In Lemma 2.1,  $(w_1\mu - w_2\nu)^+$  is the unique dominant  $W_{\mathfrak{k}}$ -conjugate of  $w_1\mu - w_2\nu$ , which means  $(w_1\mu - w_2\nu)^+$  is as in the third column of Table 2. The proof requires some representation theory and we leave it for [14].

In the computations needed to prove the Dirac inequality, we will repeatedly use the following elementary lemma.

LEMMA 2.2. Let  $\mu$  and  $\nu$  be two  $n$ -tuples with strictly decreasing coordinates and let  $\rho$  be as in Table 1.

(1) Suppose there are  $u, v, 1 \leq u < v \leq n$ , such that

$$\mu_u = \nu_u, \dots, \mu_{v-1} = \nu_{v-1}; \mu_v < \nu_v.$$

Let  $\mu', \nu'$  be obtained from  $\mu, \nu$  by moving the  $v$ -th coordinate to the  $u$ -th place and shifting the coordinates in between to the right, i.e.,

$$\mu' = (\mu_1, \dots, \mu_{u-1}, \mu_v, \mu_u, \dots, \mu_{v-1}, \mu_{v+1}, \dots, \mu_n)$$

and likewise for  $\nu'$ . Then

$$\|\mu + \rho\|^2 - \|\nu + \rho\|^2 > \|\mu' + \rho\|^2 - \|\nu' + \rho\|^2. \tag{2.1}$$

(2) Suppose there are  $u, v, 1 \leq u < v \leq n$ , such that

$$\mu_u > \nu_u; \mu_{u+1} = \nu_{u+1}, \dots, \mu_v = \nu_v.$$

Let  $\mu', \nu'$  be obtained from  $\mu, \nu$  by moving the  $u$ -th coordinate to the  $v$ -th place and shifting the coordinates in between to the left, i.e.,

$$\mu' = (\mu_1, \dots, \mu_{u-1}, \mu_{u+1}, \dots, \mu_v, \mu_u, \mu_{v+1}, \dots, \mu_n)$$

and likewise for  $\nu'$ . Then

$$\|\mu + \rho\|^2 - \|\nu + \rho\|^2 > \|\mu' + \rho\|^2 - \|\nu' + \rho\|^2. \tag{2.2}$$

(3) Let  $\mu$  be a  $n$ -tuple, such that for some  $s, t \geq 1$  and for some  $u$  between 1 and  $n - s - t$ ,

$$\mu = (\mu_1, \dots, \mu_u, \underbrace{x+1}_s, \underbrace{x}_t, \mu_{u+s+t+1}, \dots, \mu_n).$$

Let

$$\mu' = (\mu_1, \dots, \mu_u, \underbrace{x}_t, \underbrace{x+1}_s, \mu_{u+s+t+1}, \dots, \mu_n).$$

Then

$$\|\mu + \rho\|^2 > \|\mu' + \rho\|^2$$

*Proof.* (1) The difference of the two sides of (2.1) is

$$\begin{aligned} & [(\mu_v + \rho_v)^2 - (\nu_v + \rho_v)^2] - [(\mu_v + \rho_u)^2 - (\nu_v + \rho_u)^2] \\ &= (\mu_v - \nu_v)(\mu_v + \nu_v + 2\rho_v) - (\mu_v - \nu_v)(\mu_v + \nu_v + 2\rho_u) \\ &= (\mu_v - \nu_v)(2\rho_v - 2\rho_u) \\ &= 2(\nu_v - \mu_v)(\rho_u - \rho_v). \end{aligned}$$

Since  $\nu_v > \mu_v$  by assumption, and since  $\rho_u > \rho_v$ , the claim follows.

The proof of (2) is analogous to the proof of (1).

(3) We prove the required inequality by successively switching pairs  $x + 1, x$  as follows. Let

$$\mu'' = (\mu_1, \dots, \mu_u, \underbrace{x, x+1}_{s-1}, \underbrace{x, x+1}_{t-1}, \mu_{u+s+t+1}, \dots, \mu_n).$$

Then we see, factoring the differences of squares, that

$$\begin{aligned} & \|\mu + \rho\|^2 - \|\mu'' + \rho\|^2 \\ &= (x + 1 + \rho_{u+1})^2 - (x + \rho_{u+1})^2 + (x + \rho_{u+s+t})^2 - (x + 1 + \rho_{u+s+t})^2 \\ &= (2x + 1 + 2\rho_{u+1}) - (2x + 1 + 2\rho_{u+s+t}) \\ &= 2(\rho_{u+1} - \rho_{u+s+t}) > 0. \end{aligned}$$

Now we continue with the next pair until we reach  $\mu'$ . The claim follows.  $\square$

### 3. Dirac inequalities

#### 3.1. Dirac inequality for $\mathfrak{sp}(2n, \mathbb{R}), n \geq 1$

The basic Schmid  $\mathfrak{k}$ -submodules of  $S(\mathfrak{p}^-)$  have lowest weights  $-s_i$ , where

$$s_i = (\underbrace{2, \dots, 2}_i, 0, \dots, 0), \quad i = 1, \dots, n.$$

The highest weight  $(\mathfrak{g}, K)$ -modules have highest weight of the form  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ , where  $\lambda_i - \lambda_j \in \mathbb{N}_0, i > j$ . Here  $\mathbb{N}_0$  denotes the set consisting of all non-negative integers.

In this case  $\rho = (n, n - 1, \dots, 2, 1)$ . The basic necessary condition for unitarity is the Dirac inequality

$$\|(\lambda - s_1)^+ + \rho\|^2 \geq \|\lambda + \rho\|^2. \tag{3.1}$$

To understand this inequality better, let  $q \leq r$  be integers in  $[1, n]$  such that

$$\lambda = (\underbrace{\lambda_1, \dots, \lambda_1}_q, \underbrace{\lambda_1 - 1, \dots, \lambda_1 - 1}_{r-q}, \lambda_{r+1}, \dots, \lambda_n), \tag{3.2}$$

with  $\lambda_1 - 2 \geq \lambda_{r+1} \geq \dots \geq \lambda_n$ . Then

$$(\lambda - s_1)^+ = (\underbrace{\lambda_1, \dots, \lambda_1}_{q-1}, \underbrace{\lambda_1 - 1, \dots, \lambda_1 - 1}_{r-q}, \lambda_1 - 2, \lambda_{r+1}, \dots, \lambda_n) = \lambda - (\varepsilon_q + \varepsilon_r).$$

The inequality (3.1) now becomes equivalent to  $\|\lambda + \rho - \gamma\|^2 \geq \|\lambda + \rho\|^2$ , or to

$$2\langle \lambda + \rho, \varepsilon_q + \varepsilon_r \rangle \leq \|\gamma\|^2,$$

where  $\gamma = \varepsilon_q + \varepsilon_r$ . If  $q \neq r$ , then  $\lambda_q = \lambda_1$ ,  $\lambda_r = \lambda_1 - 1$  and  $\|\gamma\|^2 = 2$ , and the inequality becomes

$$\lambda_1 \leq -n + \frac{r+q}{2}. \tag{3.3}$$

If  $q = r$ , then  $\lambda_q = \lambda_r = \lambda_1$  and  $\|\gamma\|^2 = 4$ , and the inequality is again (3.3).

Now we are going to see in which cases the Dirac inequality holds for  $s_i$ ,  $i \in \{2, \dots, n\}$ . We start by examining what happens for the first  $q$  basic Schmid modules  $s = s_1, \dots, s_q$ . We have

$$\begin{aligned} (\lambda - s_i)^+ &= \left( \underbrace{\lambda_1}_{q-i}, \underbrace{\lambda_1 - 1}_i, \underbrace{\lambda_1 - 1}_{r-q-i}, \underbrace{\lambda_1 - 2}_i, \lambda_{r+1}, \dots, \lambda_n \right) \\ \lambda &= \left( \underbrace{\lambda_1}_{q-i}, \underbrace{\lambda_1}_i, \underbrace{\lambda_1 - 1}_{r-q-i}, \underbrace{\lambda_1 - 1}_i, \lambda_{r+1}, \dots, \lambda_n \right), \end{aligned}$$

so

$$\|(\lambda - s_i)^+ + \rho\|^2 \geq \|\lambda + \rho\|^2, \tag{3.4}$$

is equivalent to non-negativity of

$$\sum_{u=q-i+1}^q [(\lambda_1 - 1 + \rho_u)^2 - (\lambda_1 + \rho_u)^2] + \sum_{v=r-i+1}^r [(\lambda_1 - 2 + \rho_v)^2 - (\lambda_1 - 1 + \rho_v)^2],$$

where  $\rho_u = n - u + 1$ ,  $\rho_v = n - v + 1$ . Factoring the differences of squares this expression becomes

$$\begin{aligned} & - \sum_{u=q-i+1}^q (2\lambda_1 - 1 + 2\rho_u) - \sum_{v=r-i+1}^r (2\lambda_1 - 3 + 2\rho_v) \\ &= -i(2\lambda_1 - 1) - 2[(n - q + i) + \dots + (n - q + 1)] - i(2\lambda_1 - 3) \\ & \quad - 2[(n - r + i) + \dots + (n - r + 1)] \\ &= -4i\lambda_1 + 4i - 2i(n - q) - 2i(n - r) - 4(i + \dots + 1) \\ &= -2i(2\lambda_1 + 2n - 2 - q - r) - 2i(i + 1). \end{aligned}$$

Dividing by  $(-2i)$ , we see that (3.4) is equivalent to

$$2\lambda_1 + 2n - 2 - q - r + i + 1 \leq 0,$$

or

$$\lambda_1 \leq -n + \frac{r+q-i+1}{2}. \tag{3.5}$$

Moreover, it is clear from the above argument that (3.4) holds strictly if and only if (3.5) holds strictly. Note also that for  $i = 1$ , (3.5) is exactly our basic inequality (3.3), while for  $i = q$  we get

$$\lambda_1 \leq -n + \frac{r+1}{2}.$$

THEOREM 3.1. *Let  $\lambda$  be as in (3.2). Then:*

1. *If for some integer  $i \in [1, q]$*

$$\lambda_1 < -n + \frac{r+q-i+1}{2},$$

*then the Dirac inequality holds strictly for any Schmid module  $s = (2b_1, \dots, 2b_n)$ ,  $b_j \in \mathbb{Z}$ ,  $b_1 \geq \dots \geq b_n \geq 0$  with at most  $i$  nonzero components, i.e.*

$$\|(\lambda - s)^+ + \rho\|^2 > \|\lambda + \rho\|^2. \tag{3.6}$$

2. *If*

$$\lambda_1 < -n + \frac{r+1}{2},$$

*then the Dirac inequality holds strictly for any Schmid module  $s$ .*

*Proof.* Let

$$s = (2b_1, \dots, 2b_j, 0, \dots, 0), \quad j \leq i. \tag{3.7}$$

If  $s = s_j$  is a basic Schmid module, then we have already seen that strict (3.4) holds for  $s$ , because strict (3.4) holds by assumption. So let us assume that  $s$  is not a basic Schmid module, i.e.,  $b_1 \geq 2$ . We will show how to reduce the claim to the same claim for  $s'$  with smaller  $b_1$  and then use induction.

