

## IMPROVEMENTS OF SOME FUNCTIONAL JENSEN TYPE INEQUALITIES VIA SUPERQUADRACITY

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*Abstract.* In this paper we obtain improvements of some functional inequalities of Jensen and Hölder type. For this purpose we use superquadratic functions and positive linear functionals.

### 1. Introduction

Throughout in mathematics the inequalities by Jensen and Hölder are included in the most familiar and useful inequalities. Most of the classical inequalities can be produced from these two inequalities. There are so many generalizations and versions of these inequalities available in the literature. In case of positive linear functionals Jensen's inequality is known as Jessen's inequality. For superquadratic functions, Jessen's and Hölder's inequalities are generalized by Banić et al., in [3]. Throughout in this paper we use  $A$  for a positive linear functional, defined in the following way, see [6].

**DEFINITION 1.** Suppose that  $L$  is a linear class of real-valued functions defined on a nonempty set  $E$ , i.e.,  $L$  satisfied the following properties:

- (1) If  $f, g \in L$  and  $a, b \in \mathbb{R}$ , then  $(af + bg) \in L$ .
- (2)  $\mathbf{1} \in L$ , where  $\mathbf{1}(t) = 1$  for all  $t \in E$ .

A positive linear functional  $A : L \rightarrow \mathbb{R}$  is a functional having the following properties:

- (1) If  $f, g \in L$  and  $a, b \in \mathbb{R}$ , then  $A(af + bg) = aA(f) + bA(g)$ .
- (2) If  $f \in L$  and  $f(t) \geq 0$  for all  $t \in E$ , then  $A(f) \geq 0$ .

Additionally if the condition  $A(\mathbf{1}) = 1$  is satisfied, we say that  $A$  is a positive normalized linear functional.

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REMARK 1. Sums and Lebesgue integrals are the most familiar examples of positive linear functionals. In [2] it is shown that time scales integrals including the Cauchy, Riemann, Lebesgue, multiple Riemann, multiple Lebesgue delta, nabla and diamond- $\alpha$  time scales integrals also satisfy the properties of positive linear functionals.

Superquadratic functions are defined in the following way, see [1].

DEFINITION 2. A function  $\alpha : [0, \infty) \rightarrow \mathbb{R}$  is called superquadratic if there exists a function  $C : [0, \infty) \rightarrow \mathbb{R}$  such that

$$\alpha(y) - \alpha(x) - \alpha(|y - x|) \geq C(x)(y - x) \quad \text{for all } x, y \geq 0.$$

We say that  $\alpha$  is subquadratic if  $-\alpha$  is superquadratic.

A superquadratic function may or may not be convex but if it is a nonnegative function then it is a convex function. As an example, the function  $\alpha(x) = x^p$  is superquadratic for  $p \geq 2$  and subquadratic for  $p \in (0, 2]$ .

Some properties of a superquadratic function are collected in the following lemma form [1].

LEMMA 1. Let  $\alpha$  be a superquadratic function with  $C(x)$  as in Definition 2. Then

- (i)  $\alpha(0) \leq 0$ ;
- (ii) if  $\alpha(0) = \alpha'(0) = 0$ , then  $C(x) = \alpha'(x)$  whenever  $\alpha$  is differentiable at  $x > 0$ ;
- (iii) if  $\alpha \geq 0$ , then  $\alpha$  is convex and  $\alpha(0) = \alpha'(0) = 0$ .

Jessen’s inequality for superquadratic functions is investigated in [3] and the following result is given there.

THEOREM 1. Suppose that  $L$  and  $A$  are defined as in Definition 1,  $\alpha : [0, \infty) \rightarrow \mathbb{R}$  is a superquadratic function,  $w, f \in L$  are non-negative functions,  $A(w) > 0$ ,  $wf$ ,  $w\alpha(f)$ ,  $w\alpha\left(\left|f - \frac{A(wf)}{A(w)} \cdot \mathbf{1}\right|\right) \in L$ . Then we get

$$\alpha\left(\frac{A(wf)}{A(w)}\right) \leq \frac{A(w\alpha(f))}{A(w)} - \frac{A\left(w\alpha\left(\left|f - \frac{A(wf)}{A(w)} \cdot \mathbf{1}\right|\right)\right)}{A(w)}.$$

If  $\alpha : [0, \infty) \rightarrow \mathbb{R}$  is a superquadratic function,  $x_1, x_2$  are two non-negative real numbers and  $\lambda \in [0, 1]$ , then we obtain the following discrete form of Jensen’s inequality, which is used in our main results:

$$\alpha(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda \alpha(x_1) + (1 - \lambda)\alpha(x_2) - \lambda \alpha((1 - \lambda)|x_1 - x_2|) - (1 - \lambda)\alpha(\lambda|x_1 - x_2|). \quad (1)$$

The next two theorems, as a refined version of Jensen’s inequality for superquadratic functions, are given by Nikolova et al., in [5].

