Mathematical
Inequalities
& Applications
Volume 26, Number 3 (2023), 655–683 doi:10.7153/mia-2023-26-40

COMMUTATORS GENERATED BY BMO–FUNCTIONS AND THE FRACTIONAL INTEGRALS ON ORLICZ–MORREY SPACES

TAKESHI IIDA

(*Communicated by J. Soria*)

Abstract. We study the boundedness of commutators generated by BMO functions and the fractional integral operator on Orlicz-Morrey spaces of the second kind. To show this problem, we investigate the Fefferman-Stein type inequality concerning the Hardy-Littlewood maximal function and the sharp maximal operator and the boundedness of the Orlicz-fractional maximal operator in the Orlicz-Morrey spaces.

1. Introduction

To study the local behaviour of solutions to second-order elliptic partial differential equations, Morrey [20] introduced the classical Morrey spaces. Orlicz [23, 24] originally introduced the Orlicz spaces L^{Φ} as a generalisation of Lebesgue spaces L^p. The authors of the paper [6] consider the boundedness properties of weak (sub)solutions to the Dirichlet problem in the framework of the Orlicz spaces. To unify the properties of Orlicz and Morrey norms, the authors of papers [13, 21, 31] introduced three types of Orlicz-Morrey spaces.

The detailed study of the boundedness properties of the generalised fractional integral operators and Orlicz maximal operator on Orlicz-Morrey spaces of the second kind is in the paper [31]. In the paper [15], the author investigated the boundedness of the Orlicz-fractional maximal operator on the Morrey spaces. Based on these studies, we study the Orlicz-fractional maximal operator on Orlicz-Morrey spaces of the second kind.

In this paper, the symbol C denotes a positive constant. Whenever we evaluate the operator, the constant C may change from one constant to another. In this paper, the symbol "$\lesssim$" implies that we omit the constant of the corresponding inequality. For example, let X and Y be general function spaces equipped with quasi-norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. We write $\|Tf\|_Y \lesssim \|f\|_X$ if T is bounded from X to Y, where T is a general operator. In this case, the implicit constants of the inequality are independent of f.

We define the fractional integral operator I_α.

Mathematics subject classification (2020): 26A33, 42B25, 42B35.

Keywords and phrases: Orlicz-Morrey spaces, commutators, fractional integral operator, Orlicz maximal operator, Orlicz-fractional maximal operator.

Paper MIA-26-40

DEFINITION 1.1. Let $0<\alpha<n$. Define

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

We define the commutator $[b,I_\alpha]$ as follows:

DEFINITION 1.2. Let $0<\alpha<n$. For $b \in BMO(\mathbb{R}^n)$, define

$$[b,I_\alpha]f(x) := \int_{\mathbb{R}^n} \frac{b(x)-b(y)}{|x-y|^{n-\alpha}} f(y)dy.$$

Here, we choose f so that the right-hand side makes sense.

This paper aims to develop a theory of commutators generated by BMO functions and the fractional integral operator $[b,I_\alpha]$ on "second kind" Orlicz-Morrey spaces. The authors of [27] treated the relatively similar problem for the boundedness of $[b,I_\alpha]$ on Orlicz-Morrey spaces, we illustrate it in another problem setting.

Mathematically, we consider the problem set-up as follows. First, we know that the boundedness of $[b,I_\alpha]$ on Morrey spaces is due to the papers [8, 17].

PROPOSITION 1.3. *Let $0<\alpha<n$, $1 \leqslant p \leqslant p_0 < \infty$ and $1<q \leqslant q_0 < \infty$. Assume that $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{p}{q} = \frac{p_0}{q_0}$. If $b \in BMO$, then we have*

$$\|[b,I_\alpha]f\|_{\mathcal{M}^{q_0}_q} \lesssim \|b\|_{BMO} \|f\|_{\mathcal{M}^{p_0}_p}.$$

Here, $\mathcal{M}^{p_0}_p$ and $\mathcal{M}^{q_0}_q$ are defined below, respectively.

The main problem of this paper is to extend Morrey spaces in Proposition 1.3 to Orlicz-Morrey spaces. We describe the conclusions first.

THEOREM 1.4. *Let $0<\alpha<n$, $1<p_0<q_0<\infty$, Greek letters Φ and Ψ denote Young functions with $\Phi(t) \lesssim t^{p_0}$, $\Psi(t) \lesssim t^{q_0}$ and $b \in BMO$. Moreover, $\Psi \in \triangle_2 \cap \nabla'$. Assume that $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and*

$$\int_1^t \left(\frac{t}{s} \log\left(e + \frac{t}{s}\right)\right)^{\frac{q_0}{p_0}} \Psi'(s)ds \lesssim \Phi(t)^{\frac{q_0}{p_0}} \quad \textit{for} \quad t>1, \tag{1}$$

where the implicit constant is independent of t. Then,

$$\|[b,I_\alpha]f\|_{\mathcal{M}^{q_0}_\Psi} \lesssim \|b\|_{BMO} \|f\|_{\mathcal{M}^{p_0}_\Phi}.$$

Here, $\mathcal{M}^{p_0}_\Phi$ and $\mathcal{M}^{q_0}_\Psi$ are defined below, which are called the Orlicz-Morrey spaces, respectively.

REMARK 1.5. Implicit constants of (3), (4), (18), (19), (20), (21), (22) and (25) below are also independent of t. In the case of $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $\frac{q_0}{p_0} = \frac{q}{p}$, condition (1) holds. Let $P > 0$ and $\theta \in \mathbb{R}$. By a straightforward computation, for every $\theta > 0$,

$$\lim_{x \to \infty} \frac{(\log(e+x))^{\theta}}{x^P} = 0. \tag{2}$$

Applying (2) for $\theta = \frac{q_0}{p_0}$ and $P = q - \frac{q}{p} - \varepsilon > 0$ $\left(\text{here, } 0 < \varepsilon < q - \frac{q}{p}\right)$, we can show that condition (1) occurs:

$$\int_1^t \left(\frac{t}{s} \log\left(e + \frac{t}{s}\right)\right)^{\frac{q_0}{p_0}} (s^q)' ds \lesssim t^{p \cdot \frac{q_0}{p_0}} \quad \text{for} \quad t > 1. \tag{3}$$

Since condition (3) is satisfied, Theorem 1.4 recovers Proposition 1.3.

We obtain the following corollary.

COROLLARY 1.6. *Let $0 < \alpha < n$, $1 < p \leqslant p_0 < \infty$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\beta \in BMO$. If Young functions Φ and Ψ correspond to Example 3.7 below, then,*

$$\|[\beta, I_\alpha] f\|_{\mathcal{M}_{\Psi}^{q_0}} \lesssim \|\beta\|_{BMO} \|f\|_{\mathcal{M}_{\Phi}^{p_0}}.$$

Proof. Applying (2) for $\theta = \frac{q_0}{p_0} a + 1$ and $P = p - 1 > 0$, we obtain

$$t \log(e+t) \lesssim \frac{t^p}{(\log(e+t))^{\frac{p_0}{q_0} a}} \quad \text{for} \quad t > 1. \tag{4}$$

Estimate (4) shows that the triplet of Young functions $(t \log(e+t), \Psi, \Phi)$ satisfies (1). $\square$

We state these notations and definitions precisely. For an index $p > 1$, let $p' := \frac{p}{p-1}$. For a set E, $|E|$ denotes the Lebesgue measure of E. We assume that all cubes have their sides parallel to the coordinate axes. For a cube $Q \subset \mathbb{R}^n$, $\ell(Q)$ denotes the side-length and cQ to denote the cube with the same centre as Q but with side-length $c\ell(Q)$. The symbols $m_Q(f)$, f_Q and $⨍_Q f(x)dx$ denote the integral average of a measurable function f over Q:

$$m_Q(f) = f_Q = ⨍_Q f(x) dx = \frac{1}{|Q|} \int_Q f(x) dx.$$

For $0 \leqslant \alpha < n$, the symbol M_α denotes the fractional maximal operator and the symbol M denotes M_0, which we refer to as the Hardy-Littlewood maximal operator.

Next, we define the Morrey spaces $\mathcal{M}_p^{p_0}$ as follows:

DEFINITION 1.7. Let $0 < p \leqslant p_0 < \infty$. For $f \in L^p_{\mathrm{loc}}$,

$$\|f\|_{\mathscr{M}^{p_0}_p} := \sup_{Q \subset \mathbb{R}^n} |Q|^{\frac{1}{p_0}} \left(\fint_Q |f(x)|^p dx \right)^{\frac{1}{p}}. \tag{5}$$

By (5), we define the corresponding Morrey space concerning the norm:

$$\mathscr{M}^{p_0}_p := \left\{ f \in L^p_{\mathrm{loc}} : \|f\|_{\mathscr{M}^{p_0}_p} < \infty \right\}.$$

We define the sharp maximal operator $M^\sharp$ and BMO spaces as follows:

DEFINITION 1.8. For a locally integral function f, define the sharp maximal function $M^\sharp$ by

$$M^\sharp f(x) := \sup_{Q \subset \mathbb{R}^n} \fint_Q |f(y) - f_Q|\, dy \cdot \chi_Q(x).$$

DEFINITION 1.9. One says that a locally integral function f is an element of $\mathrm{BMO}(\mathbb{R}^n)$ if it satisfies

$$\|f\|_{BMO} := \sup_{Q \subset \mathbb{R}^n} \left(\fint_Q |f(y) - f_Q|\, dy \right).$$

We define a Young function and the complementary (Young) function (see [26, p. 6]).

DEFINITION 1.10. For a convex function $\Phi : [0,\infty) \to [0,\infty)$ which satisfies $\Phi(t) \to \infty$ as $t \to \infty$, one can associate another convex function $\overline{\Phi}$ having similar properties:

$$\overline{\Phi}(t) := \sup_{s>0} (st - \Phi(s)). \tag{6}$$

Then, we refer Φ and $\overline{\Phi}$ to the Young function and the complementary (Young) function to Φ, respectively.

By (6), a pair of Young functions $\left(\Phi, \overline{\Phi}\right)$ satisfies the following inequality, which we refer to as generalized Young's inequality:

$$st \leqslant \Phi(s) + \overline{\Phi}(t) \quad (s,t \geqslant 0). \tag{7}$$

A pair of Young functions $\left(\Phi, \overline{\Phi}\right)$ satisfies the following inequality (see [5, p. 99]):

$$\Phi^{-1}(t) \left(\overline{\Phi}\right)^{-1}(t) \cong t. \tag{8}$$

Let $A(t)$ and $B(t)$ are Young functions. We write $A(t) \cong B(t)$ if there are constants c_1, c_2 such that $c_1 A(t) \leqslant B(t) \leqslant c_2 A(t)$ for $t \geqslant 1$. Moreover, we postulate the following conditions in terms of a Young function (see [26, pp. 13, 22, 28]).