Let  $k$ ,  $1 \leq k \leq j$ , be such that  $b_1 = \dots = b_k > b_{k+1}$ ; so

$$s = (\underbrace{2b_1, \dots, 2b_1}_k, 2b_{k+1}, \dots, 2b_j, 0, \dots, 0).$$

Let

$$s' = s - s_k = (\underbrace{2b_1 - 2, \dots, 2b_1 - 2}_k, 2b_{k+1}, \dots, 2b_j, 0, \dots, 0).$$

We claim that

$$\|(\lambda - s)^+ + \rho\|^2 \geq \|(\lambda - s')^+ + \rho\|^2. \tag{3.8}$$

If we prove this, then we can do induction on  $b_1$  and conclude that (3.4) holds strictly for all  $s$  as in (3.7) ( $b_1 = 1$  corresponds to  $s = s_j$ , the case already handled).

To prove (3.8), we first note that

$$\begin{aligned} (\lambda - s)^+ &= (\underbrace{\lambda_1}_{q-j}, \underbrace{\lambda_1 - 1}_{r-q}, \lambda_1 - 2b_j, \dots, \lambda_1 - 2b_{k+1}, \underbrace{\lambda_1 - 2b_1}_k, \lambda_{r+1}, \dots, \lambda_n)^+; \\ (\lambda - s')^+ &= (\underbrace{\lambda_1}_{q-j}, \underbrace{\lambda_1 - 1}_{r-q}, \lambda_1 - 2b_j, \dots, \lambda_1 - 2b_{k+1}, \underbrace{\lambda_1 - 2b_1 + 2}_k, \lambda_{r+1}, \dots, \lambda_n)^+, \end{aligned}$$

where the first  $r$  coordinates in each expression are already arranged in descending order, and  $\lambda_{r+1}, \dots, \lambda_n$  have to be put into proper places.

By Lemma 2.2, to prove (3.8) it is enough to prove

$$\|\eta + \rho\|^2 \geq \|\eta' + \rho\|^2, \tag{3.9}$$

where

$$\begin{aligned} \eta &= (\underbrace{\lambda_1}_{q-j}, \underbrace{\lambda_1 - 1}_{r-q}, \lambda_1 - 2b_j, \dots, \lambda_1 - 2b_{k+1}, \underbrace{\lambda_1 - 2b_1}_k, \lambda_{r+1}, \dots, \lambda_n); \\ \eta' &= (\underbrace{\lambda_1}_{q-j}, \underbrace{\lambda_1 - 1}_{r-q}, \lambda_1 - 2b_j, \dots, \lambda_1 - 2b_{k+1}, \underbrace{\lambda_1 - 2b_1 + 2}_k, \lambda_{r+1}, \dots, \lambda_n). \end{aligned}$$

In more detail, to see that (3.8) follows from (3.9), we first use Lemma 2.2(3) to move the coordinates  $\lambda_1 - 2b_1$  of  $(\lambda - s)^+$  to the left past those of  $\lambda_{r+1}, \dots, \lambda_n$  that are equal to  $\lambda_1 - 2b_1 + 1$ . Now the coordinates  $\lambda_1 - 2b_1$  of  $(\lambda - s)^+$  are positioned exactly above the coordinates  $\lambda_1 - 2b_1 + 2$  of  $(\lambda - s')^+$ , and we can use Lemma 2.2(1) to move these coordinates simultaneously to the left of those of  $\lambda_{r+1}, \dots, \lambda_n$  that are  $\geq \lambda_1 - 2b_1 + 2$ .

To prove (3.9), we compute factoring the differences of squares,

$$\begin{aligned} \|\eta + \rho\|^2 - \|\eta' + \rho\|^2 &= \sum_{j=r-k+1}^r [(\lambda_1 - 2b_1 + \rho_j)^2 - (\lambda_1 - 2b_1 + 2 + \rho_j)^2] \\ &= \sum_{j=r-k+1}^r (-2)(2\lambda_1 - 4b_1 + 2 + 2\rho_j) \\ &= -2k(2\lambda_1 - 4b_1 + 2) - 4[(n - r + k) + \dots + (n - r + 1)] \\ &= -2k(2\lambda_1 - 4b_1 + 2 + 2n - 2r + k + 1). \end{aligned}$$

This expression is positive if and only if

$$2\lambda_1 + 2n - 4b_1 - 2r + k + 3 < 0. \tag{3.10}$$

Since by assumption

$$\lambda_1 < -n + \frac{r + q - i + 1}{2},$$

and since  $b_1 \geq 2$ , we see that (3.10) will follow if we prove

$$r + q - i + 1 - 8 - 2r + k + 3 < 0,$$

or

$$-r + q - i + k - 4 < 0.$$

The last inequality is however obvious since  $q \leq r$  and  $k \leq i$ .

So we have proved (3.9), and as explained above, this finishes the proof of Theorem 3.1(1).

We now prove Theorem 3.1(2). If we take  $i = q$  in the already proved Theorem 3.1(1), we see that (3.6) holds strictly for all  $s$  having at most  $q$  nonzero components.

Assume now that  $q < r$ , and that

$$s = (2b_1, \dots, 2b_i, 0, \dots, 0), \quad b_i \geq 1, \quad q < i \leq r. \tag{3.11}$$

Let  $s' = (s - s_q)^+$  and let  $\lambda' = (\lambda - s_q)^+$ , where  $s_q$  is the  $q$ th basic Schmid module. We claim that

$$\|(\lambda - s)^+ + \rho\|^2 \geq \|(\lambda' - s')^+ + \rho\|^2 \tag{3.12}$$

and that

$$\|\lambda + \rho\|^2 < \|\lambda' + \rho\|^2. \tag{3.13}$$

If we prove these two inequalities, then it follows that

$$\|(\lambda - s)^+ + \rho\|^2 - \|\lambda + \rho\|^2 > \|(\lambda' - s')^+ + \rho\|^2 - \|\lambda' + \rho\|^2, \tag{3.14}$$

so the strict inequality (3.6) for  $\lambda$  and  $s$  will follow if we can prove the strict (3.6) for  $\lambda'$  and  $s'$ .

Since

$$\lambda' = (\underbrace{\lambda_i - 1}_{r-q}, \underbrace{\lambda_i - 2}_q, \lambda_{r+1}, \dots, \lambda_n),$$

the analogue  $r'$  of  $r$  for  $\lambda'$  satisfies  $r' \geq r$ . Moreover,

$$\lambda'_1 = \lambda_1 - 1 < \lambda_1 < -n + \frac{r+1}{2} \leq -n + \frac{r'+1}{2}.$$

Furthermore, the analogue  $q'$  of  $q$  for  $\lambda'$  satisfies  $q' = r - q < r \leq r'$ , and the number of nonzero coordinates of  $s'$  is  $\leq i \leq r'$ . Also, the sum of coordinates of  $s'$  is smaller than the sum of coordinates of  $s$ , and as we keep reducing as above, the number of nonzero coordinates will also become smaller. So we can use induction and keep reducing  $s$  until we come to the situation  $i \leq q$ , which we already handled. Thus to prove Theorem 3.1(2) for  $s$  as in (3.11), it suffices to prove (3.12) and (3.13).

To prove (3.12), we apply Lemma 2.1 as follows. Let

$$\mu = \lambda' = (\lambda - s_q)^+, \quad \nu = s' = (s - s_q)^+.$$

Let  $w_1, w_2 \in W_{\mathfrak{k}}$  be such that  $\lambda - s_q = w_1\mu$  and  $s - s_q = w_2\nu$ . Then Lemma 2.1 implies

$$\|(\lambda - s)^+ + \rho\|^2 = \|(w_1\mu - w_2\nu)^+ + \rho\|^2 \geq \|(\mu - \nu)^+ + \rho\|^2 = \|(\lambda' - s')^+ + \rho\|^2,$$

and this is exactly (3.12).

To prove (3.13), we compute

$$\begin{aligned} & \|\lambda + \rho\|^2 - \|\lambda' + \rho\|^2 \\ &= \sum_{j=1}^q (\lambda_1 + \rho_j)^2 + \sum_{j=q+1}^r (\lambda_1 - 1 + \rho_j)^2 - \sum_{j=1}^{r-q} (\lambda_1 - 1 + \rho_j)^2 - \sum_{j=r-q+1}^r (\lambda_1 - 2 + \rho_j)^2 \end{aligned}$$

$$\begin{aligned}
 &= q\lambda_1^2 + \sum_{j=1}^q (2\lambda_1\rho_j + \rho_j^2) + (r-q)(\lambda_1 - 1)^2 + \sum_{j=q+1}^r [2(\lambda_1 - 1)\rho_j + \rho_j^2] \\
 &\quad - (r-q)(\lambda_1 - 1)^2 - \sum_{j=1}^{r-q} [2(\lambda_1 - 1)\rho_j + \rho_j^2] - q(\lambda_1 - 2)^2 - \sum_{j=r-q+1}^r [2(\lambda_1 - 2)\rho_j + \rho_j^2] \\
 &= q[\lambda_1^2 - (\lambda_1 - 2)^2] - 2 \sum_{j=q+1}^r \rho_j + 2 \sum_{j=1}^{r-q} \rho_j + 4 \sum_{j=r-q+1}^r \rho_j \\
 &= q(4\lambda_1 - 4) - 2 \sum_{j=q+1}^r \rho_j + 2 \sum_{j=1}^r \rho_j + 2 \sum_{j=r-q+1}^r \rho_j \\
 &= 4q(\lambda_1 - 1) + 2 \sum_{j=1}^q \rho_j + 2 \sum_{j=r-q+1}^r \rho_j \\
 &= 4q(\lambda_1 - 1) + [2q(n - q) + 2(q + \dots + 1)] + [2q(n - r) + 2(q + \dots + 1)] \\
 &= 2q[(2\lambda_1 - 2) + (n - q) + (n - r) + (q + 1)] = 2q(2\lambda_1 + 2n - r - 1).
 \end{aligned}$$

Since  $\lambda_1 < -n + \frac{r+1}{2}$ , the last expression is clearly  $< 0$ , and this implies (3.13). So we have proved Theorem 3.1(2) for  $s$  as in (3.11).

Finally, suppose that  $r < n$  and that

$$s = (2b_1, \dots, 2b_i, 0, \dots, 0), \quad b_i \geq 1, \quad i > r. \tag{3.15}$$

Let  $s' = s - 2\varepsilon_i$  and let  $\lambda' = (\lambda - 2\varepsilon_i)^+$ . We claim that

$$\|(\lambda - s)^+ + \rho\|^2 \geq \|(\lambda' - s')^+ + \rho\|^2 \tag{3.16}$$

and that

$$\|\lambda + \rho\|^2 \leq \|\lambda' + \rho\|^2. \tag{3.17}$$

These two equations imply that

$$\|(\lambda - s)^+ + \rho\|^2 - \|\lambda + \rho\|^2 \geq \|(\lambda' - s')^+ + \rho\|^2 - \|\lambda' + \rho\|^2.$$

Moreover, since  $\lambda'$  has the same  $r$  as  $\lambda$ , and also  $\lambda'_1 = \lambda_1$ , we have  $\lambda'_1 < -n + \frac{r+1}{2}$ . Now we can use induction and keep decreasing the last nonzero coordinate of  $s$  until we come to the situation  $i \leq r$ . In this case, we already proved the strict Dirac inequality. So it suffices to prove (3.16) and (3.17).