**THEOREM 2.** Let  $\theta, \Phi, w, f$  are non-negative measurable functions such that  $\theta(x) + \Phi(x) = 1$  for all  $x \in \Lambda$  and  $\int_{\Lambda} w(x) dx > 0$ . If  $\alpha : [0, \infty) \rightarrow \mathbb{R}$  is a superquadratic function, then we get

$$\begin{aligned} & \alpha \left( \frac{\int_{\Lambda} w(x)f(x) dx}{\int_{\Lambda} w(x) dx} \right) \\ & \leq \frac{\int_{\Lambda} \theta(x)w(x) dx}{\int_{\Lambda} w(x) dx} \alpha \left( \frac{\int_{\Lambda} \theta(x)w(x)f(x) dx}{\int_{\Lambda} \theta(x)w(x) dx} \right) \\ & \quad + \frac{\int_{\Lambda} \Phi(x)w(x) dx}{\int_{\Lambda} w(x) dx} \alpha \left( \frac{\int_{\Lambda} \Phi(x)w(x)f(x) dx}{\int_{\Lambda} \Phi(x)w(x) dx} \right) \\ & \quad - \frac{\int_{\Lambda} \theta(x)w(x) dx}{\int_{\Lambda} w(x) dx} \alpha \left( \left| \frac{\int_{\Lambda} \theta(x)w(x)f(x) dx}{\int_{\Lambda} \theta(x)w(x) dx} - \frac{\int_{\Lambda} w(x)f(x) dx}{\int_{\Lambda} w(x) dx} \right| \right) \\ & \quad - \frac{\int_{\Lambda} \Phi(x)w(x) dx}{\int_{\Lambda} w(x) dx} \alpha \left( \left| \frac{\int_{\Lambda} \Phi(x)w(x)f(x) dx}{\int_{\Lambda} \Phi(x)w(x) dx} - \frac{\int_{\Lambda} w(x)f(x) dx}{\int_{\Lambda} w(x) dx} \right| \right). \end{aligned} \quad (2)$$

Moreover, (2) holds in reversed direction if  $\alpha$  is subquadratic.

**THEOREM 3.** Let all the conditions of Theorem 2 be satisfied. Additionally if  $\alpha$  is non-decreasing such that  $\alpha(a+b) \leq c(\alpha(a) + \alpha(b))$  for some  $c > 0$ , then we get

$$\begin{aligned} & \alpha \left( \frac{\int_{\Lambda} w(x)f(x) dx}{\int_{\Lambda} w(x) dx} \right) \\ & \leq \frac{\int_{\Lambda} \theta(x)w(x) dx}{\int_{\Lambda} w(x) dx} \alpha \left( \frac{\int_{\Lambda} \theta(x)w(x)f(x) dx}{\int_{\Lambda} \theta(x)w(x) dx} \right) \\ & \quad + \frac{\int_{\Lambda} \Phi(x)w(x) dx}{\int_{\Lambda} w(x) dx} \alpha \left( \frac{\int_{\Lambda} \Phi(x)w(x)f(x) dx}{\int_{\Lambda} \Phi(x)w(x) dx} \right) \\ & \quad - \frac{\int_{\Lambda} \theta(x)w(x) dx}{\int_{\Lambda} w(x) dx} \alpha \left( \left| \frac{\int_{\Lambda} \theta(x)w(x)f(x) dx}{\int_{\Lambda} \theta(x)w(x) dx} - \frac{\int_{\Lambda} w(x)f(x) dx}{\int_{\Lambda} w(x) dx} \right| \right) \\ & \quad - \frac{\int_{\Lambda} \Phi(x)w(x) dx}{\int_{\Lambda} w(x) dx} \alpha \left( \left| \frac{\int_{\Lambda} \Phi(x)w(x)f(x) dx}{\int_{\Lambda} \Phi(x)w(x) dx} - \frac{\int_{\Lambda} w(x)f(x) dx}{\int_{\Lambda} w(x) dx} \right| \right) \\ & \leq \frac{\int_{\Lambda} w(x)\alpha(f(x)) dx}{\int_{\Lambda} w(x) dx} - \frac{\int_{\Lambda} w(x)\alpha(x) \left( \left| f(x) - \frac{\int_{\Lambda} w(x)f(x) dx}{\int_{\Lambda} w(x) dx} \right| \right) dx}{c \int_{\Lambda} w(x) dx}. \end{aligned}$$

An improvement of Hölder's inequality using superquadratic functions is given in [3]. In the following, we give a slightly modified theorem using weight functions.

**THEOREM 4.** Suppose that  $L$  and  $A$  are defined as in Definition 1,  $s \geq 2$  and  $t$  is defined by  $\frac{1}{s} + \frac{1}{t} = 1$ . If  $p, g, h \in L$ , are non-negative functions,  $A(ph^t) > 0$ , and  $pgh, pg^s, ph^t, p \left| g - \frac{A(pgh)}{A(ph^t)} h^{t-1} \right|^s \in L$ , then

$$A(pgh) \leq \left( A(pg^s) - A \left( p \left| g - \frac{A(pgh)}{A(ph^t)} h^{t-1} \right|^s \right) \right)^{\frac{1}{s}} A^{\frac{1}{t}} (ph^t). \quad (3)$$

A new improvement of Hölder’s inequality for sums and integrals is given by İşcan, see [4]:

**THEOREM 5.** For  $k \in \{1, \dots, n\}$ , let  $g_k, h_k > 0$ . If  $s \geq 1$  and  $t$  is defined such that  $\frac{1}{s} + \frac{1}{t} = 1$ , then we have

$$\sum_{k=1}^n g_k h_k \leq \frac{1}{n} \left\{ \left( \sum_{k=1}^n k g_k^s \right)^{1/s} \left( \sum_{k=1}^n k h_k^t \right)^{1/t} + \left( \sum_{k=1}^n (n-k) g_k^s \right)^{1/s} \left( \sum_{k=1}^n (n-k) h_k^t \right)^{1/t} \right\}. \tag{4}$$

**THEOREM 6.** Let  $s \geq 1$  and  $t$  be defined by  $\frac{1}{s} + \frac{1}{t} = 1$ . If  $g, h : [a, b] \rightarrow \mathbb{R}$  are such that  $gh, g^s$ , and  $h^t$  are integrable functions on  $[a, b]$ , then we have

$$\int_a^b g(x)h(x)dx \leq \frac{1}{b-a} \left\{ \left( \int_a^b (b-x)g^s(x)dx \right)^{1/s} \left( \int_a^b (b-x)h^t(x)dx \right)^{1/t} + \left( \int_a^b (x-a)g^s(x)dx \right)^{1/s} \left( \int_a^b (x-a)h^t(x)dx \right)^{1/t} \right\}. \tag{5}$$

### 2. Refinement of Jensen type inequality

In the following, our first result gives the generalization of Theorem 2.

**THEOREM 7.** Suppose that  $L$  and  $A$  are defined as in Definition 1. If  $\alpha : [0, \infty) \rightarrow \mathbb{R}$  is a superquadratic function,  $\theta, \Phi, w, f \in L$  are non-negative functions,  $\theta(x) + \Phi(x) = 1$  for all  $x \in E$ ,  $A(w), A(\theta w), A(\Phi w) > 0$ , and  $wf, \theta w, \theta wf, \Phi w, \Phi wf, \in L$ , then we get

$$\begin{aligned} \alpha \left( \frac{A(wf)}{A(w)} \right) &\leq \frac{A(\theta w)}{A(w)} \alpha \left( \frac{A(\theta wf)}{A(\theta w)} \right) + \frac{A(\Phi w)}{A(w)} \alpha \left( \frac{A(\Phi wf)}{A(\Phi w)} \right) \\ &\quad - \frac{A(\theta w)}{A(w)} \alpha \left( \left| \frac{A(\theta wf)}{A(\theta w)} - \frac{A(wf)}{A(w)} \right| \right) \\ &\quad - \frac{A(\Phi w)}{A(w)} \alpha \left( \left| \frac{A(\Phi wf)}{A(\Phi w)} - \frac{A(wf)}{A(w)} \right| \right). \end{aligned} \tag{6}$$

*Proof.* Set  $x_1 = \frac{A(\theta wf)}{A(\theta w)}$ ,  $x_2 = \frac{A(\Phi wf)}{A(\Phi w)}$ , and  $\lambda = \frac{A(\theta w)}{A(w)}$ . Then  $1 - \lambda = \frac{A(\Phi w)}{A(w)}$  and  $\lambda x_1 + (1 - \lambda)x_2 = \frac{A(wf)}{A(w)}$ . From (1) it follows that

$$\begin{aligned} \alpha \left( \frac{A(wf)}{A(w)} \right) &\leq \frac{A(\theta w)}{A(w)} \alpha \left( \frac{A(\theta wf)}{A(\theta w)} \right) + \frac{A(\Phi w)}{A(w)} \alpha \left( \frac{A(\Phi wf)}{A(\Phi w)} \right) \\ &\quad - \frac{A(\theta w)}{A(w)} \alpha \left( \frac{A(\Phi w)}{A(w)} \left| \frac{A(\theta wf)}{A(\theta w)} - \frac{A(\Phi wf)}{A(\Phi w)} \right| \right) \\ &\quad - \frac{A(\Phi w)}{A(w)} \alpha \left( \frac{A(\theta w)}{A(w)} \left| \frac{A(\theta wf)}{A(\theta w)} - \frac{A(\Phi wf)}{A(\Phi w)} \right| \right). \end{aligned} \tag{7}$$

Moreover

$$\frac{A(\theta w)}{A(w)} \left| \frac{A(\theta wf)}{A(\theta w)} - \frac{A(\Phi wf)}{A(\Phi w)} \right| = \left| \frac{A(\Phi wf)}{A(\Phi w)} - \frac{A(wf)}{A(w)} \right| \quad (8)$$

and

$$\frac{A(\Phi w)}{A(w)} \left| \frac{A(\theta wf)}{A(\theta w)} - \frac{A(\Phi wf)}{A(\Phi w)} \right| = \left| \frac{A(\theta wf)}{A(\theta w)} - \frac{A(wf)}{A(w)} \right|. \quad (9)$$

Hence the inequality (6) follows from the inequalities (7), (8) and (9).  $\square$

REMARK 2. In Theorem 7 assume that  $L$  is a class of integrable functions defined on a nonempty interval  $[a, b]$  and  $A(f) = \int_a^b f(x)dx$ . Then inequality (2) follows from the inequality (6).