DEFINITION 1.11. Let Φ be a Young function. Then, we define the conditions as follows:

(i) Φ is an N-function if Φ is continuous, $\Phi(t)=0$ if and only if $t=0$, $\lim_{t\to 0+0}\Phi(t)/t=0$ and $\lim_{t\to\infty}\Phi(t)/t=\infty$.

(ii) $\Phi\in\triangle_2$, if there exists $K>0$ such that for every $t>1$, $\Phi(2t)\leqslant K\Phi(t)$.

(iii) $\Phi\in\nabla_2$, if there exists $K>1$ such that for every $t>1$, $\Phi(t)\leqslant\dfrac{1}{2K}\Phi(Kt)$.

(iv) $\Phi\in\triangle'$, if there exists $C>0$ such that $\Phi(xy)\leqslant C\Phi(x)\Phi(y)$ for $x,y\geqslant 0$.

(v) $\Phi\in\nabla'$, if there exists $C>0$ such that $\Phi(x)\Phi(y)\leqslant\Phi(Cxy)$ for $x,y\geqslant 0$.

This paper assumes that all Young functions are N-functions.

REMARK 1.12. According to [26, Lemma 1 in p. 28],

$$\triangle'\subsetneq\triangle_2. \tag{9}$$

The condition (v) implies that for $\Phi\in\nabla'$, if $x,y\geqslant 0$, then

$$\Phi\left(\frac{y}{x}\right)\lesssim\frac{\Phi(y)}{\Phi(x)}. \tag{10}$$

We refer to (10) as the property of variable separation for a Young function.

In this paper, one says that a Young function Ψ dominates a Young function Φ and denotes this by $\Phi(t)\lesssim\Psi(t)$, if there exists $C>0$ such that for all $t\geqslant 1$, $\Phi(t)\leqslant\Psi(Ct)$. We refer to $\Phi\in\triangle$ and $\Phi\in\triangle'$ as "Φ is doubling" and "Φ is submultiplicative", respectively (see [5, p. 98]). The examples of Young function with doubling and submultiplicative are in [5, p. 98].

EXAMPLE 1.13. Let $\Phi(t)=t^a[\log(e+t)]^b$.

(i) If $a\geqslant 1$, then $\Phi\in\triangle_2$.

(ii) If $a\geqslant 1$ and $b\geqslant 0$, then $\Phi\in\triangle'$.

REMARK 1.14. We have the relation between Young functions Φ and $\overline{\Phi}$ (see [26, p. 26 and p. 30]):

(i) $\Phi\in\triangle_2$ if and only if $\overline{\Phi}\in\nabla_2$.

(ii) $\Phi\in\triangle'$ if and only if $\overline{\Phi}\in\nabla'$.

By (9),

$$\nabla'\subset\nabla_2. \tag{11}$$

EXAMPLE 1.15. Let $\Phi(t) = \dfrac{t^a}{[\log(e+t)]^b}$. If $a \geqslant 1$ and $b \geqslant 0$, then

$$\Phi \in \triangle_2 \cap \nabla'.$$

Since,

$$\Phi^{-1}(t) \cong t^{\frac{1}{a}} [\log(e+t)]^{\frac{b}{a}}$$

occurs (see [5, p. 105]), equivalence (8) gives,

$$\left(\overline{\Phi}\right)^{-1}(t) \cong t^{\frac{1}{a'}} [\log(e+t)]^{-\frac{b}{a}}.$$

This implies that

$$\overline{\Phi}(t) \cong t^{a'} [\log(e+t)]^{\frac{a'}{a}b}.$$

Example 1.13 and Remark 1.14 show $\Phi \in \triangle_2 \cap \nabla'$.

There is the following property in [5, 29]. For a Young function Φ,

$$\frac{\Phi(t)}{t} \cong \Phi'(t) \ (t \geqq 0). \tag{12}$$

We define the normalized Luxemburg norm of f on Q (see [5, p. 98]).

DEFINITION 1.16. Let Φ be a Young function. For a cube Q,

$$\|f\|_{\Phi,Q} := \inf\left\{\lambda > 0 : \fint_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leqslant 1\right\}.$$

REMARK 1.17. Fix Q a cube in $\mathbb{R}^n$.

(i) If $\Phi(t) = t^p$, then $\|f\|_{\Phi,Q} = \left(\fint_Q |f(x)|^p dx\right)^{\frac{1}{p}}$.

(ii) We apply (7) to $s = \dfrac{|f(x)|}{\|f\|_{\Phi,Q}}$ and $t = \dfrac{|g(x)|}{\|g\|_{\overline{\Phi},Q}}$. Then, we obtain the following inequality, which we refer to as generalised Hölder's inequality (see [5, p. 99]).

$$\fint_Q |f(x)g(x)|dx \leqslant 2\,\|f\|_{\Phi,Q}\,\|g\|_{\overline{\Phi},Q}. \tag{13}$$

By the norm $\|\cdot\|_{\Phi,Q}$, we define the Orlicz-Morrey spaces. We know that various Orlicz-type spaces. For example, the authors of papers [2, 18, 19] introduced many Orlicz-type spaces: Beurling-Orlicz spaces, weak Beurling-Orlicz spaces, Herz-Orlicz spaces, weak Herz-Orlicz spaces, central Morrey-Orlicz spaces and weak central Morrey-Orlicz spaces. Furthermore, they studied the boundedness of the operators M, I_α and Calderón-Zygmund singular integrals on these function spaces. In this paper, we consider the boundedness of the operators $M_{B,\alpha}$ and $[b,I_\alpha]$ on the ordinary Orlicz-Morrey space. Papers [7, 10, 13] categorise Orlicz-Morrey spaces as broad types. Roughly classifying, the kinds of Orlicz-Morrey spaces have three classes. In the present paper, we deal with the second kind as the Orlicz-Morrey spaces (see [31, p. 523]).

DEFINITION 1.18. Let $1 < p_0 < \infty$ and be Φ be a Young function. Then, we define the norm of Orlicz-Morrey spaces $\mathscr{M}_\Phi^{p_0}$ of the second kind as follows: For a locally integral function f,

$$\|f\|_{\mathscr{M}_\Phi^{p_0}} := \sup_{Q\subset\mathbb{R}^n} |Q|^{\frac{1}{p_0}} \|f\|_{\Phi,Q}, \tag{14}$$

where the cube Q ranges over all compact cubes whose edges are parallel to coordinate axes. By (14), we define the corresponding Orlicz-Morrey spaces $\mathscr{M}_\Phi^{p_0}$, that is,

$$\mathscr{M}_\Phi^{p_0} := \left\{ f : \text{for every cube } Q\subset\mathbb{R}^n, \|f\|_{\Phi,Q} < \infty \quad \text{and} \quad \|f\|_{\mathscr{M}_\Phi^{p_0}} < \infty \right\}.$$

REMARK 1.19.

(i) Let Φ be a Young function and let $1 \leqslant r_0 < \infty$. Then, $\mathscr{M}_\Phi^{r_0} \neq \{0\}$ if and only if $\Phi(t) \lesssim t^{r_0}$ for $t \geqslant 1$. To define the Orlicz-Morrey space $\mathscr{M}_\Phi^{p_0}$, we may assume that $\Phi(t) \lesssim t^{p_0}$ (see [15, p. 245]).

(ii) The second kind is different from the first kind and the third kind (see [7, p. 2]).

(iii) For a Young function Φ, we define the norm $\|\cdot\|_{\mathscr{M}_{\Phi,\mathscr{D}}^{p_0}}$ as follows:

$$\|f\|_{\mathscr{M}_{\Phi,\mathscr{D}}^{p_0}} := \sup_{Q\in\mathscr{D}(\mathbb{R}^n)} |Q|^{\frac{1}{p_0}} \|f\|_{\Phi,Q}.$$

Here, $\mathscr{D}(\mathbb{R}^n)$ is the set of dyadic cubes in $\mathbb{R}^n$. Then, two norms $\|\cdot\|_{\mathscr{M}_{\Phi,\mathscr{D}}^{p_0}}$ and $\|\cdot\|_{\mathscr{M}_\Phi^{p_0}}$ are equivalent:

$$\|f\|_{\mathscr{M}_{\Phi,\mathscr{D}}^{p_0}} \cong \|f\|_{\mathscr{M}_\Phi^{p_0}}. \tag{15}$$

In this paper, we omit the proof of (15) (see [16, 25]).

By the norm $\|\cdot\|_{\Phi,Q}$, we define the Orlicz-fractional maximal operator as follows:

DEFINITION 1.20. Let $0 \leqslant \alpha < n$. For a Young function B,

$$M_{B,\alpha} f(x) := \sup_{Q \subset \mathbb{R}^n} \ell(Q)^\alpha \|f\|_{B,Q} \cdot \chi_Q(x).$$

Moreover, the symbol M_B denotes $M_{B,0}$, which we refer to as the Orlicz maximal operator.

REMARK 1.21. For a Young function Φ, every cube Q and a locally integral function f, by (13),

$$\fint_Q |f(y)|\, dy \lesssim \|f\|_{\Phi,Q}. \tag{16}$$

The rest of this paper is organized as follows. In Sections 2 and 3, we list known results and main results, respectively. In Section 4, we give some lemmas. Lastly, in Section 5, we give the proofs of main Theorems.

2. Known results

We know the author of paper [1] showed the boundedness of I_α on Morrey spaces.

PROPOSITION 2.1. *Let $0 < \alpha < n$, $1 < p \leqslant p_0 < \infty$ and $1 < q \leqslant q_0 < \infty$. Assume that $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{p}{q} = \frac{p_0}{q_0}$. Then,*

$$\|I_\alpha f\|_{\mathscr{M}_q^{q_0}} \leqslant C \|f\|_{\mathscr{M}_p^{p_0}}.$$

In the paper [25, pp. 138–139], Peréz introduced B_p-condition, which characterizes the boundedness of Orlicz maximal operator $M_B : L^p \to L^p$.

PROPOSITION 2.2. *Let $1 < p < \infty$ and B be a Young function. Then, the following are equivalent.*

(i) $B \in B_p$, *that is, there exists $c > 0$ such that* $\displaystyle\int_c^\infty \frac{B(t)}{t^{p+1}} dt < \infty$.

(ii) $M_B : L^p \to L^p$.

In papers [4, p. 428] and [16, pp. 375–376], the authors introduced the necessary and sufficient condition for $M_{B,\alpha} : L^p \to L^q$.