To prove (3.16), we use Lemma 2.1. Namely, we set  $\mu = \lambda'$  and  $v = s'$ , and choose  $w \in W_{\mathbb{F}}$  such that  $w\mu = \lambda - 2\varepsilon_i$ . Then Lemma 2.1 for  $w_1 = w$ ,  $w_2 = 1$  implies

$$\|(\lambda - 2\varepsilon_i - s')^+ + \rho\|^2 \geq \|(\lambda' - s')^+ + \rho\|^2.$$

This is equivalent to (3.16) because clearly  $\lambda - 2\varepsilon_i - s' = \lambda - s$ .

To prove (3.17), we first note that

$$\lambda' = (\lambda - 2\varepsilon_i)^+ = \lambda - \varepsilon_j - \varepsilon_k$$

for some  $j, k$  satisfying  $i \leq j \leq k \leq n$ . So (3.17) becomes

$$\|\lambda + \rho\|^2 \leq \|\lambda + \rho - (\varepsilon_j + \varepsilon_k)\|^2 = \|\lambda + \rho\|^2 - 2\langle \lambda + \rho, \varepsilon_j + \varepsilon_k \rangle + \|\varepsilon_j + \varepsilon_k\|^2,$$

and this is equivalent to

$$2\langle \lambda + \rho, \varepsilon_j + \varepsilon_k \rangle \leq \|\varepsilon_j + \varepsilon_k\|^2.$$

We claim that in fact the left side of the last inequality is negative, i.e., that

$$\lambda_j + \lambda_k + \rho_j + \rho_k < 0.$$

Since  $\lambda_j$  and  $\lambda_k$  are both  $\leq \lambda_1 - 2$ , and since  $\rho_j$  and  $\rho_k$  are both  $\leq n - r$  (because  $j, k \geq r$ ), it is enough to prove that

$$2\lambda_1 - 4 + 2n - 2r < 0.$$

By assumption,  $\lambda$  is in the continuous part of its line, i.e.,  $2\lambda + 2n < r + 1$ , so the last inequality is obvious. This finishes the proof of Theorem 3.1(2).  $\square$

### 3.2. Dirac inequality for $so^*(2n)$ , $n \geq 4$

The basic Schmid  $\mathfrak{k}$ -submodules of  $S(\mathfrak{p}^-)$  have lowest weights  $-s_i$ , where

$$s_i = (\underbrace{1, \dots, 1}_{2i}, 0, \dots, 0), \quad i = 1, \dots, [n/2].$$

Moreover, all irreducible  $\mathfrak{k}$ -submodules of  $S(\mathfrak{p}^-)$  have lowest weights  $-s$ , where

$$s = (b_1, b_1, b_2, b_2, \dots, b_j, b_j, 0, \dots, 0) \tag{3.18}$$

for some  $j$ ,  $1 \leq j \leq [n/2]$ , and some positive integers  $b_1 \geq b_2 \geq \dots \geq b_j$ .

The highest weight  $(\mathfrak{g}, K)$ -modules have highest weights of the form

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n), \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n, \quad \lambda_i - \lambda_j \in \mathbb{Z}, \quad 1 \leq i, j \leq n.$$

In this case  $\rho = (n - 1, n - 2, \dots, 1, 0)$ .

The basic necessary condition for unitarity is, as before, the Dirac inequality

$$\|(\lambda - s_1)^+ + \rho\|^2 \geq \|\lambda + \rho\|^2. \tag{3.19}$$

To make this inequality more precise, we as before write

$$(\lambda - s_1)^+ = \lambda - \gamma.$$

Then (3.19) becomes

$$\|\lambda - \gamma + \rho\|^2 \geq \|\lambda + \rho\|^2,$$

and since  $\|\gamma\|^2 = 2$ , this is equivalent to

$$\langle \lambda + \rho, \gamma \rangle \leq 1.$$

There are two basic cases:

*Case 1:*  $\lambda_1 > \lambda_2$ . Let  $q \in [2, n]$  be such that  $\lambda_2 = \dots = \lambda_q$  and, in case  $q < n$ ,  $\lambda_q > \lambda_{q+1}$ . Then  $\gamma = \varepsilon_1 + \varepsilon_q$ , and since  $\lambda_q = \lambda_2$ , the basic inequality becomes

$$\lambda_1 + \lambda_2 \leq -2n + q + 2. \tag{3.20}$$

*Case 2:*  $\lambda_1 = \lambda_2$ . Let  $p \in [2, n]$  be such that  $\lambda_1 = \lambda_2 = \dots = \lambda_p$  and, in case  $p < n$ ,  $\lambda_p > \lambda_{p+1}$ . Then  $\gamma = \varepsilon_{p-1} + \varepsilon_p$ , and since  $\lambda_{p-1} = \lambda_p = \lambda_1$ , the basic inequality becomes

$$\lambda_1 \leq -n + p. \tag{3.21}$$

**THEOREM 3.2.** *Let  $\lambda$  be as in Case 1, and suppose that (3.20) holds strictly. Let  $s \neq 0$  be as in (3.18). Then*

$$\|(\lambda - s)^+ + \rho\|^2 > \|\lambda + \rho\|^2.$$

To prove the theorem, assume first that  $2j \leq q$ . Let  $k \leq j$  be the largest integer such that  $b_1 = \dots = b_k$ . Let

$$s' = \underbrace{(b_1 - 1, \dots, b_1 - 1)}_{2k}, b_{k+1}, b_{k+1}, \dots, b_j, b_j, 0, \dots, 0.$$

We claim that

$$\|(\lambda - s)^+ + \rho\|^2 > \|(\lambda - s')^+ + \rho\|^2.$$

The claim implies the theorem for  $2j \leq q$  by induction on  $b_1 + \dots + b_j$ .

To prove the claim, we first note that  $(\lambda - s)^+$  and  $(\lambda - s')^+$  both contain coordinates

$$\underbrace{\lambda_2, \dots, \lambda_2}_{q-2j}, \lambda_2 - b_j, \dots, \lambda_2 - b_{k+1}, \lambda_{q+1}, \dots, \lambda_n.$$

The remaining coordinates of  $(\lambda - s)^+$  are

$$\lambda_1 - b_1, \underbrace{\lambda_2 - b_1, \dots, \lambda_2 - b_1}_{2k-1},$$

while the remaining coordinates of  $(\lambda - s')^+$  are

$$\lambda_1 - b_1 + 1, \underbrace{\lambda_2 - b_1 + 1, \dots, \lambda_2 - b_1 + 1}_{2k-1}.$$

Using Lemma 2.2 (1), we can move the equal coordinates to the right. We use that to move  $\lambda_{q+1}, \dots, \lambda_n$  all the way to the right, and to move those of  $\lambda_2, \dots, \lambda_2, \lambda_2 - b_j, \dots, \lambda_2 - b_{k+1}$  that are left of  $\lambda_1 - b_1$  to the right of  $\lambda_1 - b_1$ . Note that  $\lambda_2, \dots, \lambda_2, \lambda_2 - b_j, \dots, \lambda_2 - b_{k+1}$  are left of  $\lambda_2 - b_1$  or  $\lambda_2 - b_1 + 1$  and we leave them in this position.

Thus we conclude that it is enough to prove that

$$\|\mu + \rho\|^2 > \|\nu + \rho\|^2, \quad (3.22)$$

where

$$\begin{aligned} \mu &= (\lambda_1 - b_1, \underbrace{\lambda_2, \lambda_2 - b_j, \dots, \lambda_2 - b_{k+1}}_{q-2j}, \underbrace{\lambda_2 - b_1, \lambda_{q+1}, \dots, \lambda_n}_{2k-1}); \\ \nu &= (\lambda_1 - b_1 + 1, \underbrace{\lambda_2, \lambda_2 - b_j, \dots, \lambda_2 - b_{k+1}}_{q-2j}, \underbrace{\lambda_2 - b_1 + 1, \lambda_{q+1}, \dots, \lambda_n}_{2k-1}). \end{aligned}$$

Proving (3.22) is equivalent to proving that the expression

$$\begin{aligned} &[(\lambda_1 - b_1 + \rho_1)^2 - (\lambda_1 - b_1 + 1 + \rho_1)^2] + [(\lambda_2 - b_1 + \rho_{q-2k+2})^2 \\ &\quad - (\lambda_2 - b_1 + 1 + \rho_{q-2k+2})^2] + \dots + [(\lambda_2 - b_1 + \rho_q)^2 - (\lambda_2 - b_1 + 1 + \rho_q)^2] \end{aligned}$$

is positive. Factoring each difference of squares, we see this is equivalent to the expression

$$(2\lambda_1 - 2b_1 + 1 + 2\rho_1) + (2\lambda_2 - 2b_1 + 1 + 2\rho_{q-2k+2}) + \dots + (2\lambda_2 - 2b_1 + 1 + 2\rho_q)$$

being negative. Dividing by two and simplifying we see that we should prove

$$\lambda_1 + (2k-1)\lambda_2 - 2kb_1 + k + \rho_1 + \rho_{q-2k+2} + \dots + \rho_q < 0. \quad (3.23)$$

We compute

$$\begin{aligned} \rho_{q-2k+2} + \dots + \rho_q &= (n - q + 2k - 2) + \dots + (n - q) \\ &= (2k - 1)(n - q) + 1 + 2 + \dots + (2k - 2) \\ &= (2k - 1)(n - q) + (k - 1)(2k - 1). \end{aligned}$$

On the other hand, since (3.20) holds strictly for  $\lambda$  and since  $\lambda_1 > \lambda_2$ , we see that

$$\lambda_1 + (2k-1)\lambda_2 < k(\lambda_1 + \lambda_2) < k(-2n + q + 2).$$

Thus we see that to prove (3.23) it is enough to prove that

$$k(-2n + q + 2) - 2kb_1 + k + n - 1 + (2k-1)(n-q) + (k-1)(2k-1) \leq 0.$$

Simplifying and taking into account that  $b_1 \geq 1$ , we see that the last inequality follows if we prove

$$(q-2k)(-k+1) \leq 0,$$

but this is obvious since  $2k \leq 2j \leq q$  and since  $k \geq 1$ .

It remains to prove the theorem when  $2j > q$ . In that case we set

$$\lambda' = (\lambda - s_j)^+; \quad s' = s - s_j.$$

We are going to prove

$$\|(\lambda - s)^+ + \rho\|^2 - \|\lambda + \rho\|^2 > \|(\lambda' - s')^+ + \rho\|^2 - \|\lambda' + \rho\|^2. \tag{3.24}$$

This implies that the statement of the theorem holds for  $\lambda$  and  $s$  if it holds for  $\lambda'$  and  $s'$ . On the other hand, since  $\lambda'$  starts with coordinates

$$\lambda_1 - 1, \underbrace{\lambda_2 - 1, \dots, \lambda_2 - 1}_{q-1},$$

we see that  $\lambda'_1 < \lambda_1$ ,  $\lambda'_2 < \lambda_2$  and  $q' \geq q$ , so (3.20) holds strictly for  $\lambda'$ . Now if  $2j' \leq q'$  we already proved the theorem for  $\lambda'$  and  $s'$ , and if  $2j' > q'$  we note that  $s'$  has last coordinate lower than  $s$  and so we can assume the theorem holds for  $\lambda'$  and  $s'$  by induction.