Next theorem is a generalization of Theorem 3.

THEOREM 8. *Let all the conditions of Theorem 7 be satisfied. Additionally if  $\alpha$  is non-decreasing such that*

$$\alpha(a+b) \leq c(\alpha(a) + \alpha(b)) \quad (10)$$

for some  $c > 0$ , then we get

$$\alpha \left( \frac{A(wf)}{A(w)} \right) \leq Q \leq \frac{A(w\alpha(f))}{A(w)} - \frac{A \left( w\alpha \left( \left| f - \frac{A(wf)}{A(w)} \cdot \mathbf{1} \right| \right) \right)}{cA(w)}, \quad (11)$$

where

$$\begin{aligned} Q &= \frac{A(\theta w)}{A(w)} \alpha \left( \frac{A(\theta wf)}{A(\theta w)} \right) + \frac{A(\Phi w)}{A(w)} \alpha \left( \frac{A(\Phi wf)}{A(\Phi w)} \right) \\ &\quad - \frac{A(\theta w)}{A(w)} \alpha \left( \left| \frac{A(\theta wf)}{A(\theta w)} - \frac{A(wf)}{A(w)} \right| \right) \\ &\quad - \frac{A(\Phi w)}{A(w)} \alpha \left( \left| \frac{A(\Phi wf)}{A(\Phi w)} - \frac{A(wf)}{A(w)} \right| \right). \end{aligned} \quad (12)$$

*Proof.* The first inequality in (11) is proved in Theorem 7. Here we prove the second inequality in (11). By using Theorem 1, we get

$$\alpha \left( \frac{A(\theta wf)}{A(\theta w)} \right) \leq \frac{A(\theta w\alpha(f))}{A(\theta w)} - \frac{A \left( \theta w\alpha \left( \left| f - \frac{A(\theta wf)}{A(\theta w)} \cdot \mathbf{1} \right| \right) \right)}{A(\theta w)} \quad (13)$$

and

$$\alpha \left( \frac{A(\Phi wf)}{A(\Phi w)} \right) \leq \frac{A(\Phi w\alpha(f))}{A(\Phi w)} - \frac{A \left( \Phi w\alpha \left( \left| f - \frac{A(\Phi wf)}{A(\Phi w)} \cdot \mathbf{1} \right| \right) \right)}{A(\Phi w)}. \quad (14)$$

By combining (13), (14) and (12), we get

$$Q \leq \frac{A(w\alpha(f))}{A(w)} - (Q_1 + Q_2), \quad (15)$$

where

$$Q_1 = \frac{A(\theta w)}{A(w)} \alpha \left( \left| \frac{A(\theta wf)}{A(\theta w)} - \frac{A(wf)}{A(w)} \right| \right) + \frac{A(\Phi w)}{A(w)} \alpha \left( \left| \frac{A(\Phi wf)}{A(\Phi w)} - \frac{A(wf)}{A(w)} \right| \right)$$

and

$$Q_2 = \frac{1}{A(w)} A \left( \theta w \alpha \left( \left| f - \frac{A(\theta wf)}{A(\theta w)} \cdot \mathbf{1} \right| \right) \right) + \frac{1}{A(w)} A \left( \Phi w \alpha \left( \left| f - \frac{A(wf)}{A(\Phi w)} \cdot \mathbf{1} \right| \right) \right).$$

Now, by using the triangle inequality, the nondecreasing property of  $\alpha$ , and (10), we get

$$\begin{aligned} & \frac{1}{A(w)} A \left( \theta w \alpha \left( \left| f - \frac{A(wf)}{A(w)} \cdot \mathbf{1} \right| \right) \right) \\ & \leq c \left( \frac{1}{A(w)} A \left( \theta w \alpha \left( \left| f - \frac{A(\theta wf)}{A(\theta w)} \cdot \mathbf{1} \right| \right) \right) + \frac{A(\theta w)}{A(w)} \alpha \left( \left| \frac{A(\theta wf)}{A(\theta w)} - \frac{A(wf)}{A(w)} \right| \right) \right) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{A(w)} A \left( \Phi w \alpha \left( \left| f - \frac{A(wf)}{A(w)} \cdot \mathbf{1} \right| \right) \right) \\ & \leq c \left( \frac{1}{A(w)} A \left( \Phi w \alpha \left( \left| f - \frac{A(\Phi wf)}{A(\Phi w)} \cdot \mathbf{1} \right| \right) \right) + \frac{A(\Phi w)}{A(w)} \alpha \left( \left| \frac{A(\Phi wf)}{A(\Phi w)} - \frac{A(wf)}{A(w)} \right| \right) \right). \end{aligned}$$

By adding the above two inequalities, we get

$$\frac{1}{A(w)} A \left( w \alpha \left( \left| f - \frac{A(wf)}{A(w)} \cdot \mathbf{1} \right| \right) \right) \leq Q_1 + Q_2. \tag{16}$$