PROPOSITION 2.3. *Let $0 \leqslant \alpha < n$ and $1 < p < \frac{n}{\alpha}$. Then, the following are equivalent.*

(i) $B^{\frac{q}{p}} \in B_q$ *with* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

(ii) $M_{B,\alpha} : L^p \to L^q$

The following result is due to the paper [15, p. 249], which concerns the boundedness of the operator $M_{B,\alpha}$ on Morrey space.

PROPOSITION 2.4. *Let* $0 \leqslant \alpha < n$, $1 < p \leqslant p_0 < \frac{n}{\alpha}$, $0 < q \leqslant q_0 < \infty$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ *and* $\frac{q}{q_0} = \frac{p}{p_0}$. *If* $B^{\frac{q}{p}} \in B_q$, *then* $M_{B,\alpha} : \mathscr{M}_p^{p_0} \to \mathscr{M}_q^{q_0}$.

In the paper [31, p. 535], the authors show the local boundedness of M_B:

PROPOSITION 2.5. *Let* B *and* Φ *be Young functions. Then, the following are equivalent.*

(i) For a cube Q,

$$\|M_B(f\chi_Q)\|_{\Phi,Q} \lesssim \|f\|_{\Phi,Q}, \tag{17}$$

where the implicit constant in inequality (17) *is independent of* Q *and* f.

(ii) The functions Φ *and* B *satisfy*

$$\int_1^t B\left(\frac{t}{s}\right)\Phi'(s)ds \lesssim \Phi(t) \quad \text{for} \quad t > 1. \tag{18}$$

REMARK 2.6. Let $1 < p < \infty$ and $\Phi(t) = t^p$. Changing of variables $\ell = \frac{t}{s}$, we can replace (18) to the following: For $t > 1$,

$$\int_1^t B\left(\frac{t}{s}\right)(s^p)'\,ds \cong t^p \int_1^t \frac{B(\ell)}{\ell^{p+1}}d\ell.$$

Proposition 2.5 essentially shows the boundedness of M_B in Orlicz-Morrey spaces.

We invoke the following result, which we refer to as the Fefferman-Stein inequality (see [9, p. 153]).

PROPOSITION 2.7. *Suppose that* $f \in L^{p_0}(\mathbb{R}^n)$, *for some* p_0. *Assume that* $1 < p < \infty$, $1 \leqslant p_0 \leqslant p$, *and that* $M^\sharp f \in L^p(\mathbb{R}^n)$, *Then* $Mf \in L^p(\mathbb{R}^n)$, *and we have the a priori inequality*

$$\|Mf\|_{L^p} \lesssim \left\|M^\sharp f\right\|_{L^p},$$

here, the implicit constant is independent of f.

To show Theorem 1.4, we use the following pointwise inequality in the paper [3].

PROPOSITION 2.8. *Let* $0 < \alpha < n$, $1 < p_0 < \infty$, *If* $B(t) = t\log(e+t)$ *and* $b \in BMO$. *Then, for a non-negative locally integral function* f,

$$M^\sharp([b,I_\alpha]f)(x) \lesssim \|b\|_{BMO}(I_\alpha f(x) + M_{B,\alpha}f(x)).$$

To show Theorem 1.4, we study Propositions 2.1–2.8 in the framework of Orlicz-Morrey spaces.

3. Main results

We have the following result which relates to [14, Theorem 1.2] (see also [30, Example 49]). At least, we can drop the assumption $\Phi \in \nabla_2$.

THEOREM 3.1. *Let $0 < \alpha < n$, $1 < p_0 < \infty$ and Φ be a Young function with $\Phi(t) \lesssim t^{p_0}$. If $\Phi \in \triangle_2$, then, for a locally integral function f,*

$$\|I_\alpha f\|_{\mathscr{M}_\Phi^{p_0}} \lesssim \|M_\alpha f\|_{\mathscr{M}_\Phi^{p_0}}.$$

REMARK 3.2. In Theorem 3.1, we may replace the condition $\Phi \in \triangle_2$ with

$$\int_1^t \frac{\overline{\Phi}(s)}{s^2} ds \lesssim \frac{\overline{\Phi}(t)}{t} \quad \text{for} \quad t > 1. \tag{19}$$

Condition (19) corresponds to the case of $B(t) = t$ in condition (18).

By (16), for a Young function B and a locally integral function f, $M_\alpha f \leqslant M_{B,\alpha} f$ holds. Hence, the boundedness of the operator $M_{B,\alpha}$ controls the boundedness of I_α on Orlicz-Morrey spaces. Furthermore, we can show that the boundedness of the operator $M_{B,\alpha}$ in Orlicz-Morrey spaces:

THEOREM 3.3. *Let $0 \leqslant \alpha < n$, $1 < p_0 < q_0 < \infty$ which satisfy $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$. Let B, Φ and Ψ be Young functions with $\Phi(t) \lesssim t^{p_0}$ and $\Psi(t) \lesssim t^{q_0}$, respectively. If*

$$\int_1^t B\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}} \Psi'(s) ds \lesssim \Phi(t)^{\frac{q_0}{p_0}} \quad \textit{for} \quad t > 1, \tag{20}$$

then the following holds:

$$\|M_{B,\alpha} f\|_{\mathscr{M}_\Psi^{q_0}} \lesssim \|f\|_{\mathscr{M}_\Phi^{p_0}}.$$

REMARK 3.4. By (12), we may replace $\Psi'(s)$ with $\frac{\Psi(s)}{s}$ in condition (20).

REMARK 3.5. Theorem 3.3 recovers Proposition 2.4, which is the case of $\Phi(t) = t^p$, $\Psi(t) = t^q$ for $1 < p < q < \infty$: Changing of $\ell = \frac{t}{s}$ for $t > 1$ in (20), we obtain

$$\int_1^t B\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}} \Psi'(s) ds \cong t^q \int_1^t \frac{B(\ell)^{\frac{q_0}{p_0}}}{\ell^{q+1}} dt.$$

Moreover, if $\frac{q}{p} = \frac{q_0}{p_0}$,

$$t^q \int_1^t \frac{B(\ell)^{\frac{q_0}{p_0}}}{\ell^{q+1}} dt = t^{p \cdot \frac{q_0}{p_0}} \int_1^t \frac{B(\ell)^{\frac{q}{p}}}{\ell^{q+1}} d\ell.$$

This implies that $B^{\frac{q}{p}} \in B_q$ holds.

By the property of variable separation (10), we obtain the following Proposition 3.6.

PROPOSITION 3.6. *Let B, Φ and Ψ be Young functions. We consider the following conditions (i) and (ii).*

(i) $\Psi \in \nabla'$ *and*

$$\Psi(t)^{\frac{p_0}{q_0}} \left(\int_1^t \frac{B(s)^{\frac{q_0}{p_0}}}{\Psi(s)} \frac{ds}{s} \right)^{\frac{p_0}{q_0}} \lesssim \Phi(t) \quad \textit{for} \quad t > 1. \tag{21}$$

(ii) $B^{\frac{q_0}{p_0}} \in \nabla'$ *and*

$$B(t) \left(\int_1^t \frac{\Psi(s)}{B(s)^{\frac{q_0}{p_0}}} \frac{ds}{s} \right)^{\frac{p_0}{q_0}} \lesssim \Phi(t) \quad \textit{for} \quad t > 1. \tag{22}$$

If the triplet of Young functions (B, Ψ, Φ) satisfies either condition (i) or condition (ii), then (20) *occurs.*

Proof.

(i) Changing of $\ell = \frac{t}{s}$ for $t > 1$, we obtain

$$\int_1^t B\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}} \frac{\Psi(s)}{s} ds = \int_1^t B(\ell)^{\frac{q_0}{p_0}} \Psi\left(\frac{t}{\ell}\right) \frac{d\ell}{\ell}.$$

Since $\Psi \in \nabla'$, using (10), we learn

$$\int_1^t B(\ell)^{\frac{q_0}{p_0}} \Psi\left(\frac{t}{\ell}\right) \frac{d\ell}{\ell} \lesssim \Psi(t) \int_1^t \frac{B(\ell)^{\frac{q_0}{p_0}}}{\Psi(\ell)} \frac{d\ell}{\ell} = \left\{ \Psi(t)^{\frac{p_0}{q_0}} \left(\int_1^t \frac{B(\ell)^{\frac{q_0}{p_0}}}{\Psi(\ell)} \frac{d\ell}{\ell} \right)^{\frac{p_0}{q_0}} \right\}^{\frac{q_0}{p_0}}.$$

Assuming that (21) holds, we see that (20) occurs.

(ii) Since $B^{\frac{q_0}{p_0}} \in \nabla'$, using (10),

$$\int_1^t B\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}} \Psi(s) \frac{ds}{s} \lesssim B(t)^{\frac{p_0}{q_0}} \left(\int_1^t \frac{\Psi(s)}{B(s)^{\frac{q_0}{p_0}}} \frac{ds}{s} \right) = \left\{ B(t) \left(\int_1^t \frac{\Psi(s)}{B(s)^{\frac{q_0}{p_0}}} \frac{ds}{s} \right)^{\frac{p_0}{q_0}} \right\}^{\frac{q_0}{p_0}}.$$

Assuming that (22) holds, we see that (20) occurs. □

Using Proposition 3.6, we can construct concrete examples that satisfy condition (20).

EXAMPLE 3.7. For simplicity, let a and $b \geqslant 0$. We classify Φ in the following five cases, given that

$$B(t) = \frac{t^p}{(\log(e+t))^{\frac{p_0}{q_0}a}} \quad \text{and} \quad \Psi(t) = \frac{t^{p\cdot\frac{q_0}{p_0}}}{(\log(e+t))^b}:$$

(i) In case of $a > b+1$, let

$$\Phi(t) = \frac{t^p}{[\log(e+t)]^{\frac{p_0}{q_0}b}}.$$

(ii) In case of $a = b+1$, let

$$\Phi(t) = t^p\left[\frac{\log(\log(e+t))}{[\log(e+t)]^b}\right]^{\frac{p_0}{q_0}}.$$

(iii) In case of $b-1 < a < b+1$, let

$$\Phi(t) = \frac{t^p}{[\log(e+t)]^{\frac{p_0}{q_0}\max\{a,b\}-1}}.$$

(iv) In case of $a = b-1$, let

$$\Phi(t) = t^p\left[\frac{\log(\log(e+t))}{[\log(e+t)]^a}\right]^{\frac{p_0}{q_0}}.$$

(v) In case of $a < b-1$, let

$$\Phi(t) = \frac{t^p}{[\log(e+t)]^{\frac{p_0}{q_0}a}}.$$

Then, the triplet of Young functions (B,Ψ,Φ) satisfies condition (20).