To prove (3.24), we first note that Lemma 2.1 immediately implies

$$\|(\lambda - s)^+ + \rho\|^2 \geq \|(\lambda' - s')^+ + \rho\|^2,$$

by setting  $\mu = \lambda'$  and  $\nu = s'$ , and picking  $w_1$  such that  $\lambda - s_j = w_1\mu$  and  $w_2 = 1$ . So it is enough to prove that

$$\|\lambda + \rho\|^2 < \|\lambda' + \rho\|^2. \tag{3.25}$$

The difference  $\|\lambda + \rho\|^2 - \|\lambda' + \rho\|^2$  is the sum of expressions

$$(\lambda_i + \rho_i)^2 - (\lambda_i - 1 + \rho_i)^2 = 2\lambda_i - 1 + 2\rho_i,$$

where  $i$  runs over  $1, 2, \dots, q$  and over some  $2j - q$  values greater than  $q$ . For  $i > q$ , we use the strict (3.20) and  $\lambda_1 > \lambda_2$  to conclude

$$2\lambda_i - 1 + 2\rho_i < 2\lambda_2 - 1 + 2\rho_q < (-2n + q + 2) - 1 + 2n - 2q = -q + 1 < 0.$$

Furthermore, we claim that

$$\sum_{i=1}^q (2\lambda_i - 1 + 2\rho_i) = 2\lambda_1 + (2q - 2)\lambda_2 - q + 2(\rho_1 + \dots + \rho_q)$$

is also negative; this will then imply (3.25). Using the strict (3.20) and  $\lambda_1 > \lambda_2$ , we see that

$$2\lambda_1 + (2q - 2)\lambda_2 < q(\lambda_1 + \lambda_2) < q(-2n + q + 2) = -2qn + q^2 + 2q.$$

On the other hand,

$$2(\rho_1 + \dots + \rho_q) = 2qn - 2(1 + \dots + q) = 2qn - q(q + 1).$$

So

$$\sum_{i=1}^q (2\lambda_i - 1 + 2\rho_i) < (-2nq + q^2 + 2q) - q + (2qn - q^2 - q) = 0.$$

This finishes the proof of Theorem 3.2.

We now turn to Case 2, i.e.,

$$\lambda = (\underbrace{\lambda_1, \dots, \lambda_1}_p, \lambda_{p+1}, \dots, \lambda_n)$$

for some  $p \in [2, n]$ , with  $\lambda_1 > \lambda_{p+1}$  if  $p < n$ . Besides the basic inequality (3.21), we also examine when

$$\|(\lambda - s_i)^+ + \rho\|^2 \geq \|\lambda + \rho\|^2 \tag{3.26}$$

for a basic Schmid module  $s_i$  with  $2i \leq p$ . Since  $(\lambda - s_i)^+ = \lambda - (\varepsilon_{p-2i+1} + \dots + \varepsilon_p)$ , (3.26) is equivalent to

$$2\langle \lambda + \rho, \varepsilon_{p-2i+1} + \dots + \varepsilon_p \rangle \leq \|\varepsilon_{p-2i+1} + \dots + \varepsilon_p\|^2 = 2i. \tag{3.27}$$

Furthermore,

$$\begin{aligned} \langle \lambda + \rho, \varepsilon_{p-2i+1} + \dots + \varepsilon_p \rangle &= 2i\lambda_1 + 2i(n - p) + (1 + \dots + (2i - 1)) \\ &= 2i \left( \lambda_1 + n - p + i - \frac{1}{2} \right). \end{aligned}$$

We substitute this into (3.27) and divide the resulting inequality by  $4i$ . It follows that (3.26) is equivalent to

$$\lambda_1 \leq -n + p - i + 1. \tag{3.28}$$

In particular, for  $i = 1$  this is the basic inequality (3.21).

**THEOREM 3.3.** *Let  $\lambda$  be in Case 2, i.e.,*

$$\lambda = (\underbrace{\lambda_1, \dots, \lambda_1}_p, \lambda_{p+1}, \dots, \lambda_n)$$

for some  $p \in [2, n]$ , with  $\lambda_1 > \lambda_{p+1}$  if  $p < n$ . Then:

1. If for some integer  $i \in [1, \lfloor \frac{p}{2} \rfloor]$

$$\lambda_1 < -n + p - i + 1,$$

then the inequality

$$\|(\lambda - s)^+ + \rho\|^2 > \|\lambda + \rho\|^2 \tag{3.29}$$

holds for any Schmid module  $s$  with at most  $2i$  nonzero components.

2. If

$$\lambda_1 < -n + p - \left\lfloor \frac{p}{2} \right\rfloor + 1 = -n + \left\lceil \frac{p+1}{2} \right\rceil + 1, \tag{3.30}$$

then (3.29) holds strictly for any Schmid module  $s$ .

*Proof.* (1) Let  $i \in [1, \lfloor \frac{p}{2} \rfloor]$  be an integer such that

$$\lambda_1 < -n + p - i + 1, \tag{3.31}$$

and let

$$s = (b_1, b_1, \dots, b_j, b_j, 0, \dots, 0)$$

with  $j \leq i$  and  $b_1 \geq \dots \geq b_j > 0$ .

We need to show that (3.29) holds strictly for  $\lambda$  and  $s$ .

Let  $k \in [1, j]$  be such that

$$b_1 = \dots = b_k > b_{k+1}.$$

Let  $s' = s - s_k$ . It is enough to prove

$$\|(\lambda - s)^+ + \rho\|^2 > \|(\lambda - s')^+ + \rho\|^2; \tag{3.32}$$

the statement then follows by induction on  $b_1$ . (If  $b_1 = 1$ , then  $s' = 0$ , and (3.32) is the same as the strict (3.29).)

We first note that  $(\lambda - s)^+$  contains coordinates

$$\underbrace{\lambda_1, \dots, \lambda_1}_{p-2j}, \underbrace{\lambda_1 - b_i, \dots, \lambda_1 - b_{k+1}}_{2j-2k}, \underbrace{\lambda_1 - b_1, \dots, \lambda_1 - b_1}_{2k}$$

in that order, and then also  $\lambda_{p+1}, \dots, \lambda_n$ , which may be interlaced with these coordinates. Similarly,  $(\lambda - s')^+$  contains coordinates

$$\underbrace{\lambda_1, \dots, \lambda_1}_{p-2j}, \underbrace{\lambda_1 - b_i, \dots, \lambda_1 - b_{k+1}}_{2j-2k}, \underbrace{\lambda_1 - b_1 + 1, \dots, \lambda_1 - b_1 + 1}_{2k}$$

in that order, and then also  $\lambda_{p+1}, \dots, \lambda_n$ , which may be interlaced with these coordinates.

Using Lemma 2.2 (1), we may assume that  $\lambda_{p+1}, \dots, \lambda_n$  are all the way to the right in both  $(\lambda - s)^+$  and  $(\lambda - s')^+$ . Thus it suffices to show that

$$\sum_{r=p-2k+1}^p [(\lambda_1 - b_1 + \rho_r)^2 - (\lambda_1 - b_1 + 1 + \rho_r)^2] > 0.$$

By factoring differences of squares, this is equivalent to

$$\sum_{r=p-2k+1}^p (2\lambda_1 - 2b_1 + 1 + 2\rho_r) < 0. \tag{3.33}$$

Since

$$\begin{aligned} \sum_{r=p-2k+1}^p \rho_r &= (n - p + 2k - 1) + \dots + (n - p) \\ &= 2k(n - p - 1) + (2k + \dots + 1) = 2k(n - p - 1) + k(2k + 1), \end{aligned}$$

(3.33) is equivalent to

$$2k(2\lambda_1 - 2b_1 + 1) + 4k(n - p - 1) + 2k(2k + 1) < 0,$$

which upon dividing by  $4k$  becomes

$$\lambda_1 - b_1 + n - p + k < 0.$$

Since  $b_1 \geq 1$  and since  $k \leq j \leq i$ , this follows from our assumption (3.31). This finishes the proof of (1).

(2) If  $s$  has at most  $p$  nonzero components, then the statement follows from (1) by specializing to  $i = \lfloor \frac{p}{2} \rfloor$ . So we can assume that

$$s = (b_1, b_1, \dots, b_i, b_i, 0, \dots, 0),$$

with  $2i > p$ .

Let  $\lambda' = (\lambda - s_i)^+$ . Then  $\lambda'_1, \dots, \lambda'_p$  are all equal to  $\lambda_1 - 1$ , while for  $r > p$ ,  $\lambda'_r$  is equal to either  $\lambda_r$  or  $\lambda_r - 1$ . In particular,  $\lambda'$  is still in Case 2, with  $p' \geq p$  and with  $\lambda'_1 = \lambda_1 - 1$ , and so we have

$$\lambda'_1 < -n + \left\lfloor \frac{p' + 1}{2} \right\rfloor + 1.$$

We claim that

$$\|(\lambda - s)^+ + \rho\|^2 - \|\lambda + \rho\|^2 > \|(\lambda' - s')^+ + \rho\|^2 - \|\lambda' + \rho\|^2. \quad (3.34)$$

If we prove (3.34), then we can use induction. Namely  $s'$  either has at most  $p$  and hence at most  $p'$  nonzero coordinates, so the statement is true by Theorem 3.3 (1), or we can use the fact that the first coordinate  $b'_1$  of  $s'$  is strictly smaller than the first coordinate  $b_1$  of  $s$  and do induction on  $b_1$ . (As before, if  $b_1 = 1$ , then  $s' = 0$  and (3.34) is the same as the strict (3.29).)

To prove (3.34), we first note that

$$\|(\lambda - s)^+ + \rho\|^2 \geq \|(\lambda' - s')^+ + \rho\|^2.$$

This follows from Lemma 2.1, if we set  $\mu = \lambda'$ ,  $\nu = s'$  and  $w_2 = 1$ , and pick  $w_1$  such that  $\lambda - s_i = w_1 \lambda'$ . It thus suffices to prove that

$$\|\lambda + \rho\|^2 < \|\lambda' + \rho\|^2. \quad (3.35)$$

To prove (3.35), we note that  $\|\lambda + \rho\|^2 - \|\lambda' + \rho\|^2$  is the sum of expressions of the form

$$(\lambda_r + \rho_r)^2 - (\lambda_r - 1 + \rho_r)^2 = 2\lambda_r - 1 + 2\rho_r,$$

with summation over  $r = 1, \dots, p$  and over some  $r > p$ . Note that since  $2i > p$ , there is at least one summand with  $r > p$ .