Hence inequalities in (11) follows from (15) and (16).  $\square$

**COROLLARY 1.** For  $k \in \{1, \dots, n\}$ , let  $w_k, f_k > 0$ . If  $\alpha : [0, \infty) \rightarrow \mathbb{R}$  is a superquadratic function,  $\sum_{k=1}^n w_k > 0$ , and

$$\alpha(a + b) \leq c(\alpha(a) + \alpha(b))$$

for some  $c > 0$ , then we get

$$\alpha \left( \frac{\sum_{k=1}^n w_k f_k}{\sum_{k=1}^n w_k} \right) \leq Q_3 \leq \frac{\sum_{k=1}^n w_k \alpha(f_k)}{\sum_{k=1}^n w_k} - \frac{\sum_{k=1}^n w_k \alpha \left( \left| f_k - \frac{\sum_{k=1}^n w_k f_k}{\sum_{k=1}^n w_k} \right| \right)}{c \sum_{k=1}^n w_k},$$

where

$$Q_3 = \frac{1}{n \sum_{k=1}^n w_k} \left[ \sum_{k=1}^n k w_k \alpha \left( \frac{\sum_{k=1}^n k w_k f_k}{\sum_{k=1}^n k w_k} \right) + \sum_{k=1}^n (n-k) w_k \alpha \left( \frac{\sum_{k=1}^n (n-k) w_k f_k}{\sum_{k=1}^n (n-k) w_k} \right) \right. \\ \left. - \sum_{k=1}^n k w_k \alpha \left( \left| \frac{\sum_{k=1}^n k w_k f_k}{\sum_{k=1}^n k w_k} - \frac{\sum_{k=1}^n w_k f_k}{\sum_{k=1}^n w_k} \right| \right) \right. \\ \left. - \sum_{k=1}^n (n-k) w_k \alpha \left( \left| \frac{\sum_{k=1}^n (n-k) w_k f_k}{\sum_{k=1}^n (n-k) w_k} - \frac{\sum_{k=1}^n w_k f_k}{\sum_{k=1}^n w_k} \right| \right) \right].$$

*Proof.* The result follows from Theorem 8 by taking  $E = \{1, \dots, n\}$ ,  $\theta(k) = \frac{k}{n}$ ,  $\Phi(k) = \frac{n-k}{n}$ ,  $w(k) = w_k$ , and  $f(k) = f_k$ .  $\square$

### 3. Refinements of Hölder's inequality

In this section we obtain some improvements of Hölder's inequality for positive linear functionals. Our results are new in case of sums and integrals as well.

**THEOREM 9.** *Suppose that  $L$  and  $A$  are defined as in Definition 1,  $s \geq 2$  and  $t$  is defined by  $\frac{1}{s} + \frac{1}{t} = 1$ . If  $\theta, \Phi, p, g, h \in L$  are non-negative functions,  $\theta(x) + \Phi(x) = 1$  for all  $x \in E$ ,  $A(ph^t) > 0$ ,  $pgh$ ,  $\theta p g^s$ ,  $\Phi p g^s$ ,  $\theta ph^t$ ,  $\Phi ph^t$ ,  $\theta p \left| g - h^{t-1} \frac{A(\theta p g h)}{A(\theta p h^t)} \right|^s$ ,  $\Phi p \left| g - \frac{A(\Phi p g h)}{A(\Phi p h^t)} h^{t-1} \right|^s \in L$ , then we get*

$$A(pgh) \leq \left( A(\theta p g^s) - A \left( \theta p \left| g - \frac{A(\theta p g h)}{A(\theta p h^t)} h^{t-1} \right|^s \right) \right)^{\frac{1}{s}} A^{\frac{1}{t}}(\theta p h^t) \\ + \left( A(\Phi p g^s) - A \left( \Phi p \left| g - \frac{A(\Phi p g h)}{A(\Phi p h^t)} h^{t-1} \right|^s \right) \right)^{\frac{1}{s}} A^{\frac{1}{t}}(\Phi p h^t). \quad (17)$$

*Proof.* By using Hölder's inequality (3), we obtain

$$A(pgh) = A(\theta p g h + \Phi p g h) = A(\theta p g h) + A(\Phi p g h) \\ \leq \left( A(\theta p g^s) - A \left( \theta p \left| g - \frac{A(\theta p g h)}{A(\theta p h^t)} h^{t-1} \right|^s \right) \right)^{\frac{1}{s}} A^{\frac{1}{t}}(\theta p h^t) \\ + \left( A(\Phi p g^s) - A \left( \Phi p \left| g - \frac{A(\Phi p g h)}{A(\Phi p h^t)} h^{t-1} \right|^s \right) \right)^{\frac{1}{s}} A^{\frac{1}{t}}(\Phi p h^t),$$

which is the required inequality.  $\square$

REMARK 3. If  $E = \{1, \dots, n\}$ ,  $\theta(k) = \frac{k}{n}$ ,  $\Phi(k) = \frac{n-k}{n}$ ,  $p(k) = 1$  for all  $k \in E$ ,  $g(k) = g_k$ ,  $h(k) = h_k$  and  $A(g) = \sum_{k=1}^n g_k$ , then the inequality (4) follows from (17).