Proof. Let

$$F_1(t) := \Psi(t)^{\frac{p_0}{q_0}}\left(\int_1^t \frac{B(s)^{\frac{q_0}{p_0}}}{\Psi(s)}\frac{ds}{s}\right)^{\frac{p_0}{q_0}} = \frac{t^p}{(\log(e+t))^{\frac{p_0}{q_0}b}}\left(\int_1^t \log(e+s)^{b-a}\frac{ds}{s}\right)^{\frac{p_0}{q_0}}$$

and

$$F_2(t) := B(t)\left(\int_1^t \frac{\Psi(s)}{B(s)^{\frac{q_0}{p_0}}}\frac{ds}{s}\right)^{\frac{p_0}{q_0}} = \frac{t^p}{(\log(e+t))^{\frac{p_0}{q_0}a}}\left(\int_1^t \log(e+s)^{a-b}\frac{ds}{s}\right)^{\frac{p_0}{q_0}}.$$

Changing of $x = \log(e+s)$, we obtain

$$F_1(t) = \frac{t^p}{(\log(e+t))^{\frac{p_0}{q_0}b}} \left(\int_{\log(e+1)}^{\log(e+t)} x^{b-a} \frac{e^x}{e^x - e} dx \right)^{\frac{p_0}{q_0}}$$

and

$$F_2(t) = \frac{t^p}{(\log(e+t))^{\frac{p_0}{q_0}a}} \left(\int_{\log(e+1)}^{\log(e+t)} x^{a-b} \frac{e^x}{e^x - e} dx \right)^{\frac{p_0}{q_0}}.$$

Since $\dfrac{e^x}{e^x - e}$ is a decreasing function,

$$F_1(t) \lesssim \frac{t^p}{(\log(e+t))^{\frac{p_0}{q_0}b}} \left(\int_{\log(e+1)}^{\log(e+t)} x^{b-a} dx \right)^{\frac{p_0}{q_0}}, \tag{23}$$

and

$$F_2(t) \lesssim \frac{t^p}{(\log(e+t))^{\frac{p_0}{q_0}a}} \left(\int_{\log(e+1)}^{\log(e+t)} x^{a-b} dx \right)^{\frac{p_0}{q_0}}. \tag{24}$$

In the cases of (i) and (ii), we use estimate (23). In the cases of (iv) and (v), we use estimate (24). We consider the case of (iii).

(A) In the case of $a \leqslant b$, by (23),

$$F_1(t) \lesssim \frac{t^p}{(\log(e+t))^{\frac{p_0}{q_0}b}} \left[(\log(e+t))^{b-a+1} \right]^{\frac{p_0}{q_0}} = \frac{t^p}{(\log(e+t))^{(a-1)\frac{p_0}{q_0}}},$$

and, by (24),

$$F_2(t) \lesssim \frac{t^p}{(\log(e+t))^{\frac{p_0}{q_0}a}} \left[(\log(e+t))^{a-b+1} \right]^{\frac{p_0}{q_0}} = \frac{t^p}{(\log(e+t))^{(b-1)\frac{p_0}{q_0}}}.$$

Since $a \leqslant b$,

$$\frac{t^p}{(\log(e+t))^{(b-1)\frac{p_0}{q_0}}} \leqslant \frac{t^p}{(\log(e+t))^{(a-1)\frac{p_0}{q_0}}}.$$

(B) In the case of $b < a$, let $a' = b$ and $b' = a$. By the argument of (A) for $a' < b'$, we obtain

$$\begin{aligned} \frac{t^p}{(\log(e+t))^{(a-1)\frac{p_0}{q_0}}} &= \frac{t^p}{(\log(e+t))^{(b'-1)\frac{p_0}{q_0}}} \leqslant \frac{t^p}{(\log(e+t))^{(a'-1)\frac{p_0}{q_0}}} \\ &= \frac{t^p}{(\log(e+t))^{(b-1)\frac{p_0}{q_0}}}. \end{aligned}$$

Thanks to the cases (A) and (B), we may take

$$\Phi(t)=\frac{t^p}{(\log(e+t))^{(\max\{a,b\}-1)\frac{p_0}{q_0}}}. \quad \square$$

Next, we investigate the relation between (18) and (20).

PROPOSITION 3.8. *Let $1<p_0\leqslant q_0<\infty$, Φ_0 be a Young function which satisfies $\Phi_0\in\nabla'$ and $\theta(t)$ be a monotone increasing function for $t>1$. We take $\Phi(t)=\Phi_0(t)\theta(t)$ and $\Psi(t)=\Phi\left(t^{\frac{q_0}{p_0}}\right)=\Phi_0\left(t^{\frac{q_0}{p_0}}\right)\theta\left(t^{\frac{q_0}{p_0}}\right)$. Assume that for all $t>1$,*

$$\Psi(t)=\Phi_0\left(t^{\frac{q_0}{p_0}}\right)\theta\left(t^{\frac{q_0}{p_0}}\right)\lesssim\Phi_0(t)^{\frac{q_0}{p_0}}\theta(t) \tag{25}$$

and the condition (18). Then the condition (20) holds.

Proof of Proposition 3.8. Lemma 4.6 below implies that if (18) holds, then $B\left(\frac{t}{s}\right)\lesssim\Phi\left(\frac{t}{s}\right)$. For all $1<s<t$, by (12),

$$\begin{aligned}\int_1^t B\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}}\Psi'(s)ds &\lesssim\int_1^t\Phi\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}-1}B\left(\frac{t}{s}\right)\Psi(s)\frac{ds}{s}\\ &\cong\int_1^t\Phi\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}-1}B\left(\frac{t}{s}\right)\Psi(s)\frac{ds}{s}\\ &=\int_1^t\Phi\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}-1}B\left(\frac{t}{s}\right)\Phi_0\left(s^{\frac{q_0}{p_0}}\right)\theta\left(s^{\frac{q_0}{p_0}}\right)\frac{ds}{s}.\end{aligned} \tag{26}$$

By (25),

$$\begin{aligned}&\int_1^t\Phi\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}-1}B\left(\frac{t}{s}\right)\Phi_0\left(s^{\frac{q_0}{p_0}}\right)\theta\left(s^{\frac{q_0}{p_0}}\right)\frac{ds}{s}\\ &\lesssim\int_1^t\Phi_0\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}-1}\theta\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}-1}B\left(\frac{t}{s}\right)\Phi_0(s)^{\frac{q_0}{p_0}}\theta(s)\frac{ds}{s}.\end{aligned} \tag{27}$$

By (10),

$$\begin{aligned}&\int_1^t\Phi_0\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}-1}\theta\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}-1}B\left(\frac{t}{s}\right)\Phi_0(s)^{\frac{q_0}{p_0}}\theta(s)\frac{ds}{s}\\ &\lesssim\int_1^t\left(\frac{\Phi_0(t)}{\Phi_0(s)}\right)^{\frac{q_0}{p_0}-1}\theta\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}-1}B\left(\frac{t}{s}\right)\Phi_0(s)^{\frac{q_0}{p_0}}\theta(s)\frac{ds}{s}\\ &=\Phi_0(t)^{\frac{q_0}{p_0}-1}\int_1^t\theta\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}-1}B\left(\frac{t}{s}\right)\Phi_0(s)\theta(s)\frac{ds}{s}.\end{aligned} \tag{28}$$

Since θ is a monotone increasing function, for all $1 < s < t$,

$$\begin{aligned}&\Phi_0(t)^{\frac{q_0}{p_0}-1}\int_1^t \theta\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}-1} B\left(\frac{t}{s}\right)\Phi_0(s)\,\theta(s)\,\frac{ds}{s}\\ &\leqslant \Phi_0(t)^{\frac{q_0}{p_0}-1}\theta(t)^{\frac{q_0}{p_0}-1}\int_1^t B\left(\frac{t}{s}\right)\Phi_0(s)\,\theta(s)\,\frac{ds}{s}\\ &\cong \Phi(t)^{\frac{q_0}{p_0}-1}\int_1^t B\left(\frac{t}{s}\right)\Phi'(s)\,ds.\end{aligned} \tag{29}$$

By (18),

$$\Phi(t)^{\frac{q_0}{p_0}-1}\int_1^t B\left(\frac{t}{s}\right)\Phi'(s)\,ds \lesssim \Phi(t)^{\frac{q_0}{p_0}-1}\Phi(t) = \Phi(t)^{\frac{q_0}{p_0}}. \tag{30}$$

Estimates (26)–(30) show Proposition 3.8. $\square$

EXAMPLE 3.9. In Proposition 3.8, we take $\Phi_0(t) = t^p$ for $1 < p < p_0 < q_0 < \infty$. Then, we can list monotone-increasing functions $\theta(t)$ which satisfy (25) as follows: For $t > 1$,

(i) In the case $\theta(t) \equiv 1$, $\Phi(t) = t^p$ and $\Psi(t) = t^{p\cdot\frac{q_0}{p_0}}$.

(ii) In the case $\theta(t) = \{\log(e+t)\}^\gamma$ for $\gamma \geqslant 0$, $\Phi(t) = t^p\{\log(e+t)\}^\gamma$ and

$$\Psi(t) = t^{p\cdot\frac{q_0}{p_0}}\left\{\log\left(e+t^{\frac{q_0}{p_0}}\right)\right\}^\gamma \lesssim t^{p\cdot\frac{q_0}{p_0}}\{\log(e+t)\}^\gamma.$$

(iii) In the case $\theta(t) = \left\{\log\left(\frac{q_0}{p_0}+\log(t)\right)\right\}^\gamma$ for $\gamma \geqslant 0$, $\Phi(t) = t^p\left\{\log\left(\frac{q_0}{p_0}+\log(t)\right)\right\}^\gamma$ and

$$\Psi(t) = t^{p\cdot\frac{q_0}{p_0}}\left\{\log\left(\frac{q_0}{p_0}+\log\left(t^{\frac{q_0}{p_0}}\right)\right)\right\}^\gamma \lesssim t^{p\cdot\frac{q_0}{p_0}}\left\{\log\left(\frac{q_0}{p_0}+\log(t)\right)\right\}^\gamma.$$

In Theorem 3.3, if $\Phi(t) = t^p\log(e+t)$ and $\Psi(t) \cong t^{p\cdot\frac{q_0}{p_0}}\log(e+t)$, then we obtain the following result.

COROLLARY 3.10. *Let $0 \leqslant \alpha < n$, $1 < p < p_0 < q_0 < \infty$ and B be a Young function. Assume that $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and*

$$\int_1^t B\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}} s^{p\cdot\frac{q_0}{p_0}}\log(e+s)\frac{ds}{s} \lesssim (t^p\log(e+t))^{\frac{q_0}{p_0}} \quad \textit{for} \quad t > 1,$$

then the following holds:

$$\|M_{B,\alpha}f\|_{\mathscr{M}_\Psi^{q_0}} \lesssim \|f\|_{\mathscr{M}_\Phi^{p_0}}.$$

To show Theorem 1.4, we extend L^p spaces in Proposition 2.7 to Orlicz-Morrey spaces.