Since  $\lambda_1, \dots, \lambda_r$  are all equal to  $\lambda_1$ , and since

$$\sum_{r=1}^p \rho_r = (n-1) + \dots + (n-p) = pn - \frac{p(p+1)}{2},$$

we see that

$$\sum_{r=1}^p (2\lambda_r - 1 + 2\rho_r) = 2p\lambda_1 - p + 2pn - p(p+1) = p(2\lambda_1 - 2 + 2n - p).$$

Since  $\lambda$  satisfies (3.30), it follows that

$$\begin{aligned} \sum_{r=1}^p (2\lambda_r - 1 + 2\rho_r) &< p \left[ \left( -2n + 2p - 2 \left\lfloor \frac{p}{2} \right\rfloor + 2 \right) - 2 + 2n - p \right] \\ &= p \left( p - 2 \left\lfloor \frac{p}{2} \right\rfloor \right) \leq p. \end{aligned} \tag{3.36}$$

On the other hand, for any  $r > p$  we have

$$\lambda_r \leq \lambda_1 - 1 < -n + p - \left\lfloor \frac{p}{2} \right\rfloor,$$

and  $\rho_r \leq \rho_{p+1} = n - p - 1$ . It follows that

$$2\lambda_r - 1 + 2\rho_r < -2n + 2p - 2 \left\lfloor \frac{p}{2} \right\rfloor - 1 + 2(n - p - 1) = -2 \left\lfloor \frac{p}{2} \right\rfloor - 3 < -p. \tag{3.37}$$

Since  $\|\lambda + \rho\|^2 - \|\lambda' + \rho\|^2$  is the sum of expressions  $2\lambda_r - 1 + 2\rho_r$  over  $r = 1, \dots, p$  and over some (at least one)  $r > p$ , we see from (3.36) and (3.37) that

$$\|\lambda + \rho\|^2 - \|\lambda' + \rho\|^2 < p - p = 0,$$

as claimed.  $\square$

### 3.3. Dirac inequality for $\mathfrak{su}(p, q)$ , $p \leq q$ , $p \geq 1$ , $q \geq 1$

The basic Schmid  $\mathfrak{k}$ -submodules of  $S(\mathfrak{p}^-)$  have lowest weights  $-s_i$ , where

$$s_i = (\underbrace{1, \dots, 1}_i, 0, \dots, 0 \mid 0, \dots, 0, \underbrace{-1, \dots, -1}_i)$$

for  $i = 1, \dots, p$ . Moreover, all irreducible  $\mathfrak{k}$ -submodules of  $S(\mathfrak{p}^-)$  have lowest weights  $-s$ , where

$$s = (b_1, \dots, b_p \mid 0, \dots, 0, -b_p, \dots, -b_1), \tag{3.38}$$

where  $b_1 \geq \dots \geq b_p \geq 0$  are integers.

The highest weight  $(\mathfrak{g}, K)$ -modules have highest weights of the form

$$\lambda = (\lambda_1, \dots, \lambda_p \mid \lambda_{p+1}, \dots, \lambda_n),$$

where components  $\lambda_1, \dots, \lambda_n$  satisfy

$$\lambda_1 \geq \dots \geq \lambda_p; \quad \lambda_{p+1} \geq \dots \geq \lambda_n,$$

and  $\lambda_i - \lambda_j$  is an integer for any  $i, j \in \{1, \dots, p\}$  or  $i, j \in \{p+1, \dots, n\}$ .

In this case  $\rho = \frac{1}{2}(n-1, n-3, \dots, -n+3, -n+1)$ .

The basic necessary condition for unitarity is, as before, the Dirac inequality

$$\|(\lambda - s_1)^+ + \rho\|^2 \geq \|\lambda + \rho\|^2. \quad (3.39)$$

To understand this inequality better, let  $p' \leq p$  and  $q' \leq q$  be the maximal positive integers such that

$$\lambda_1 = \dots = \lambda_{p'} \quad \text{and} \quad \lambda_{n-q'+1} = \dots = \lambda_n.$$

Then

$$(\lambda - s_1)^+ = \lambda - (\varepsilon_{p'} - \varepsilon_{n-q'+1}).$$

The inequality (3.39) now becomes equivalent to  $\|\lambda + \rho - \gamma\|^2 \geq \|\lambda + \rho\|^2$ , or to  $2\langle \lambda + \rho, \gamma \rangle \leq \|\gamma\|^2 = 2$ , or to  $\langle \lambda + \rho, \varepsilon_{p'} - \varepsilon_{n-q'+1} \rangle \leq 1$ . Since  $\lambda_{p'} = \lambda_1$ ,  $\lambda_{n-q'+1} = \lambda_n$ , and since  $\rho_{p'} - \rho_{n-q'+1} = n - q' + 1 - p'$ , we see that (3.39) is equivalent to

$$\lambda_1 - \lambda_n \leq -n + p' + q'. \quad (3.40)$$

As in the other cases, we start by examining the condition on  $\lambda$  which ensures that the Dirac inequality

$$\|(\lambda - s_i)^+ + \rho\|^2 \geq \|\lambda + \rho\|^2 \quad (3.41)$$

holds, where  $i = 1, \dots, \min(p', q')$ . We already know that for  $i = 1$  this is just the basic inequality (3.39), or equivalently (3.40).

To examine when (3.41) holds, we first note that

$$(\lambda - s_i)^+ = \lambda - t_i, \quad \text{where} \quad t_i = (\varepsilon_{p'-i+1} + \dots + \varepsilon_{p'}) - (\varepsilon_{n-q'+1} + \dots + \varepsilon_{n-q'+i}).$$

Since  $\|t_i\|^2 = 2i$ , it follows that (3.41) is equivalent to

$$\langle \lambda + \rho, t_i \rangle \leq i. \quad (3.42)$$

We note that  $\lambda_{p'-i+1}, \dots, \lambda_{p'}$  are all equal to  $\lambda_1$  while  $\lambda_{n-q'+1}, \dots, \lambda_{n-q'+i}$  are all equal to  $\lambda_n$ , and that  $\rho_{p'-i+j} - \rho_{n-q'+j} = n - p' - q' + i$  for any  $j \in [1, i]$ . Plugging this into (3.42) and dividing by  $i$ , we see that (3.42) (and hence (3.41)) is equivalent to

$$\lambda_1 - \lambda_n \leq -n + p' + q' - i + 1. \quad (3.43)$$

**THEOREM 3.4.** *Let*

$$\lambda = \underbrace{(\lambda_1, \dots, \lambda_1)}_{p'} | \lambda_{p'+1}, \dots, \lambda_p | \lambda_{p+1}, \dots, \lambda_{n-q'} | \underbrace{(\lambda_n, \dots, \lambda_n)}_{q'}.$$

*Then:*

1. If  $\lambda$  satisfies (3.43) strictly for some integer  $i \in [1, \min(p', q')]$ , then the inequality

$$\|(\lambda - s)^+ + \rho\|^2 \geq \|\lambda + \rho^2\| \tag{3.44}$$

holds strictly for any Schmid module

$$s = (b_1, \dots, b_j, 0, \dots, 0 \mid 0, \dots, 0, -b_j, \dots, -b_1) \tag{3.45}$$

with  $j \leq i$ .

2. If

$$\lambda_1 - \lambda_n < -n + p' + q' - \min(p', q') + 1 = -n + \max(p', q') + 1, \tag{3.46}$$

then (3.44) holds strictly for any Schmid module  $s$ .

*Proof.* Both parts of the theorem follow from the following discussion. Let  $s$  be any Schmid module:

$$s = (b_1, \dots, b_p \mid -b_q, \dots, -b_{p+1}, -b_p, \dots, -b_1)$$

where  $b_1 \geq \dots \geq b_p \geq 0$  are integers, not all of them zero, and  $b_{p+1} = \dots = b_q = 0$ . Let  $k \in [1, p]$  be the maximal integer such that  $b_1 = \dots = b_k$ , and define

$$s' = s - s_k = (\underbrace{b_1 - 1, b_{k+1}, \dots, b_p}_k \mid -b_q, \dots, -b_{k+1}, \underbrace{-b_1 + 1}_k).$$

For both parts of the theorem, it will suffice to prove

$$\|(\lambda - s)^+ + \rho\|^2 > \|(\lambda - s')^+ + \rho\|^2. \tag{3.47}$$

The statements then follow by induction on  $b_1$ ; if  $b_1 = 1$ , then  $s' = 0$ , and (3.47) is the same as the strict (3.44).

To prove (3.47), we examine separately the two sides of all expressions involved, the left respectively right side (of the bar).

If  $k \leq p'$ , the left side of  $(\lambda - s)^+$  contains coordinates

$$\underbrace{\lambda_1 - b_{p'}, \dots, \lambda_1 - b_{k+1}}_{p'-k}, \underbrace{\lambda_1 - b_1, \dots, \lambda_1 - b_1}_k \tag{3.48}$$

in that order, and also  $\lambda_{p'+1} - b_{p'+1}, \dots, \lambda_p - b_p$ , arranged in descending order and appropriately interlaced with the coordinates (3.48).

On the other hand, the left side of  $(\lambda - s')^+$  contains the same coordinates, except that it contains  $k$  coordinates equal to  $\lambda_1 - b_1 + 1$  in place of  $k$  coordinates equal to  $\lambda_1 - b_1$ .

Using a version of Lemma 2.2, we may assume that  $\lambda_{p'+1} - b_{p'+1}, \dots, \lambda_p - b_p$  are all the way to the right in the left sides of both  $(\lambda - s)^+$  and  $(\lambda - s')^+$ .

Thus the contribution of the left sides to  $\|(\lambda - s)^+ + \rho\|^2 - \|(\lambda - s')^+ + \rho\|^2$  is

$$\begin{aligned} & \sum_{u=p'-k+1}^{p'} [(\lambda_1 - b_1 + \rho_u)^2 - (\lambda_1 - b_1 + 1 + \rho_u)^2] \\ &= - \sum_{u=p'-k+1}^{p'} (2\lambda_1 - 2b_1 + 1 + 2\rho_u) = - \left( 2k\lambda_1 - 2kb_1 + k + \sum_{u=p'-k+1}^{p'} 2\rho_u \right) \\ &\geq -2k\lambda_1 + k - \sum_{u=p'-k+1}^{p'} 2\rho_u \end{aligned} \tag{3.49}$$

(for the last inequality we used  $b_1 \geq 1$ ).

If  $k > p'$ , the left side of  $(\lambda - s)^+$  contains coordinates

$$\underbrace{\lambda_1 - b_1, \dots, \lambda_1 - b_1}_{p'}, \lambda_{p'+1} - b_1, \dots, \lambda_k - b_1 \tag{3.50}$$

in that order, and also  $\lambda_{k+1} - b_{k+1}, \dots, \lambda_p - b_p$ , arranged in descending order and appropriately interlaced with the coordinates (3.50).

On the other hand, the left side of  $(\lambda - s')^+$  contains coordinates

$$\underbrace{\lambda_1 - b_1 + 1, \dots, \lambda_1 - b_1 + 1}_{p'}, \lambda_{p'+1} - b_1 + 1, \dots, \lambda_k - b_1 + 1 \tag{3.51}$$

in that order, and also  $\lambda_{k+1} - b_{k+1}, \dots, \lambda_p - b_p$ , arranged in descending order and appropriately interlaced with the coordinates (3.51). These last coordinates are the same as in  $(\lambda - s)^+$ , and also at the same places.