Similarly, if we take  $L$  a class of integrable functions defined on a nonempty interval  $[a, b]$ ,  $A(g) = \int_a^b g(x)dx$ ,  $p(x) = 1$  for all  $x \in E = [a, b]$ ,  $\theta(x) = \frac{b-x}{b-a}$ , and  $\Phi(x) = \frac{x-a}{b-a}$ , then the inequality (5) follows from (17).

THEOREM 10. *Let all the conditions of Theorem 9 be satisfied. If*

$$\left| g - \frac{A(pgh)}{A(ph^t)} h^{t-1} \right|^s \leq \theta \left| g - \frac{A(\theta pgh)}{A(\theta ph^t)} h^{t-1} \right|^s + \Phi \left| g - \frac{A(\Phi pgh)}{A(\Phi ph^t)} h^{t-1} \right|^s, \quad (18)$$

then we get

$$\begin{aligned} A(pgh) &\leq \left( A(\theta p g^s) - A \left( \theta p \left| g - \frac{A(\theta pgh)}{A(\theta ph^t)} h^{t-1} \right|^s \right) \right)^{\frac{1}{s}} A^{\frac{1}{t}}(\theta ph^t) \\ &\quad + \left( A(\Phi p g^s) - A \left( \Phi p \left| g - \frac{A(\Phi pgh)}{A(\Phi ph^t)} h^{t-1} \right|^s \right) \right)^{\frac{1}{s}} A^{\frac{1}{t}}(\Phi ph^t) \\ &\leq \left( A(p g^s) - A \left( p \left| g - \frac{A(pgh)}{A(ph^t)} h^{t-1} \right|^s \right) \right)^{\frac{1}{s}} A^{\frac{1}{t}}(ph^t). \end{aligned} \quad (19)$$

*Proof.* First inequality in (19) is proved in Theorem 9 and the second one follows from the inequality (18) and the discrete Hölder's inequality:

$$u_1 v_1 + u_2 v_2 \leq (u_1^s + u_2^s)^{\frac{1}{s}} (v_1^t + v_2^t)^{\frac{1}{t}}, \quad u_1, u_2, v_1, v_2 \geq 0,$$

by substituting

$$\begin{aligned} u_1 &= \left( A(\theta p g^s) - A \left( \theta p \left| g - \frac{A(\theta pgh)}{A(\theta ph^t)} h^{t-1} \right|^s \right) \right)^{\frac{1}{s}}, \\ u_2 &= \left( A(\Phi p g^s) - A \left( \Phi p \left| g - \frac{A(\Phi pgh)}{A(\Phi ph^t)} h^{t-1} \right|^s \right) \right)^{\frac{1}{s}}, \\ v_1 &= A^{\frac{1}{t}}(\theta ph^t), \quad v_2 = A^{\frac{1}{t}}(\Phi ph^t). \quad \square \end{aligned}$$

The discrete and continuous version of Theorem 10 gives the following refinements of Hölder's inequality for sums and integrals.

COROLLARY 2. For  $k \in \{1, \dots, n\}$ , let  $p_k, g_k, h_k > 0$ ,  $s \geq 2$  and  $t$  be defined by  $\frac{1}{s} + \frac{1}{t} = 1$ . If

$$\left| g_k - \frac{\sum_{k=1}^n p_k g_k h_k}{\sum_{k=1}^n p_k h_k^t} h_k^{t-1} \right|^s \leq \frac{1}{n} \left[ k \left| g_k - \frac{\sum_{k=1}^n k p_k g_k h_k}{\sum_{k=1}^n k p_k h_k^t} h_k^{t-1} \right|^s + (n-k) \left| g_k - \frac{\sum_{k=1}^n (n-k) p_k g_k h_k}{\sum_{k=1}^n (n-k) p_k h_k^t} h_k^{t-1} \right|^s \right],$$

then we get

$$\sum_{k=1}^n p_k g_k h_k \leq R_1 \leq \left\{ \sum_{k=1}^n p_k g_k^s - \sum_{k=1}^n p_k \left| g_k - \frac{\sum_{k=1}^n p_k g_k h_k}{\sum_{k=1}^n p_k h_k^t} h_k^{t-1} \right|^s \right\}^{1/s} \left( \sum_{k=1}^n p_k h_k^t \right)^{1/t},$$

where

$$\begin{aligned} R_1 &= \frac{1}{n} \left\{ \sum_{k=1}^n k p_k g_k^s - \sum_{k=1}^n k p_k \left| g_k - \frac{\sum_{k=1}^n k p_k g_k h_k}{\sum_{k=1}^n k p_k h_k^t} h_k^{t-1} \right|^s \right\}^{1/s} \left( \sum_{k=1}^n k p_k h_k^t \right)^{1/t} \\ &+ \frac{1}{n} \left\{ \sum_{k=1}^n (n-k) p_k g_k^s - \sum_{k=1}^n (n-k) p_k \left| g_k - \frac{\sum_{k=1}^n (n-k) p_k g_k h_k}{\sum_{k=1}^n (n-k) p_k h_k^t} h_k^{t-1} \right|^s \right\}^{1/s} \\ &\times \left( \sum_{k=1}^n (n-k) p_k h_k^t \right)^{1/t}. \end{aligned}$$