THEOREM 3.11. *Let* $1 < p_0 < \infty$ *and* Φ *be a Young function and* $\Phi(t) \lesssim t^{p_0}$*. If* $\Phi \in \nabla'$*, then*

$$\|Mf\|_{\mathscr{M}_{\Phi}^{p_0}} \lesssim \left\|M^{\sharp} f\right\|_{\mathscr{M}_{\Phi}^{p_0}},$$

as long as, $Mf \in \mathscr{M}_{\Phi}^{p_0}$.

4. Some Lemmas

To show Theorem 3.1, we use the standard argument (for example, see [15]). Without loss of generality, we may assume that $f(x) \geqslant 0$ a.e. $x \in \mathbb{R}^n$.

LEMMA 4.1. *Let* $0 < \alpha < n$. *For a dyadic cube* Q_0*, Let* $f_0 = f\chi_{3Q_0}$, $f_\infty = f - f\chi_{3Q_0}$ *and* $\mathscr{D}(Q_0) = \{Q \in \mathscr{D}(\mathbb{R}^n) : Q \subset Q_0\}$*. Then, for* $x \in Q_0$,

$$I_\alpha(f_0)(x) \lesssim \sum_{Q \in \mathscr{D}(Q_0)} \ell(Q)^\alpha m_{3Q}(f)\chi_Q(x) \tag{31}$$

and

$$I_\alpha(f_\infty)(x) \lesssim \sum_{k=1}^{\infty} \ell\left(3 \cdot 2^k Q_0\right)^\alpha m_{3\cdot 2^k Q_0}(f). \tag{32}$$

To analyze the right hand side of (31), we introduce the maximal cubes concerning inclusion as follows:

Let $\gamma = m_{3Q_0}(f)$ and $A = 2 \cdot 18^n$. For $k = 1, 2, 3, \ldots$, set

$$D_k := \bigcup \left\{Q \in \mathscr{D}(Q_0); m_{3Q}(f) > \gamma A^k\right\}.$$

Considering the maximal cubes concerning inclusion, we can write as follows:

$$D_k = \bigcup_j Q_{k,j},$$

where the cubes $\left\{Q_{k,j}\right\} \subset \mathscr{D}(Q_0)$ are nonoverlapping. One says that the dyadic cube $Q_{k,j}$ is maximal concerning inclusion in the set D_k, if $Q \in \mathscr{D}(Q_0)$ satisfies that $Q_{k,j} \subset Q$, then $m_{3Q}(f) \leqslant \gamma A^k$. By the maximality of $Q_{k,j}$ and the weak-L^1 boundedness of M, we have the following Lemma (see [15, Lemma 8]).

LEMMA 4.2. *Let* f *be a locally integral function. For a dyadic cube* Q_0*, taking* γ*,* A*,* $Q_{k,j}$ *and* D_k $(k, j = 1, 2, \ldots)$ *above,*

$$\gamma A^k < m_{3Q_{k,j}}(f) \leqslant 2^n \gamma A^k.$$

Let $E_0 := Q_0 \backslash D_1$ *and* $E_{k,j} := Q_{k,j} \backslash D_{k+1}$. *Then, the following properties occur:* $\{E_0\} \cup \left\{E_{k,j}\right\}$ *is a disjoint family of sets, which decomposes* Q_0, *and satisfies*

$$|Q_0| \leqslant 2|E_0| \quad \text{and} \quad \left|Q_{k,j}\right| \leqslant 2\left|E_{k,j}\right|.$$

Moreover, letting

$$\mathscr{D}_0(Q_0) := \{Q \in \mathscr{D}(Q_0) : m_{3Q}(f) \leqslant \gamma A\}$$

and

$$\mathscr{D}_{k,j}(Q_0) := \left\{Q \in \mathscr{D}(Q_0) : Q \subset Q_{k,j}, \gamma A^k < m_{3Q}(f) \leqslant \gamma A^{k+1}\right\},$$

we have $\mathscr{D}_0(Q_0)$ *and* $\mathscr{D}_{k,j}(Q_0)$ *are disjoint and*

$$\mathscr{D}(Q_0) = \mathscr{D}_0(Q_0) \cup \left(\bigcup_{k,j} \mathscr{D}_{k,j}(Q_0)\right). \tag{33}$$

We invoke the lemma in [26, Proposition 4 in p. 61].

LEMMA 4.3. *For a cube* Q *and measurable function* f, *we define the norm* $\|\cdot\|_{\mathscr{L}_Q^\Phi\left(\frac{dx}{|Q|}\right)}$ *as follows:*

$$\begin{aligned}\|f\|_{\mathscr{L}_Q^\Phi\left(\frac{dx}{|Q|}\right)} &:= \sup\left\{\left|\fint_Q f(x)g(x)dx\right| : \|g\|_{\overline{\Phi},Q} \leqslant 1\right\}\\ &= \sup\left\{\fint_Q |f(x)g(x)|\,dx : \|g\|_{\overline{\Phi},Q} \leqslant 1\right\}.\end{aligned}$$

Then,

$$\|f\|_{\Phi,Q} \leqslant \|f\|_{\mathscr{L}_Q^\Phi\left(\frac{dx}{|Q|}\right)} \leqslant 2\|f\|_{\Phi,Q}.$$

In [5, p. 118], the following estimate is essential to prove Theorem 3.3 (see also paper [15]).

LEMMA 4.4. *Given* α *and* p_0, $0 \leqslant \alpha < n$, $1 < p_0 < \frac{n}{\alpha}$ *and* $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$, *let* B *be a Young function such that* $t^{-n/\alpha}B(t)$ *is almost decreasing and that* $t^{-n/\alpha}B(t) \to 0$ *as* $t \to \infty$. *Let* Q_0 *be a cube and* $f(x) \geqslant 0$ *a.e.* $x \in \mathbb{R}^n$ *such that supp* $(f) \subset 3Q_0$. *For every* $\lambda > 0$,

$$\begin{aligned}&\left|\left\{x \in Q_0 : M_{B,\alpha}\left(f\chi_{3Q_0}\right)(x) > \lambda\right\}\right|\\ &\lesssim \frac{1}{\ell(Q_0)^{\alpha q_0}}\left(\int_{\{x \in 3Q_0 : C\ell(Q_0)^\alpha f(x) \geqslant \lambda\}} B\left(\frac{C\ell(Q_0)^\alpha f(x)}{\lambda}\right)dx\right)^{\frac{q_0}{p_0}}.\end{aligned}$$

There is the following lemma in the paper [31, p. 537].

LEMMA 4.5. *If* $\Phi \in \nabla_2$, *then for every* Q_0,

$$\left\|M\left(f\chi_{Q_0}\right)\right\|_{\Phi,Q_0} \lesssim \|f\|_{\Phi,Q_0}. \tag{34}$$

Estimate (34) *gives that if* $\Phi \in \nabla_2$ *and* $\Phi(t) \lesssim t^{p_0}$, *then*

$$\|Mf\|_{\mathscr{M}_\Phi^{p_0}} \lesssim \|f\|_{\mathscr{M}_\Phi^{p_0}}.$$

To prove Theorem 3.3, we verify the following lemma.

LEMMA 4.6. *If condition* (20) *holds, then* $B(t) \lesssim \Phi(t)$ *for* $t > 1$ *occurs.*

Proof. By the convexity of Ψ, for all $t > 1$, $\Psi'(t) \geqslant \Psi'(1) > 0$ holds.

$$\int_1^{2t} B\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}} \Psi'(s)ds \geqslant \Psi'(1)\int_1^{2t} B\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}} ds. \tag{35}$$

Since $2t > 2$,

$$\Psi'(1)\int_1^{2t} B\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}} ds \geqslant \Psi'(1)\int_1^{2} B\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}} ds. \tag{36}$$

For $1 < s < 2$, $B\left(\frac{t}{s}\right) \geqslant B\left(\frac{t}{2}\right)$ holds.

$$\Psi'(1)\int_1^{2} B\left(\frac{t}{s}\right)^{\frac{q_0}{p_0}} ds \geqslant \Psi'(1)B\left(\frac{t}{2}\right)^{\frac{q_0}{p_0}}. \tag{37}$$

Assumption (20) and estimates (35)–(37) give $B(t) \lesssim \Phi(t)$ for $t > 1$. $\square$

To show Theorem 3.3, we invoke the triangle inequality on L^p, which we refer to as Minkowski's integral inequality (for example, see [11, p. 13]):

LEMMA 4.7. *Let* $1 < p < \infty$ *and* Q_0 *be a cube. Then,*

$$\left(\int_0^\infty \left(\int_{Q_0} |F(x,\lambda)|\,dx\right)^p d\lambda\right)^{\frac{1}{p}} \leqslant \int_{Q_0}\left(\int_0^\infty |F(x,\lambda)|^p\,d\lambda\right)^{\frac{1}{p}} dx.$$

We invoke the following result from [28, p. 1101]. In Theorem 3.11, we state the result as the case $d\mu = dx$ and $\varepsilon = 1$. Additionally, we suppose $\Phi^{-1}(\delta) << 1$. We explicitly allude to this assumption whenever we need it.

LEMMA 4.8. *For a cube* Q_0, *we define* $\mathscr{Q}_0$ *and* $\mathscr{Q}_1$ *as*

$$\mathscr{Q}_0 := \{R : \textit{cube},\ R \textit{ meets } Q_0 \textit{ and is not contained in } 8Q_0\},$$
$$\mathscr{Q}_1 := \{R : \textit{cube},\ R \textit{ meets } Q_0 \textit{ and is contained in } 8Q_0\}.$$

Fix $\tilde{M} := \sup_{R\in\mathscr{Q}_0} m_R(|f|)$ *and assume that* $\lambda > \tilde{M}$. *Then there exists* $C_1 > 0$ *such that for any sufficiently small* $\delta > 0$, *we have*

$$\left|\left\{x \in Q_0 :\ Mf(x) > 2\lambda, M^\sharp f(x) \leqslant \delta\lambda\right\}\right| \leqslant C_1\delta\,|\{x \in 8Q_0 :\ Mf(x) > \lambda\}|.$$

We use the following properties to show Theorem 3.11, which is a direct consequence of $\{t>0:\Phi(t)\leqslant a, \Psi(t)\leqslant b\}\subset\{t>0:\Phi(t)+\Psi(t)\ \leqslant a+b\}$ for $a,b>0$.