Using (the extension of) Lemma 2.2, we may assume that  $\lambda_{k+1} - b_{k+1}, \dots, \lambda_p - b_p$  are all the way to the right in the left sides of both  $(\lambda - s)^+$  and  $(\lambda - s')^+$ . Thus the contribution of the left sides to  $\|(\lambda - s)^+ + \rho\|^2 - \|(\lambda - s')^+ + \rho\|^2$  is

$$\begin{aligned} & \sum_{u=1}^{p'} [(\lambda_1 - b_1 + \rho_u)^2 - (\lambda_1 - b_1 + 1 + \rho_u)^2] \\ &+ \sum_{u=p'+1}^k [(\lambda_u - b_1 + \rho_u)^2 - (\lambda_u - b_1 + 1 + \rho_u)^2] \\ &= - \sum_{u=1}^{p'} (2\lambda_1 - 2b_1 + 1 + 2\rho_u) - \sum_{u=p'+1}^k (2\lambda_u - 2b_1 + 1 + 2\rho_u). \end{aligned} \tag{3.52}$$

Since  $\lambda_u \leq \lambda_1 - 1$  for  $u > p'$ , and since  $b_1 \geq 1$ , we conclude that the expression (3.52) is

$$\geq -2k\lambda_1 + 3k - 2p' - \sum_{u=1}^k 2\rho_u. \tag{3.53}$$

We now consider the right sides. We first assume  $k \leq q'$ . Then the right side of  $(\lambda - s)^+$  contains coordinates

$$\underbrace{\lambda_n + b_1, \dots, \lambda_n + b_1}_k, \lambda_n + b_{k+1}, \dots, \lambda_n + b_{q'}, \tag{3.54}$$

in that order, and also  $\lambda_{p+1} + b_q, \dots, \lambda_{n-q'} + b_{q'+1}$ , arranged in descending order and appropriately interlaced with coordinates (3.54). On the other hand, the right side of  $(\lambda - s')^+$  contains the same coordinates, except that it contains  $k$  coordinates equal to  $\lambda_n + b_1 - 1$  in place of  $k$  coordinates equal to  $\lambda_n + b_1$ .

Using (the extension of) Lemma 2.2, we may assume that  $\lambda_{p+1} + b_q, \dots, \lambda_{n-q'} + b_{q'+1}$  are all the way to the left in the right groups of both  $(\lambda - s)^+$  and  $(\lambda - s')^+$ . Thus the contribution of the right sides to  $\|(\lambda - s)^+ + \rho\|^2 - \|(\lambda - s')^+ + \rho\|^2$  is

$$\begin{aligned} & \sum_{v=n-q'+1}^{n-q'+k} [(\lambda_n + b_1 + \rho_v)^2 - (\lambda_n + b_1 - 1 + \rho_v)^2] \\ = & \sum_{v=n-q'+1}^{n-q'+k} (2\lambda_n + 2b_1 - 1 + 2\rho_v) = 2k\lambda_n + 2kb_1 - k + \sum_{v=n-q'+1}^{n-q'+k} 2\rho_v \\ \geq & 2k\lambda_n + k + \sum_{v=n-q'+1}^{n-q'+k} 2\rho_v \end{aligned} \tag{3.55}$$

(for the last inequality we used  $b_1 \geq 1$ ).

If  $k > q'$ , the right side of  $(\lambda - s)^+$  contains coordinates

$$\lambda_{n-k+1} + b_1, \dots, \lambda_{n-q'} + b_1, \underbrace{\lambda_n + b_1, \dots, \lambda_n + b_1}_{q'}, \tag{3.56}$$

in that order, and also  $\lambda_{p+1} + b_q, \dots, \lambda_{n-k} + b_{k+1}$ , arranged in descending order and appropriately interlaced with coordinates (3.56).

On the other hand, the right side of  $(\lambda - s')^+$  contains coordinates

$$\lambda_{n-k+1} + b_1 - 1, \dots, \lambda_{n-q'} + b_1 - 1, \underbrace{\lambda_n + b_1 - 1, \dots, \lambda_n + b_1 - 1}_{q'}, \tag{3.57}$$

in that order, and also  $\lambda_{p+1} + b_q, \dots, \lambda_{n-k} + b_{k+1}$ , arranged in descending order and appropriately interlaced with the coordinates (3.57). These last coordinates are the same as in  $(\lambda - s)^+$ , and also at the same places.

Using (the extension of) Lemma 2.2, we may assume that  $\lambda_{p+1} + b_q, \dots, \lambda_{n-k} + b_{k+1}$  are all the way to the left in the right groups of both  $(\lambda - s)^+$  and  $(\lambda - s')^+$ . Thus

the contribution of the right sides to  $\|(\lambda - s)^+ + \rho\|^2 - \|(\lambda - s')^+ + \rho\|^2$  is

$$\begin{aligned} & \sum_{v=n-k+1}^{n-q'} [(\lambda_v + b_1 + \rho_v)^2 - (\lambda_v + b_1 - 1 + \rho_v)^2] \\ & \quad + \sum_{v=n-q'+1}^n [(\lambda_n + b_1 + \rho_v)^2 - (\lambda_n + b_1 - 1 + \rho_v)^2] \\ & = \sum_{v=n-k+1}^{n-q'} (2\lambda_v + 2b_1 - 1 + 2\rho_v) + \sum_{v=n-q'+1}^n (2\lambda_n + 2b_1 - 1 + 2\rho_v). \end{aligned} \quad (3.58)$$

Since  $\lambda_v \geq \lambda_n + 1$  for  $v \leq n - q'$ , and since  $b_1 \geq 1$ , we conclude that the expression (3.58) is

$$\geq 2k\lambda_n + 3k - 2q' + \sum_{v=n-k+1}^n 2\rho_v. \quad (3.59)$$

Let us now assume that  $k \leq \min(p', q')$ ; this is always true under the assumptions of the first part of the theorem. Using (3.49) and (3.55), we see that to prove the required inequality (3.47) it is enough to prove that

$$2k(\lambda_1 - \lambda_n) - 2k + \sum_{u=p'-k+1}^{p'} 2\rho_u - \sum_{v=n-q'+1}^{n-q'+k} 2\rho_v < 0. \quad (3.60)$$

Since for any integer  $t \in [1, k]$  we have

$$\rho_{p'-k+t} - \rho_{n-q'+t} = n - p' - q' + k,$$

we see that

$$\sum_{u=p'-k+1}^{p'} 2\rho_u - \sum_{v=n-q'+1}^{n-q'+k} 2\rho_v = 2k(n - p' - q' + k).$$

We substitute this into (3.60) and divide by  $2k$ . It follows that (3.60) is equivalent to

$$\lambda_1 - \lambda_n - 1 + n - p' - q' + k < 0. \quad (3.61)$$

Under the assumptions of the first part of the theorem, it follows that the left side of (3.61) is

$$< (-n + p' + q' - i + 1) - 1 + n - p' - q' + k = -i + k \leq 0,$$

so this finishes the proof of Theorem 3.4 (1).

Under the assumptions of the second part of the theorem, and our current assumption that  $k \leq \min(p', q')$ , it follows that the left side of (3.61) is

$$< (-n + \max(p', q') + 1) - 1 + n - p' - q' + k \leq \max(p', q') - p' - q' + \min(p', q') = 0,$$

so this finishes the proof of Theorem 3.4 (2) in case  $k \leq \min(p', q')$ .

Let us now assume that  $p' < k \leq q'$ . Using (3.53) and (3.55), we see that to prove the required inequality (3.47) it is enough to prove that

$$2k(\lambda_1 - \lambda_n) - 4k + 2p' + \sum_{u=1}^k 2\rho_u - \sum_{v=n-q'+1}^{n-q'+k} 2\rho_v < 0. \quad (3.62)$$

Since for any integer  $t \in [1, k]$  we have

$$\rho_t - \rho_{n-q'+t} = n - q',$$

we see that

$$\sum_{u=1}^k 2\rho_u - \sum_{v=n-q'+1}^{n-q'+k} 2\rho_v = 2k(n - q').$$

We substitute this into (3.62). It follows that (3.62) is equivalent to

$$2k(\lambda_1 - \lambda_n) - 4k + 2p' + 2k(n - q') < 0. \quad (3.63)$$

Under the assumptions of Theorem 3.4 (2), remembering that in the present case  $\max(p', q') = q'$ , we see that the left side of (3.61) is

$$< 2k(-n + q' + 1) - 4k + 2p' + 2k(n - q') = -2k + 2p',$$

and this is  $< 0$  since in the present case  $k > p'$ . This finishes the proof of Theorem 3.4 (2) in case  $p' < k \leq q'$ .

Let us now assume that  $q' < k \leq p'$ . Using (3.49) and (3.59), we see that to prove the required inequality (3.47) it is enough to prove that

$$2k(\lambda_1 - \lambda_n) - 4k + 2q' + \sum_{u=p'-k+1}^{p'} 2\rho_u - \sum_{v=n-k+1}^n 2\rho_v < 0. \quad (3.64)$$

Since for any integer  $t \in [1, k]$  we have

$$\rho_{p'-k+t} - \rho_{n-k+t} = n - p',$$

we see that

$$\sum_{u=p'-k+1}^{p'} 2\rho_u - \sum_{v=n-k+1}^n 2\rho_v = 2k(n - p').$$

We substitute this into (3.64). It follows that (3.64) is equivalent to

$$2k(\lambda_1 - \lambda_n) - 4k + 2q' + 2k(n - p') < 0. \quad (3.65)$$

Under the assumptions of Theorem 3.4 (2), remembering that in the present case  $\max(p', q') = p'$ , we see that the left side of (3.65) is

$$< 2k(-n + p' + 1) - 4k + 2q' + 2k(n - p') = -2k + 2q',$$

and this is  $< 0$  since in the present case  $k > q'$ . This finishes the proof of Theorem 3.4 (2) in case  $q' < k \leq p'$ .

Finally, let us assume that  $k > \max(p', q')$ . Using (3.53) and (3.59), we see that to prove the required inequality (3.47) it is enough to prove that

$$2k(\lambda_1 - \lambda_n) - 6k + 2p' + 2q' + \sum_{u=1}^k 2\rho_u - \sum_{v=n-k+1}^n 2\rho_v < 0. \quad (3.66)$$

Since for any integer  $t \in [1, k]$  we have

$$\rho_t - \rho_{n-k+t} = n - k,$$

we see that

$$\sum_{u=1}^k 2\rho_u - \sum_{v=n-k+1}^n 2\rho_v = 2k(n - k).$$

We substitute this into (3.66). It follows that (3.66) is equivalent to

$$2k(\lambda_1 - \lambda_n) - 6k + 2p' + 2q' + 2k(n - k) < 0. \quad (3.67)$$

Under the assumptions of the second part of the theorem, it follows that the left side of (3.67) is

$$\begin{aligned} &< 2k(-n + \max(p', q') + 1) - 6k + 2p' + 2q' + 2k(n - k) \\ &= 2k(\max(p', q') - k) - 4k + 2p' + 2q', \end{aligned}$$

and this is  $< 0$ , since in the present case  $k > \max(p', q')$ . This finishes the proof of Theorem 3.4 (2) in case  $k > \max(p', q')$ , and hence the proof of Theorem 3.4 is completed.  $\square$

### 3.4. Dirac inequality for $\mathfrak{so}(2, 2n - 2)$ , $n \geq 3$

The basic Schmid  $\mathfrak{k}$ -submodules of  $S(\mathfrak{p}^-)$  have lowest weights  $-s_1$  or  $-s_2$ , where

$$s_1 = (1, 1, 0, \dots, 0), \quad s_2 = (2, 0, 0, \dots, 0).$$

Moreover, all irreducible  $\mathfrak{k}$ -submodules of  $S(\mathfrak{p}^-)$  have lowest weights  $-s_{a,b}$ , where  $s_{a,b} = (2b + a, a, 0, \dots, 0)$ .