COROLLARY 3. Let  $s \geq 2$  and  $t$  be defined by  $\frac{1}{s} + \frac{1}{t} = 1$  and  $p, g, h : [a, b] \rightarrow \mathbb{R}$  are such that  $pgh$ ,  $pg^s$ , and  $ph^t$  are integrable functions on  $[a, b]$ . If

$$\begin{aligned} &\left| g(x) - \frac{\int_a^b p(x)g(x)h(x)dx}{\int_a^b p(x)h^t(x)dx} h^{t-1}(x) \right|^s \\ &\leq \frac{1}{b-a} \left\{ (b-x) \left| g(x) - \frac{\int_a^b (b-x)p(x)g(x)h(x)dx}{\int_a^b (b-x)p(x)h^t(x)dx} h^{t-1}(x) \right|^s \right. \\ &\quad \left. + (x-a) \left| g(x) - \frac{\int_a^b (x-a)p(x)g(x)h(x)dx}{\int_a^b (x-a)p(x)h^t(x)dx} h^{t-1}(x) \right|^s \right\}, \end{aligned}$$

then we get

$$\begin{aligned} & \int_a^b p(x)g(x)h(x)dx \leq R_2 + R_3 \\ & \leq \left( \int_a^b p(x)g^s(x)dx - \int_a^b p \left| g(x) - \frac{\int_a^b p(x)g(x)h(x)dx}{\int_a^b p(x)h^t(x)dx} h^{t-1}(x) \right|^s dx \right)^{\frac{1}{s}} \\ & \quad \times \left( \int_a^b p(x)h^t(x)dx \right)^{\frac{1}{t}}, \end{aligned}$$

where

$$\begin{aligned} R_2 &= \frac{1}{b-a} \left\{ \int_a^b (b-x)p(x)g^s(x)dx \right. \\ & \quad \left. - \int_a^b (b-x)p(x) \left| g(x) - \frac{\int_a^b (b-x)p(x)g(x)h(x)dx}{\int_a^b (b-x)p(x)h^t(x)dx} h^{t-1}(x) \right|^s dx \right\}^{1/s} \\ & \quad \times \left( \sum_{k=1}^n (b-x)p(x)h^t(x) \right)^{1/t} \end{aligned}$$

and

$$\begin{aligned} R_3 &= \frac{1}{b-a} \left\{ \int_a^b (x-a)p(x)g^s(x)dx \right. \\ & \quad \left. - \int_a^b (x-a)p(x) \left| g(x) - \frac{\int_a^b (x-a)p(x)g(x)h(x)dx}{\int_a^b (x-a)p(x)h^t(x)dx} h^{t-1}(x) \right|^s dx \right\}^{1/s} \\ & \quad \times \left( \sum_{k=1}^n (x-a)p(x)h^t(x) \right)^{1/t}. \end{aligned}$$

**THEOREM 11.** *Let all the conditions of Theorem 9 be satisfied. Additionally if*

$$(a + b)^s \leq c(a^s + b^s)$$

*holds for some  $c > 0$ , then we get*

$$A(pgh) \leq R_4 \leq \left[ A(pg^s) - \frac{1}{c} A \left( p \left| g - \frac{A(pgh)}{A(ph^t)} h^{t-1} \right|^s \right) \right]^{1/s} A^{1/t}(ph^t), \tag{20}$$

where

$$\begin{aligned} R_4 &= \left[ \frac{A^s(\Theta pgh)}{A^{(s-1)}(\Theta ph^t)} + \frac{A^s(\Phi pgh)}{A^{(s-1)}(\Phi ph^t)} - A(\Theta ph^t) \left| \frac{A(\Theta pgh)}{A(\Theta ph^t)} - \frac{A(pgh)}{A(ph^t)} \right|^s \right. \\ & \quad \left. - A(\Phi ph^t) \left| \frac{A(\Phi pgh)}{A(\Phi ph^t)} - \frac{A(pgh)}{A(ph^t)} \right|^s \right]^{1/s} A^{1/t}(ph^t). \end{aligned}$$

*Proof.* Let  $\alpha(x) = x^s$ ,  $w(x) = p(x)h^t(x)$  and  $f(x) = g(x)h^{-t/s}(x)$ . Then the inequality (11) becomes

$$\begin{aligned} & \left( \frac{A(pgh)}{A(ph^t)} \right)^s \\ & \leq \frac{1}{A(ph^t)} \frac{A^s(\theta pgh)}{A^{(s-1)}(\theta ph^t)} + \frac{1}{A(ph^t)} \frac{A^s(\Phi pgh)}{A^{(s-1)}(\Phi ph^t)} \\ & \quad - \frac{A(\theta ph^t)}{A(ph^t)} \left| \frac{A(\theta pgh)}{A(\theta ph^t)} - \frac{A(pgh)}{A(ph^t)} \right|^s - \frac{A(\Phi ph^t)}{A(ph^t)} \left| \frac{A(\Phi pgh)}{A(\Phi ph^t)} - \frac{A(pgh)}{A(ph^t)} \right|^s \\ & \leq \frac{A(pg^s)}{A(ph^t)} - \frac{A\left(p \left| g - \frac{A(pgh)}{A(ph^t)} h^{t-1} \right|^s\right)}{cA(ph^t)}. \end{aligned}$$

Now multiplying both sides with  $A^s(ph^t)$  and then taking the power  $1/s$  on both sides, we obtain the inequalities in (20).  $\square$

Using Theorem 11, we obtain the following refinements of Hölder’s inequality for sums and integrals.