LEMMA 4.9. *For m functions $A_i(\lambda)>0$ $(i=1,\dots,m)$,*

$$\inf\left\{\lambda>0:\sum_{i=1}^{m}A_i(\lambda)\leqslant 1\right\}\leqslant\inf\left\{\lambda>0:A_i(\lambda)\leqslant\frac{1}{m},\ i=1,\dots,m\right\}. \tag{38}$$

If $0<a\leqslant b$ and a positive function $A(\lambda)$ is decreasing for $\lambda>0$, then

$$\inf\{\lambda>0:A(\lambda)\leqslant b\}\leqslant\inf\{\lambda>0:A(\lambda)\leqslant a\}. \tag{39}$$

Assume that a positive function $A_i(\lambda)$ is decreasing for $\lambda>0$ for $i=1,2,\dots,m$. If $a_i>0$ for $i=1,2,\dots,m$, then

$$\inf\{\lambda>0:A_i(\lambda)\leqslant a_i,\ i=1,2,\dots,m\}=\min\{\inf\{\lambda>0:A_i(\lambda)\leqslant a_i\}:\ i=1,\dots,m\}. \tag{40}$$

We invoke the following lemma to show Theorem 3.11 (see [22, p. 165] and [28, p. 1094]).

LEMMA 4.10. *Suppose that $1\leqslant p\leqslant p_0<\infty$ and $Mf\in\mathscr{M}_p^{p_0}$. Then,*

$$\|f\|_{\mathscr{M}_1^{p_0}}\lesssim\left\|M^\sharp f\right\|_{\mathscr{M}_p^{p_0}}.$$

Note that $Mf\in\mathscr{M}_p^{p_0}$ implies that $f\in\mathscr{M}_1^{p_0}$. As in [26, Corollary 5 in p. 26], the ∇_2-condition implies the growth of Young functions.

LEMMA 4.11. *Let $1<p_0<\infty$. If $\Phi\in\nabla_2$ and $\Phi(t)\lesssim t^{p_0}$, then there exists $\alpha>1$ such that $\alpha<p_0$ and*

$$t^\alpha\lesssim\Phi(t)\quad\text{for}\quad t>1.$$

By (11), Lemmas 4.10 and 4.11, we obtain the following Lemma.

LEMMA 4.12. *Let $1<p_0<\infty$. For a Young function $\Phi(t)\lesssim t^{p_0}$, if $\Phi\in\nabla'$ and a measurable function f such that $Mf\in\mathscr{M}_\Phi^{p_0}$, then*

$$\|f\|_{\mathscr{M}_1^{p_0}}\lesssim\left\|M^\sharp f\right\|_{\mathscr{M}_\Phi^{p_0}}.$$

Proof of Lemma 4.12. By $\Phi\in\nabla'\subset\nabla_2$, there exists $\alpha>1$ such that $\alpha<p_0$ and $t^\alpha\lesssim\Phi(t)$. For $1<\alpha\leqslant p_0<\infty$, using Lemma 4.10, we obtain

$$\|f\|_{\mathscr{M}_1^{p_0}}\lesssim\left\|M^\sharp f\right\|_{\mathscr{M}_\alpha^{p_0}}\lesssim\left\|M^\sharp f\right\|_{\mathscr{M}_\Phi^{p_0}}.\quad\square$$

LEMMA 4.13. *Let* $0<\alpha<n$, $1<p\leqslant p_0<q_0<\infty$. *For* $t>1$,

$$\int_1^t \left(\frac{t}{s}\log\left(e+\frac{t}{s}\right)\right)^{\frac{q_0}{p_0}} \left(s^{p\cdot\frac{q_0}{p_0}}\log\left(e+s^{\frac{q_0}{p_0}}\right)\right)' ds \lesssim \left(t^p\log(e+t)\right)^{\frac{q_0}{p_0}}. \tag{41}$$

Proof. Let $B(t)=t\log(e+t)$, $\Phi_0(t)=t^p$, $\theta(t)=\log(e+t)$, $\Phi(t)=\Phi_0(t)\theta(t)$ and $\Psi(t)=\Phi\left(t^{\frac{q_0}{p_0}}\right)=t^{p\cdot\frac{q_0}{p_0}}\log\left(e+t^{\frac{q_0}{p_0}}\right)$.

Applying (2) for $\theta=1$ and $P=p-1-\varepsilon>0$, we obtain, for $0<\varepsilon<p-1$,

$$\int_1^t \frac{t}{s}\log\left(e+\frac{t}{s}\right)s^p\log(e+s)\frac{ds}{s} \lesssim t^{p-\varepsilon}\int_1^t s^{\varepsilon-1}\log(e+s)ds. \tag{42}$$

By integration by parts,

$$t^{p-\varepsilon}\int_1^t s^{\varepsilon-1}\log(e+s)ds \lesssim t^{p-\varepsilon}\cdot t^{\varepsilon}\log(e+t)=t^p\log(e+t)=\Phi(t). \tag{43}$$

Estimates (42), (43) and Proposition 3.8 give (41). □

5. Proofs of Theorems

Proof of Theorem 3.1. Without loss of generality, we may assume that $f(x)\geqslant 0$ a.e. $x\in\mathbb{R}^n$. Fix a dyadic cube Q_0. Let $f_0=f\chi_{3Q_0}$ and $f_\infty=f-f\chi_{3Q_0}$. Then,

$$\|I_\alpha(f)\|_{\Phi,Q_0}\leqslant\|I_\alpha(f_0)\|_{\Phi,Q_0}+\|I_\alpha(f_\infty)\|_{\Phi,Q_0}. \tag{44}$$

Using (32) in Lemma 4.1, we learn

$$\begin{aligned}\|I_\alpha(f_\infty)\|_{\Phi,Q_0} &\lesssim \sum_{k=1}^{\infty}\left\|\ell\left(3\cdot 2^kQ_0\right)^{\alpha}m_{3\cdot 2^kQ_0}(f)\right\|_{\Phi,3\cdot 2^kQ_0}\leqslant\sum_{k=1}^{\infty}\|M_\alpha(f)\|_{\Phi,3\cdot 2^kQ_0}\\ &\leqslant\|M_\alpha(f)\|_{\mathscr{M}_\Phi^{p_0}}\sum_{k=1}^{\infty}\left|3\cdot 2^kQ_0\right|^{-\frac{1}{p_0}}\lesssim|Q_0|^{-\frac{1}{p_0}}\|M_\alpha(f)\|_{\mathscr{M}_\Phi^{p_0}}.\end{aligned} \tag{45}$$

By Lemma 4.3,

$$\begin{aligned}\|I_\alpha(f_0)\|_{\Phi,Q_0} &\leqslant \|I_\alpha(f_0)\|_{\mathscr{L}_{Q_0}^{\Phi}\left(\frac{dx}{|Q_0|}\right)}\\ &=\sup\left\{\fint_{Q_0}I_\alpha(f_0)(x)|g(x)|dx:\|g\|_{\overline{\Phi},Q_0}\leqslant 1\right\}.\end{aligned} \tag{46}$$

Let $g \geqslant 0$ a.e. $x \in \mathbb{R}^n$ such that supp $(g) \subset Q_0$ and $\|g\|_{\overline{\Phi},Q_0} \leqslant 1$. By (31) in Lemma 4.1 and (33) in Lemma 4.2,

$$\begin{aligned} \fint_{Q_0} I_\alpha(f_0)(x)g(x)dx &\lesssim \sum_{Q \in \mathscr{D}(Q_0)} \ell(Q)^\alpha m_{3Q}(f) \frac{1}{|Q_0|} \int_Q g(y)dy \\ &= \frac{1}{|Q_0|} \left(\sum_{Q \in \mathscr{D}_0(Q_0)} + \sum_{k,j} \sum_{Q \in \mathscr{D}_{k,j}(Q_0)} \right) \ell(Q)^\alpha m_{3Q}(f) \int_Q g(y)dy \\ &= I_0 + \sum_{k,j} II_{k,j}. \end{aligned} \tag{47}$$

By Lemma 4.2,

$$I_0 \lesssim \frac{1}{|Q_0|} \int_{E_0} M_\alpha(f)(x) M\left(g\chi_{Q_0}\right)(x)dx \tag{48}$$

and

$$II_{k,j} \lesssim \frac{1}{|Q_0|} \int_{E_{k,j}} M_\alpha(f)(x) M\left(g\chi_{Q_0}\right)(x)dx. \tag{49}$$

Estimates (48) and (49) imply that

$$I_0 + \sum_{k,j} II_{k,j} \lesssim \fint_{Q_0} M_\alpha(f)(x) M\left(g\chi_{Q_0}\right)(x)dx. \tag{50}$$

By (13),

$$\fint_{Q_0} M_\alpha(f)(x) M\left(g\chi_{Q_0}\right)(x)dx \lesssim \|M_\alpha(f)\|_{\Phi,Q_0} \cdot \left\|M\left(g\chi_{Q_0}\right)\right\|_{\overline{\Phi},Q_0}. \tag{51}$$

Since $\Phi \in \triangle_2$, by Remark 1.14, $\overline{\Phi} \in \nabla_2$. By Lemma 4.5 and $\|g\|_{\overline{\Phi},Q_0} \leqslant 1$,

$$\|M_\alpha(f)\|_{\Phi,Q_0} \cdot \left\|M\left(g\chi_{Q_0}\right)\right\|_{\overline{\Phi},Q_0} \lesssim \|M_\alpha(f)\|_{\Phi,Q_0} \cdot \|g\|_{\overline{\Phi},Q_0} \leqslant \|M_\alpha(f)\|_{\Phi,Q_0}. \tag{52}$$

Estimates (46) -(52) and the definition of the norm $\|\cdot\|_{\mathscr{M}_\Phi^{p_0}}$ give

$$\|I_\alpha(f_0)\|_{\Phi,Q_0} \leqslant \|M_\alpha(f_0)\|_{\Phi,Q_0} \leqslant |Q_0|^{-\frac{1}{p_0}} \|M_\alpha(f_0)\|_{\mathscr{M}_\Phi^{p_0}}. \tag{53}$$

By (15), (44), (45) and (53), we obtain the desired result. $\square$

Proof of Theorem 3.3. For every cube Q_0, let $f_0 = f\chi_{3Q_0}$ and $f_\infty = f - f_0$. Firstly, we consider $M_{B,\alpha}(f_\infty)(x)$. By a routine argument, for $x \in Q_0$,

$$M_{B,\alpha} f_\infty(x) \lesssim \sup_{Q_0 \subset Q} \ell(Q)^\alpha \|f\|_{B,Q} \lesssim |Q_0|^{\frac{\alpha}{n} - \frac{1}{p_0}} \|f\|_{\mathscr{M}_B^{p_0}}.$$