The highest weight  $(\mathfrak{g}, K)$ -modules have highest weights of the form  $\lambda = (\lambda_1, \dots, \lambda_n)$

$$\lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_{n-1} \geq |\lambda_n|.$$

where components  $\lambda_1, \dots, \lambda_n$  satisfy  $\lambda_i - \lambda_j \in \mathbb{Z}$  for  $2 \leq i, j \leq n$ .

In this case  $\rho = (n-1, n-2, \dots, 0)$ .

The basic necessary condition for unitarity is, as before, the Dirac inequality

$$\|(\lambda - s_1)^+ + \rho\|^2 \geq \|\lambda + \rho\|^2. \tag{3.68}$$

The basic Dirac inequality for a Schmid module  $s$  is

$$\|(\lambda - s)^+ + \rho\|^2 \geq \|\lambda + \rho\|^2. \tag{3.69}$$

This is equivalent to

$$2\langle \gamma | \lambda + \rho \rangle \leq \|\gamma\|^2 \tag{3.70}$$

where  $\gamma$  is defined by  $(\lambda - s_{a,b})^+ = \lambda - \gamma$ .

LEMMA 3.1. *The basic Dirac inequality for  $s = s_1$  is given by*

$$\begin{aligned} \lambda_1 &\leq 0 && \text{for } \lambda = (\lambda_1, 0, \dots, 0) \\ \lambda_1 &\leq 3/2 - n && \text{for } \lambda = (\lambda_1, 1/2, \dots, \pm 1/2) \\ \lambda_1 + \lambda_2 &\leq 2 + p - 2n && \text{for } \lambda = (\lambda_1, \lambda_2, \dots, \lambda_p, \dots, \lambda_n) \\ &&& \text{where } 1 \leq \lambda_2 = \dots = \lambda_p > \lambda_{p+1} \text{ and } 2 \leq p \leq n. \end{aligned} \tag{3.71}$$

*Proof.* Case 1:  $\lambda = (\lambda_1, 0, \dots, 0)$

In this case we have

$$\begin{aligned} (\lambda - s_1)^+ &= (\lambda_1 - 1, -1, 0, \dots, 0)^+ \\ &= (\lambda_1 - 1, 1, 0, \dots, 0) \\ &= \lambda - (\varepsilon_1 - \varepsilon_2) \end{aligned}$$

which shows that  $\gamma = \varepsilon_1 - \varepsilon_2$  and (3.69) reduces to  $\lambda_1 + n - 1 - (n - 2) \leq 1$  which is equivalent to  $\lambda_1 \leq 0$ .

Case 2:  $\lambda = (\lambda_1, 1/2, \dots, \pm 1/2)$

In this case we have

$$\begin{aligned} (\lambda - s_1)^+ &= (\lambda_1 - 1, -1/2, 1/2, \dots, 1/2, \pm 1/2)^+ \\ &= (\lambda_1 - 1, 1/2, 1/2, \dots, 1/2, \mp 1/2) \\ &= \lambda - (\varepsilon_1 \pm \varepsilon_n). \end{aligned}$$

Plugging  $\gamma = \varepsilon_1 \pm \varepsilon_n$  into (3.69) we obtain  $\lambda_1 + n - 1 + 1/2 \leq 1$  which gives

$$\lambda_1 \leq 3/2 - n. \tag{3.72}$$

Case 3:  $1 \leq \lambda_2 = \dots = \lambda_p > \lambda_{p+1}$

The basic Dirac inequality (3.69) for  $s = s_1$  has  $\gamma = \varepsilon_1 + \varepsilon_p$  and

$$\begin{aligned} 2(\lambda_1 + n - 1 + \lambda_p + n - p) &\leq 2 \\ \lambda_1 + \lambda_2 &\leq 2 + p - 2n. \quad \square \end{aligned}$$

We will refer to the first case as to the *scalar* case, the second case is the *spinor* case and the remaining one is the *general* case. In the scalar and spinor case we can actually prove Dirac inequalities directly.

**THEOREM 3.5. (Scalar case)** *Let  $\lambda = (\lambda_1, 0, \dots, 0)$  such that  $\lambda_1 < 2 - n$ . Then the Dirac inequality (3.69) holds for any Schmid module  $s$ .*

*Proof.* The Dirac inequality for the second basic Schmid  $s_2 = 2\epsilon_1$  yields  $\lambda_1 \leq 2 - n$ .

Now we have

$$\begin{aligned} (\lambda - s_{a,b})^+ &= (\lambda_1 - 2b - a, -a, 0, \dots, 0)^+ \\ &= (\lambda_1 - 2b - a, a, 0, \dots, 0) \\ &= \lambda - [(2b + a)\epsilon_1 - a\epsilon_2] \end{aligned}$$

which we will plug into the Dirac inequality (3.69)

$$\begin{aligned} 2[(2b + a)(\lambda_1 + n - 1) - a(n - 2)] &\leq (2b + a)^2 + a^2 \\ (2b + a)(\lambda_1 + n - 1) - a(n - 2) &\leq 2b^2 + 2ab + a^2 \end{aligned}$$

Using  $\lambda_1 \leq 2 - n$  we see that it is sufficient to prove that

$$\begin{aligned} (2b + a)(2 - n + n - 1) - a(n - 2) &\leq 2b^2 + 2ab + a^2 \\ 2b + (3 - n)a &\leq 2b^2 + 2ab + a^2 \\ 0 &\leq 2b(b - 1) + 2ab + a(a + n - 3). \end{aligned}$$

Since  $a, b \geq 0$  and  $n \geq 3$  we are done.  $\square$

**THEOREM 3.6. (Spinor case)** *Let  $\lambda = (\lambda_1, 1/2, \dots, \pm 1/2)$  such that  $\lambda_1 \leq 3/2 - n$ . Then the Dirac inequality (3.69) holds for any Schmid module  $s$ .*

*Proof.* We have

$$\begin{aligned} (\lambda - s_{a,b})^+ &= (\lambda_1 - 2b - a, 1/2 - a, 1/2, \dots, \pm 1/2)^+ \\ &= (\lambda_1 - 2b - a, a - 1/2, 1/2, \dots, \mp 1/2) \\ &= \lambda - [(2b + a)\epsilon_1 - (a - 1)\epsilon_2 \pm \epsilon_n] \end{aligned}$$

for  $a \geq 1$ .

Dirac inequality for the case  $a = 0$  reads

$$\begin{aligned} 2[2b(\lambda_1 + n - 1)] &\leq 4b^2 \\ \lambda_1 + n - 1 &\leq b \\ \lambda_1 &\leq (b + 1) - n \end{aligned}$$

which is satisfied (strictly) whenever  $b \geq 1$  due to (3.71). In the general case we get

$$2[(2b+a)(\lambda_1+n-1) - (a-1)(1/2+n-2) + 1/2] \leq (2b+a)^2 + (a-1)^2 + 1$$

and using (3.71) it is sufficient to show

$$2[(2b+a)(3/2-n+n-1) - (a-1)(1/2+n-2) + 1/2] \leq (2b+a)^2 + (a-1)^2 + 1.$$

This is in turn equivalent to showing nonnegativity of

$$\begin{aligned} a^2 + 2ab + an - 3a + 2b^2 - b - n + 2 &\geq 0 \\ (a+b)^2 + b(b-1) + a(n-3) + 2 - n &\geq 0 \end{aligned}$$

This is clearly decreasing in  $b$  and so we just need to prove that

$$a^2 + (n-3)a - n + 2 \geq 0.$$

The roots of this quadratic polynomial are 1 and  $2-n$  which finishes this case.  $\square$

LEMMA 3.2. Let  $s_{a,b} = (2b+a, a, 0, \dots, 0)$  be a Schmid module with  $a \geq 0$  and  $b \geq 0$ . Let  $\lambda$  satisfy (3.71) in the case  $\lambda_2 \neq 0$  and  $\lambda_1 \leq 2-n$  in the scalar case. Then we have

$$\|(\lambda - s_{a,b+1})^+ + \rho\|^2 - \|\lambda + \rho\|^2 \geq \|(\lambda - s_{a,b})^+ + \rho\|^2 - \|\lambda + \rho\|^2. \tag{3.73}$$

*Proof.* Since the weights differ only in the first coordinate, the difference of the left hand side and the right hand side is just the difference of squares on the first coordinate:

$$\begin{aligned} (\lambda_1 - 2b - 2 - a + n - 1)^2 - (\lambda_1 - 2b - a + n - 1)^2 &\geq 0 \\ -2[2(\lambda_1 - 2b - a + n - 1) - 2] &\geq 0 \\ \lambda_1 - 2b - a + n - 1 - 1 &\leq 0 \\ \lambda_1 &\leq a + 2b + 2 - n \end{aligned}$$

For scalar  $\lambda$  we immediately get  $\lambda_1 \leq 2-n \leq a+2b+2-n$ . In the spinorial case we get  $\lambda_1 \leq 3/2-n \leq a+2b+2-n$ . In the remaining case we actually have  $\lambda_2 \geq 1$  and so (3.71) implies  $\lambda_1 \leq 1+p-2n$  which is indeed less than or equal to  $a+2b-n$ .  $\square$

THEOREM 3.7. (General case) Let  $\lambda$  be as in case 3 and let (3.71) hold. Then the Dirac inequality (3.69) holds for any Schmid module  $s$ .

*Proof.* Thanks to the previous lemma we only have to prove that for  $a \geq 1$  we have

$$\|(\lambda - s_{a,0})^+ + \rho\|^2 - \|\lambda + \rho\|^2 \geq 0,$$

with strict inequality if (3.71) is strict. It follows by induction from combining the following two inequalities

$$\begin{aligned} \|(\lambda - s_{a,0})^+ + \rho\|^2 &\geq \|(\lambda' - s')^+ + \rho\|^2 \\ \|\lambda' + \rho\|^2 &\geq \|\lambda + \rho\|^2 \end{aligned}$$

where

$$\lambda' = (\lambda - s_1)^+ \quad s' = s_{a,0} - s_1 = s_{a-1,0}.$$

The first one immediately follows from the Lemma 2.1 for  $\mu = (\lambda - s_1)^+$  and  $\nu = (s_{a,0} - s_1)^+$  and the second one is the Dirac inequality for  $s_1$  which is nothing but (3.71).