**COROLLARY 4.** For  $k \in \{1, \dots, n\}$ , let  $p_k, g_k, h_k > 0$ . If  $s \geq 2$  and  $t$  is defined such that  $\frac{1}{s} + \frac{1}{t} = 1$ , and

$$(a + b)^s \leq c(a^s + b^s)$$

holds for some  $c > 0$ , then we get

$$\sum_{k=1}^n p_k g_k h_k \leq R_5 \leq \left\{ \sum_{k=1}^n p_k g_k^s - \frac{1}{c} \sum_{k=1}^n p_k \left| g_k - \frac{\sum_{k=1}^n p_k g_k h_k}{\sum_{k=1}^n p_k h_k^t} h_k^{t-1} \right|^s \right\}^{1/s} \left( \sum_{k=1}^n p_k h_k^t \right)^{1/t}, \tag{21}$$

where

$$\begin{aligned} R_5 = & \left[ \frac{\left( \sum_{k=1}^n k p_k g_k h_k \right)^s}{\left( \sum_{k=1}^n k p_k h_k^t \right)^{(s-1)}} + \frac{\left( \sum_{k=1}^n (n-k) p_k g_k h_k \right)^s}{\left( \sum_{k=1}^n (n-k) p_k h_k^t \right)^{(s-1)}} \right. \\ & - \frac{1}{n} \left\{ \sum_{k=1}^n k p_k h_k^t \left| \frac{\sum_{k=1}^n k p_k g_k h_k}{\sum_{k=1}^n k p_k h_k^t} - \frac{\sum_{k=1}^n p_k g_k h_k}{\sum_{k=1}^n p_k h_k^t} \right|^s \right. \\ & \left. \left. + \sum_{k=1}^n (n-k) p_k h_k^t \left| \frac{\sum_{k=1}^n (n-k) p_k g_k h_k}{\sum_{k=1}^n (n-k) p_k h_k^t} - \frac{\sum_{k=1}^n p_k g_k h_k}{\sum_{k=1}^n p_k h_k^t} \right|^s \right\} \right]^{1/s} \sum_{k=1}^n (p_k h_k^t)^{1/t}. \end{aligned}$$

COROLLARY 5. Let  $s \geq 2$  and  $t$  is defined such that  $\frac{1}{s} + \frac{1}{t} = 1$  and  $p, g, h : [a, b] \rightarrow \mathbb{R}$  are such that  $pg, pg^s$ , and  $ph^t$  are integrable functions on  $[a, b]$ . If

$$(a + b)^s \leq c(a^s + b^s)$$

holds for some  $c > 0$ , then we get

$$\begin{aligned} & \int_a^b p(x)g(x)h(x)dx \leq R_6 \\ & \leq \left( \int_a^b p(x)g^s(x)dx - \frac{1}{c} \int_a^b p \left| g(x) - \frac{\int_a^b p(x)g(x)h(x)dx}{\int_a^b p(x)h^t(x)dx} h^{t-1}(x) \right|^s dx \right)^{\frac{1}{s}} \\ & \quad \times \left( \int_a^b p(x)h^t(x)dx \right)^{\frac{1}{t}}, \end{aligned}$$

where

$$\begin{aligned} R_6 = & \left[ \frac{\left( \int_a^b (b-x)p(x)g(x)h(x)dt \right)^s}{\left( \int_a^b (b-x)p(x)h^t(x)dt \right)^{(s-1)}} + \frac{\left( \int_a^b (x-a)p(x)g(x)h(x)dt \right)^s}{\left( \int_a^b (x-a)p(x)h^t(x)dt \right)^{(s-1)}} \right. \\ & - \frac{1}{b-a} \left\{ \int_a^b (b-x)p(x)h^t(x)dt \left| \frac{\int_a^b (b-x)p(x)g(x)h(x)dt}{\int_a^b (b-x)p(x)h^t(x)dt} - \frac{\int_a^b p(x)g(x)h(x)dt}{\int_a^b p(x)h^t(x)dt} \right|^s \right. \\ & \left. \left. + \int_a^b (x-a)p(x)h^t(x)dt \left| \frac{\int_a^b (x-a)p(x)g(x)h(x)dt}{\int_a^b (x-a)p(x)h^t(x)dt} - \frac{\int_a^b p(x)g(x)h(x)dt}{\int_a^b p(x)h^t(x)dt} \right|^s \right\} \right]^{1/s} \\ & \times \left( \int_a^b p(x)h^t(x)dt \right)^{1/t}. \end{aligned}$$

REMARK 4. We state our results for positive linear functionals and apply them to sums and integrals. Some of our results are new even in case of sums and integrals. As most of the classical inequalities can be obtained from the Jensen and Hölder inequalities, we can also obtain improvements of these inequalities by using our new results. Further, like sums and integrals, time scales integrals are also important examples of positive linear functionals. Therefore we can use our results to get improvements for time scales integrals.

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