Lemma 4.6 gives

$$|Q_0|^{\frac{1}{q_0}} \|M_{B,\alpha} f_\infty\|_{\Psi,Q_0} \lesssim \|f\|_{\mathscr{M}_\Phi^{p_0}}. \tag{54}$$

Next, we consider $M_{B,\alpha}(f_0)(x)$. We choose one function $F(x)$ such as the following:

$$F(x) \cong \frac{|f(x)|\chi_{3Q_0}(x)}{\ell(Q_0)^\alpha \left\|f\chi_{3Q_0}\right\|_{\Phi,3Q_0}},$$

We divide $\int_{Q_0} \Psi(M_{B,\alpha}F(x))\,dx$ into two parts I and II:

$$I := \int_0^1 \Psi'(\lambda)|\{x \in Q_0 : M_{B,\alpha}F(x) > \lambda\}|\,d\lambda$$

and

$$II := \int_1^\infty \Psi'(\lambda)|\{x \in Q_0 : M_{B,\alpha}F(x) > \lambda\}|\,d\lambda.$$

By $|\{x \in Q_0 : M_{B,\alpha}F(x) > \lambda\}| \leqslant |Q_0|$, the estimate of I is simple:

$$I \leqslant \Psi(1)|Q_0|. \tag{55}$$

Next, we evaluate II as follows. Lemma 4.4 gives

$$II \lesssim \frac{1}{\ell(Q_0)^{\alpha q_0}} \int_1^\infty \left(\int_{3Q_0} \Psi'(\lambda)^{\frac{p_0}{q_0}} B\left(\frac{C\ell(Q_0)^\alpha F(x)}{\lambda}\right)\chi_{\{x:\lambda<C\ell(Q_0)^\alpha F(x)\}}dx\right)^{\frac{q_0}{p_0}} d\lambda. \tag{56}$$

Lemma 4.7 for $\frac{q_0}{p_0} > 1$ gives the following estimates:

$$\begin{aligned}&\int_1^\infty \left(\int_{3Q_0} \Psi'(\lambda)^{\frac{p_0}{q_0}} B\left(\frac{C\ell(Q_0)^\alpha F(x)}{\lambda}\right)\chi_{\{x:\lambda<C\ell(Q_0)^\alpha F(x)\}}dx\right)^{\frac{q_0}{p_0}} d\lambda \\ &\leqslant \left(\int_{3Q_0}\left(\int_1^{C\ell(Q_0)^\alpha F(x)} \Psi'(\lambda) B\left(\frac{C\ell(Q_0)^\alpha F(x)}{\lambda}\right)^{\frac{q_0}{p_0}} d\lambda\right)^{\frac{p_0}{q_0}} dx\right)^{\frac{q_0}{p_0}}.\end{aligned} \tag{57}$$

By (20),

$$\begin{aligned}&\left(\int_{3Q_0}\left(\int_1^{C\ell(Q_0)^\alpha F(x)} \Psi'(\lambda) B\left(\frac{C\ell(Q_0)^\alpha F(x)}{\lambda}\right)^{\frac{q_0}{p_0}} d\lambda\right)^{\frac{p_0}{q_0}} dx\right)^{\frac{q_0}{p_0}} \\ &\lesssim |Q_0|^{\frac{q_0}{p_0}} \left(⨍_{3Q_0} \Phi((C\ell(Q_0)^\alpha F(x))\,dx\right)^{\frac{q_0}{p_0}} \\ &\cong |Q_0|^{\frac{q_0}{p_0}} \left(⨍_{3Q_0} \Phi\left(\frac{|f(x)|\chi_{3Q_0}(x)}{\left\|f\chi_{3Q_0}\right\|_{\Phi,3Q_0}}\right) dx\right)^{\frac{q_0}{p_0}} \leqslant |Q_0|^{\frac{q_0}{p_0}}.\end{aligned} \tag{58}$$

Since $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$, estimates (56)–(58) give

$$II \lesssim |Q_0|. \tag{59}$$

Since $\int_{Q_0} \Psi(M_{B,\alpha}F(x))\,dx = I + II$, estimates (55) and (59) show

$$\|M_{B,\alpha}F\|_{\Psi,Q_0} \leqslant 1,$$

that is,

$$\|M_{B,\alpha}(f_0)\|_{\Psi,Q_0} \lesssim \ell(Q_0)^{\alpha}\,\|f\|_{\Phi,3Q_0}\,. \tag{60}$$

By (60),

$$|Q_0|^{\frac{1}{q_0}}\,\|M_{B,\alpha}(f_0)\|_{\Psi,Q_0} \lesssim \|f\|_{\mathscr{M}_{\Phi}^{p_0}}\,. \tag{61}$$

Estimates (54) and (61) give the desired result. □

Proof of Theorem 3.11. For a constant $L > 1$, let

$$\|Mf\|_{\mathscr{M}_{\Phi,L}^{p_0}} := \sup_{\substack{Q\subset\mathbb{R}^n,\\ \ell(Q)\leqslant L}} |Q|^{\frac{1}{p_0}}\,\|\min\{L,Mf\}\|_{\Phi,Q}\,.$$

Let Q_0 a cube that satisfies $\ell(Q_0) \leqslant L$. Then,

$$\begin{aligned}
&|Q_0|^{\frac{1}{p_0}}\,\|\min\{L,Mf\}\|_{\Phi,Q_0} = 2|Q_0|^{\frac{1}{p_0}}\left\|\min\left\{\frac{L}{2},\frac{Mf}{2}\right\}\right\|_{\Phi,Q_0}\\
&= 2|Q_0|^{\frac{1}{p_0}}\inf\left\{\lambda > 0 : \frac{1}{|Q_0|}\int_0^{\infty}\Phi'(t)\,|\{x\in Q_0 : \min\{L,Mf(x)\} > 2\lambda t\}|\,dt \leqslant 1\right\}.
\end{aligned}$$

For the cube Q_0, we take $\tilde{M}$ in Lemma 4.8 and consider two cases
(I) $L \leqslant 2\tilde{M}$ (II) $L > 2\tilde{M}$.

(I) Since $|\{x \in Q_0 : \min\{L,Mf(x)\} > 2\lambda t\}| \leqslant |Q_0|$, Case (I) is simple:

$$\begin{aligned}
|Q_0|^{\frac{1}{p_0}}\,\|\min\{L,Mf\}\|_{\Phi,Q_0} &\leqslant 2|Q_0|^{\frac{1}{p_0}}\inf\left\{\lambda > 0 : \int_0^{\frac{L}{2\lambda}}\Phi'(t)dt \leqslant 1\right\}\\
&= \frac{|Q_0|^{\frac{1}{p_0}}L}{\Phi^{-1}(1)} \leqslant \frac{2|Q_0|^{\frac{1}{p_0}}\tilde{M}}{\Phi^{-1}(1)}.
\end{aligned}$$

Since $\tilde{M} \leqslant |Q_0|^{-\frac{1}{p_0}}\,\|f\|_{\mathscr{M}_1^{p_0}}$ (see [28, p. 1099]),

$$\frac{|Q_0|^{\frac{1}{p_0}}\tilde{M}}{\Phi^{-1}(1)} \leqslant \frac{1}{\Phi^{-1}(1)}\,\|f\|_{\mathscr{M}_1^{p_0}}\,. \tag{62}$$

(II) Let

$$A := \int_0^{\frac{\tilde{M}}{\lambda}} \Phi'(t) \left|\{x \in Q_0 : \min\{L, Mf(x)\} > 2\lambda t\}\right| dt$$

and

$$B := \int_{\frac{\tilde{M}}{\lambda}}^{\frac{L}{2\lambda}} \Phi'(t) \left|\{x \in Q_0 : \min\{L, Mf(x)\} > 2\lambda t\}\right| dt.$$

Since if $t \geqslant \dfrac{L}{2\lambda}$, then $\min\{L, Mf(x)\} \leqslant 2\lambda t$, we have

$$\int_0^{\infty} \Phi'(t) \left|\{x \in Q_0 : \min\{L, Mf(x)\} > 2\lambda t\}\right| dt = A + B. \tag{63}$$

Since $|\{x \in Q_0 : \min\{L, Mf(x)\} > 2\lambda t\}| \leqslant |Q_0|$, the estimate of A is simple:

$$A \leqslant |Q_0| \Phi\left(\frac{\tilde{M}}{\lambda}\right). \tag{64}$$

Dividing the set $\{x \in Q_0 : \min\{L, Mf(x)\} > 2\lambda t\}$ into two parts, we evaluate B:

$$\begin{aligned} B &= \int_{\frac{\tilde{M}}{\lambda}}^{\frac{L}{2\lambda}} \Phi'(t) \left|\left\{x \in Q_0 : \min\{L, Mf(x)\} > 2\lambda t, M^\sharp f(x) \leqslant \delta\lambda t\right\}\right| dt \\ &\quad + \int_{\frac{\tilde{M}}{\lambda}}^{\frac{L}{2\lambda}} \Phi'(t) \left|\left\{x \in Q_0 : \min\{L, Mf(x)\} > 2\lambda t, M^\sharp f(x) > \delta\lambda t\right\}\right| dt \\ &= I_{L,\tilde{M},\lambda} + II_{L,\tilde{M},\lambda}. \end{aligned} \tag{65}$$

Applying Lemma 4.8 for $I_{L,\tilde{M},\lambda}$ and $\lambda t > \tilde{M}$, we learn

$$\begin{aligned} I_{L,\tilde{M},\lambda} &\leqslant \int_{\frac{\tilde{M}}{\lambda}}^{\frac{L}{2\lambda}} \Phi'(t) \left|\left\{x \in Q_0 : Mf(x) > 2\lambda t, M^\sharp f(x) \leqslant \delta\lambda t\right\}\right| dt \\ &\lesssim \delta \int_{\frac{\tilde{M}}{\lambda}}^{\frac{L}{2\lambda}} \Phi'(t) \left|\{x \in 8Q_0 : Mf(x) > \lambda t\}\right| dt. \end{aligned} \tag{66}$$

Divide equally the cube $8Q_0$ into 16^n cubes $Q_1,\ Q_2, \ldots, Q_{16^n}$ with their volume $|Q_i| = |Q_0|/2^n$:

$$8Q_0 = \bigcup_{i=1}^{16^n} Q_i.$$

Then,

$$
\begin{aligned}
&\delta \int_{\frac{\tilde{M}}{\lambda}}^{\frac{L}{2\lambda}} \Phi'(t)\, |\{x \in 8Q_0 : Mf(x) > \lambda t\}|\, dt \\
&\lesssim \delta \int_0^{\frac{L}{\lambda}} \Phi'(t) \left| \bigcup_{i=1}^{16^n} \{x \in Q_i : Mf(x) > \lambda t\} \right| dt \\
&= \delta \sum_{i=1}^{16^n} \int_0^{\frac{L}{\lambda}} \Phi'(t)\, |\{x \in Q_i : Mf(x) > \lambda t\}|\, dt \\
&\cong \delta |Q_0| \sum_{i=1}^{16^n} \fint_{Q_i} \Phi\left(\min\left\{ \frac{L}{\lambda}, \frac{Mf(x)}{\lambda} \right\} \right) dx.
\end{aligned} \tag{67}
$$