What remains is to check that  $\lambda'$  satisfies the Dirac inequality for  $s' = s_{a-1,0}$ . It can happen that  $\lambda'$  falls into the spinor or scalar case. For  $\lambda = (\lambda_1, 1, 0, \dots, 0)$  the inequality (3.71) takes the form  $\lambda_1 + 1 \leq 2 + 2 - 2n$  which means that  $\lambda'_1 = \lambda_1 - 1$  satisfies  $\lambda'_1 \leq 2 - 2n$ . Looking back at the Theorem 3.5 we see that  $\lambda'$  satisfies the Dirac inequality with any Schmid module. Analogously, for  $\lambda = (\lambda_1, 3/2, 1/2, \dots, \pm 1/2)$  we have  $\lambda_1 + \lambda_2 = \lambda_1 + 3/2 \leq 2 + 2 - 2n$  which gives  $\lambda'_1 = \lambda_1 - 1 \leq 3/2 - 2n$  which is below the unitarizability bound  $3/2 - n$  for the spinor case.

In all other cases  $\lambda'$  is of general type. If  $p > 2$ , then  $\lambda'_1 + \lambda'_2 = \lambda_1 - 1 + \lambda_2$  and using (3.71) and  $p' = p - 1$  we see that this is less than or equal to  $2 + p' - 2n$ . Hence the Dirac inequality is satisfied by the induction hypothesis. For  $p = 2$  we have similarly  $\lambda'_1 + \lambda'_2 = \lambda_1 + \lambda_2 - 2 \leq 2 + 2 - 2n \leq 2 + p' - 2n$  since  $p' \geq 2$ .  $\square$

### 3.5. Dirac inequality for $\mathfrak{so}(2, 2n - 1)$ , $n \geq 2$

The basic Schmid  $\mathfrak{k}$ -submodules of  $S(\mathfrak{p}^-)$  have lowest weights  $-s_1$  or  $-s_2$ , where

$$s_1 = (1, 1, 0, \dots, 0), \quad s_2 = (2, 0, 0, \dots, 0).$$

Moreover, all irreducible  $\mathfrak{k}$ -submodules of  $S(\mathfrak{p}^-)$  have lowest weights  $-s_{a,b}$ , where  $s_{a,b} = (2b + a, a, 0, \dots, 0)$ .

The highest weight  $(\mathfrak{g}, K)$ -modules have highest weights of the form  $\lambda = (\lambda_1, \dots, \lambda_n)$ , where

$$\lambda_2 \geq \lambda_3 \geq \dots \lambda_n \geq 0,$$

$$\lambda_i - \lambda_j \in \mathbb{Z} \text{ and } 2\lambda_i \in \mathbb{N}_0 \text{ for all } 2 \leq i, j \leq n.$$

In this case  $\rho = (n - 1/2, n - 3/2, \dots, 1/2)$ .

The basic necessary condition for unitarity is, as before, the Dirac inequality

$$\|(\lambda - s_1)^+ + \rho\|^2 \geq \|\lambda + \rho\|^2. \tag{3.74}$$

The basic Dirac inequality for a Schmid module is

$$\|(\lambda - s)^+ + \rho\|^2 \geq \|\lambda + \rho\|^2 \tag{3.75}$$

This is equivalent to

$$2\langle \gamma | \lambda + \rho \rangle \leq \|\gamma\|^2 \tag{3.76}$$

where  $\gamma$  is defined by  $(\lambda - s)^+ = \lambda - \gamma$ .

LEMMA 3.3. *The basic Dirac inequality for  $s = s_1$  is given by*

$$\begin{aligned} \lambda_1 &\leq 0 && \text{for } \lambda = (\lambda_1, 0, \dots, 0) \\ \lambda_1 &\leq 1 - n && \text{for } \lambda = (\lambda_1, 1/2, \dots, 1/2) \\ \lambda_1 + \lambda_2 &\leq 1 + p - 2n && \text{for } \lambda = (\lambda_1, \lambda_2, \dots, \lambda_p, \dots, \lambda_n) \end{aligned} \tag{3.77}$$

where  $1 \leq \lambda_2 = \dots = \lambda_p > \lambda_{p+1}$  and  $2 \leq p \leq n$ .

*Proof.* Case 1:  $\lambda = (\lambda_1, 0, \dots, 0)$

In this case we have

$$\begin{aligned}(\lambda - s_1)^+ &= (\lambda_1 - 1, -1, 0, \dots, 0)^+ \\ &= (\lambda_1 - 1, 1, 0, \dots, 0) \\ &= \lambda - (\varepsilon_1 - \varepsilon_2)\end{aligned}$$

which shows that  $\gamma = \varepsilon_1 - \varepsilon_2$  and (3.75) reduces to  $\lambda_1 + n - 1/2 - (n - 3/2) \leq 1$  which is equivalent to  $\lambda_1 \leq 0$ .

Case 2:  $\lambda = (\lambda_1, 1/2, \dots, 1/2)$

In this case we have

$$\begin{aligned}(\lambda - s_1)^+ &= (\lambda_1 - 1, -1/2, 1/2, \dots, 1/2, 1/2)^+ \\ &= (\lambda_1 - 1, 1/2, 1/2, \dots, 1/2, 1/2) \\ &= \lambda - \varepsilon_1\end{aligned}$$

Plugging  $\gamma = \varepsilon_1$  into (3.75) we obtain  $2(\lambda_1 + n - 1/2) \leq 1$  which gives

$$\lambda_1 \leq 1 - n. \quad (3.78)$$

Case 3:  $1 \leq \lambda_2 = \dots = \lambda_p > \lambda_{p+1}$

The basic Dirac inequality (3.75) for  $s = s_1$  has  $\gamma = \varepsilon_1 + \varepsilon_p$  and

$$\begin{aligned}2(\lambda_1 + n - 1/2 + \lambda_p + n - p + 1/2) &\leq 2 \\ \lambda_1 + \lambda_2 &\leq 1 + p - 2n. \quad \square\end{aligned}$$

**THEOREM 3.8. (Scalar case)** *Let  $\lambda = (\lambda_1, 0, \dots, 0)$  such that  $\lambda_1 < 3/2 - n$ . Then (3.75) holds for any Schmid module  $s$ .*

*Proof.* The Dirac inequality for the second basic Schmid yields  $\lambda_1 \leq 3/2 - n$ .

General Schmid module has highest weight  $s_{a,b} = (2b + a, a, 0, \dots, 0)$  and similarly as before we get that the Dirac inequality is equivalent to

$$(2b + a)(\lambda_1 + n + 1/2 - 1) - a(n + 1/2 - 2) \leq 2b^2 + 2ab + a^2.$$

Using  $\lambda_1 \leq 3/2 - n$  we see that it is sufficient to prove that

$$\begin{aligned}(2b + a)(3/2 - n + n + 1/2 - 1) - a(n + 1/2 - 2) &\leq 2b^2 + 2ab + a^2 \\ 2b + a + (3/2 - n)a &\leq 2b^2 + 2ab + a^2 \\ 0 &\leq 2b(b - 1) + 2ab + a(a + n - 5/2).\end{aligned}$$

Since  $a, b \geq 0$  and  $n \geq 2$  we are done.  $\square$

LEMMA 3.4. Let  $s_{a,b} = (2b + a, a, 0, \dots, 0)$  be a Schmid module with  $a \geq 0$  and  $b \geq 0$ . Let  $\lambda$  satisfy (3.77) in the case  $\lambda_2 \neq 0$  and  $\lambda_1 \leq 3/2 - n$  in the scalar case. Then we have

$$\|(\lambda - s_{a,b+1})^+ + \rho\|^2 - \|\lambda + \rho\|^2 \geq \|(\lambda - s_{a,b})^+ + \rho\|^2 - \|\lambda + \rho\|^2. \tag{3.79}$$

*Proof.* Since the weights differ only in the first coordinate, the difference of the left hand side and the right hand side is just the difference of squares on the first coordinate:

$$\begin{aligned} (\lambda_1 - 2b - 2 - a + n - 1/2)^2 - (\lambda_1 - 2b - a + n - 1/2)^2 &\geq 0 \\ -2[2(\lambda_1 - 2b - a + n - 1/2) - 2] &\geq 0 \\ \lambda_1 - 2b - a + n - 1/2 - 1 &\leq 0 \\ \lambda_1 &\leq a + 2b + 3/2 - n \end{aligned}$$

For scalar  $\lambda$  we immediately get  $\lambda_1 \leq 3/2 - n \leq a + 2b + 3/2 - n$ . In the spinorial case we get  $\lambda_1 \leq 1 - n \leq a + 2b + 3/2 - n$ . In the remaining case we actually have  $\lambda_2 \geq 1$  and so (3.77) implies  $\lambda_1 \leq p - 2n$  which is indeed less than or equal to  $a + 2b - n$ .  $\square$

THEOREM 3.9. Let  $\lambda$  be as in case 2 or as in case 3 and let (3.77) holds. Then the Dirac inequality (3.75) holds for any Schmid module  $s$ .

*Proof.* Thanks to the previous lemma we only have to prove that for  $a \geq 1$  we have

$$\|(\lambda - s_{a,0})^+ + \rho\|^2 - \|\lambda + \rho\|^2 \geq 0,$$

with strict inequality if (3.77) is strict. It follows by induction from combining the following two inequalities

$$\begin{aligned} \|(\lambda - s_{a,0})^+ + \rho\|^2 &\geq \|(\lambda' - s')^+ + \rho\|^2 \\ \|\lambda' + \rho\|^2 &\geq \|\lambda + \rho\|^2 \end{aligned}$$

where

$$\lambda' = (\lambda - s_1)^+ \quad s' = s_{a,0} - s_1 = s_{a-1,0}.$$

The first one immediately follows from the Lemma 2.1 for  $\mu = (\lambda - s_1)^+$  and  $\nu = (s_{a,0} - s_1)^+$  and the second one is the Dirac inequality for  $s_1$ .

If  $\lambda$  is in the spinor case, then  $\lambda'$  is also in the spinor case. In the general situation  $\lambda'$  can fall into all three cases. For  $\lambda = (\lambda_1, 3/2, 1/2, \dots, 1/2)$  we have spinorial  $\lambda'$  with  $\lambda'_1 = \lambda_1 - 1$  and since our  $\lambda$  satisfies (3.77) we have  $\lambda_1 \leq 3/2 - 2n$  which means that  $\lambda'_1 \leq 1/2 - 2n$ . For  $\lambda'$  of general type the exactly same reasoning as in the even case ( $\mathfrak{so}(2, 2n - 2)$ ) finishes the proof.  $\square$

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*Pavle Pandžić*

*Department of Mathematics  
Faculty of Science, University of Zagreb  
Bijenička 30, 10000 Zagreb, Croatia  
e-mail: pandzic@math.hr*

*Ana Prlić*

*Department of Mathematics  
Faculty of Science, University of Zagreb  
Bijenička 30, 10000 Zagreb, Croatia  
e-mail: anaprlic@math.hr*

*Vladimír Souček, Matematický ústav UK  
Sokolovská 83, 186 75 Praha 8, Czech Republic  
e-mail: soucek@karlin.mff.cuni.cz*

*Vít Tuček*

*Department of Mathematics  
Faculty of Science, University of Zagreb  
Bijenička 30, 10000 Zagreb, Croatia  
e-mail: vit.tucek@gmail.com*