Note that $\delta = \Phi\left(\Phi^{-1}(\delta)\right)$. Together with the fact that $\Phi \in \nabla'$ estimates (66) and (67) give

$$
I_{L,\tilde{M},\lambda} \lesssim |Q_0| \sum_{i=1}^{16^n} \fint_{Q_i} \Phi\left(\Phi^{-1}(\delta) \min\left\{ \frac{L}{\lambda}, \frac{Mf(x)}{\lambda} \right\} \right) dx. \tag{68}
$$

Next, we evaluate $II_{L,\tilde{M},\lambda}$. By $\left\{x \in Q_0 : \min\{L, Mf(x)\} > 2\lambda t, M^\sharp f(x) > \delta\lambda t\right\} \subset \left\{x \in Q_0 : M^\sharp f(x) > \delta\lambda t\right\}$,

$$
\begin{aligned}
II_{L,\tilde{M},\lambda} &\leqslant \int_{\frac{\tilde{M}}{\lambda}}^{\frac{L}{2\lambda}} \Phi'(t) \left| \left\{ x \in Q_0 : M^\sharp f(x) > \delta\lambda t \right\} \right| dt \\
&\leqslant \int_0^{\infty} \Phi'(t) \left| \left\{ x \in Q_0 : M^\sharp f(x) > \delta\lambda t \right\} \right| dt \\
&= \int_{Q_0} \Phi\left(\frac{M^\sharp f(x)}{\delta\lambda} \right) dx.
\end{aligned} \tag{69}
$$

Estimates (64), (65), (68) and (69) give

$$
\begin{aligned}
&\inf\left\{ \lambda > 0 : \frac{1}{|Q_0|} \int_0^{\infty} \Phi'(t)\, |\{x \in Q_0 : \min\{L, Mf(x)\} > 2\lambda t\}|\, dt \leqslant 1 \right\} \\
&\lesssim \inf\left\{ \lambda > 0 : \Phi\left(\frac{\tilde{M}}{\lambda} \right) + \sum_{i=1}^{16^n} \fint_{Q_i} \Phi\left(\Phi^{-1}(\delta) \min\left\{ \frac{L}{\lambda}, \frac{Mf(x)}{\lambda} \right\} \right) dx \right. \\
&\qquad\qquad \left. + \fint_{Q_0} \Phi\left(\frac{M^\sharp f(x)}{\delta\lambda} \right) dx \leqslant 1 \right\}.
\end{aligned} \tag{70}
$$

Since $0 < \frac{1}{32^n} \leqslant \frac{1}{2+16^n}$, we use estimates (38) and (39) in Lemma 4.9:

$$\begin{aligned}
&\inf\left\{\lambda>0:\Phi\left(\frac{\tilde{M}}{\lambda}\right)+\sum_{i=1}^{16^n}\fint_{Q_i}\Phi\left(\Phi^{-1}(\delta)\min\left\{\frac{L}{\lambda},\frac{Mf(x)}{\lambda}\right\}\right)dx\right.\\
&\qquad\left.+\fint_{Q_0}\Phi\left(\frac{M^\sharp f(x)}{\delta\lambda}\right)dx\leqslant 1\right\}.\\
&\leqslant\inf\left\{\lambda>0:\Phi\left(\frac{\tilde{M}}{\lambda}\right)\leqslant\frac{1}{32^n},\fint_{Q_0}\Phi\left(\frac{M^\sharp f(x)}{\delta\lambda}\right)dx\leqslant\frac{1}{32^n},\right.\\
&\qquad\fint_{Q_i}\Phi\left(\Phi^{-1}(\delta)\min\left\{\frac{L}{\lambda},\frac{Mf(x)}{\lambda}\right\}\right)dx\leqslant\frac{1}{32^n},\\
&\qquad\left.i=1,2,\dots,16^n\right\}.
\end{aligned}\tag{71}$$

Estimate (40) in Lemma 4.9 gives

$$\begin{aligned}
&\inf\left\{\lambda>0:\Phi\left(\frac{\tilde{M}}{\lambda}\right)\leqslant\frac{1}{32^n},\fint_{Q_0}\Phi\left(\frac{M^\sharp f(x)}{\delta\lambda}\right)dx\leqslant\frac{1}{32^n},\right.\\
&\qquad\left.\fint_{Q_i}\Phi\left(\Phi^{-1}(\delta)\min\left\{\frac{L}{\lambda},\frac{Mf(x)}{\lambda}\right\}\right)dx\leqslant\frac{1}{32^n},\ i=1,2,\dots,16^n\right\}\\
&=\min\left\{\inf\left\{\lambda>0:\Phi\left(\frac{\tilde{M}}{\lambda}\right)\leqslant\frac{1}{32^n}\right\},\right.\\
&\qquad\inf\left\{\lambda>0:\fint_{Q_0}\Phi\left(\frac{M^\sharp f(x)}{\delta\lambda}\right)dx\leqslant\frac{1}{32^n}\right\},\\
&\qquad\inf\left\{\lambda>0:\fint_{Q_i}\Phi\left(\Phi^{-1}(\delta)\min\left\{\frac{L}{\lambda},\frac{Mf(x)}{\lambda}\right\}\right)dx\leqslant\frac{1}{32^n}\right\},\\
&\qquad\left.i=1,2,\dots,16^n\right\}.
\end{aligned}\tag{72}$$

On the other hand, note that

$$\begin{aligned}
&\inf\left\{\lambda>0:\Phi\left(\frac{\tilde{M}}{\lambda}\right)\leqslant\frac{1}{32^n}\right\}=\Phi^{-1}\left(\frac{1}{32^n}\right)^{-1}\cdot\tilde{M},\\
&\inf\left\{\lambda>0:\fint_{Q_0}\Phi\left(\frac{M^\sharp f(x)}{\delta\lambda}\right)dx\leqslant\frac{1}{32^n}\right\}\leqslant\frac{32^n}{\delta}\left\|M^\sharp f\right\|_{\Phi,Q_0},
\end{aligned}\tag{73}$$

and for $i=1,2,\dots,16^n$,

$$\begin{aligned}
&\inf\left\{\lambda>0:\fint_{Q_i}\Phi\left(\Phi^{-1}(\delta)\min\left\{\frac{L}{\lambda},\frac{Mf(x)}{\lambda}\right\}\right)dx\leqslant\frac{1}{32^n}\right\}\\
&\leqslant 32^n\Phi^{-1}(\delta)\left\|\min\{L,Mf\}\right\|_{\Phi,Q_i}.
\end{aligned}\tag{74}$$

By estimates (73) and (74),

$$\begin{aligned}
&\inf\left\{\lambda>0:\frac{1}{|Q_0|}\int_0^{\infty}\Phi'(t)\,|\{x\in Q_0:\min\{L,Mf(x)\}>2\lambda t\}|\,dt\leqslant 1\right\}\\
&\leqslant \min\left\{\Phi^{-1}\left(\frac{1}{32^n}\right)^{-1}\cdot\tilde{M},\frac{32^n}{\delta}\left\|M^{\sharp}f\right\|_{\Phi,Q_0},32^n\Phi^{-1}(\delta)\left\|\min\{L,Mf\}\right\|_{\Phi,Q_i},\right.\\
&\qquad\qquad\left. i=1,2,\ldots,16^n\right\}\\
&\leqslant \min\left\{\Phi^{-1}\left(\frac{1}{32^n}\right)^{-1}\cdot\tilde{M},\frac{32^n}{\delta}\left\|M^{\sharp}f\right\|_{\Phi,Q_0}\right\}\\
&\quad+32^n\Phi^{-1}(\delta)\sum_{i=1}^{16^n}\left\|\min\{L,Mf\}\right\|_{\Phi,Q_i}.
\end{aligned}\tag{75}$$

Since $\tilde{M}\leqslant |Q_0|^{-\frac{1}{p_0}}\|f\|_{\mathscr{M}_1^{p_0}}$, $\left\|M^{\sharp}f\right\|_{\Phi,Q_0}\lesssim |Q_0|^{-\frac{1}{p_0}}\left\|M^{\sharp}f\right\|_{\mathscr{M}_\Phi^{p_0}}$ and $\|\min\{L,Mf\}\|_{\Phi,Q_i}\lesssim |Q_0|^{-\frac{1}{p_0}}\|Mf\|_{\mathscr{M}_{\Phi,L}^{p_0}}$, a constant $C_0>0$ exists such that

$$\begin{aligned}
&\min\left\{\Phi^{-1}\left(\frac{1}{32^n}\right)^{-1}\cdot\tilde{M},\frac{32^n}{\delta}\left\|M^{\sharp}f\right\|_{\Phi,Q_0}\right\}+32^n\Phi^{-1}(\delta)\sum_{i=1}^{16^n}\left\|\min\{L,Mf\}\right\|_{\Phi,Q_i}\\
&\leqslant C_0|Q_0|^{-\frac{1}{p_0}}\left(\min\left\{\|f\|_{\mathscr{M}_1^{p_0}},\frac{1}{\delta}\left\|M^{\sharp}f\right\|_{\mathscr{M}_\Phi^{p_0}}\right\}+\Phi^{-1}(\delta)\|Mf\|_{\mathscr{M}_{\Phi,L}^{p_0}}\right).
\end{aligned}\tag{76}$$

Estimates (63)–(76) show that

$$\begin{aligned}
|Q_0|^{\frac{1}{p_0}}\|\min\{L,Mf\}\|_{\Phi,Q_0}\leqslant C_0\min\left\{\|f\|_{\mathscr{M}_1^{p_0}},\frac{1}{\delta}\left\|M^{\sharp}f\right\|_{\mathscr{M}_\Phi^{p_0}}\right\}&\\
+C_0\Phi^{-1}(\delta)\|Mf\|_{\mathscr{M}_{\Phi,L}^{p_0}},&
\end{aligned}$$

that is,

$$\left(1-C_0\Phi^{-1}(\delta)\right)\|Mf\|_{\mathscr{M}_{\Phi,L}^{p_0}}\leqslant C_0\min\left\{\|f\|_{\mathscr{M}_1^{p_0}},\frac{1}{\delta}\left\|M^{\sharp}f\right\|_{\mathscr{M}_\Phi^{p_0}}\right\}.$$

Since we can choose $\delta>0$ such as $C_0\Phi^{-1}(\delta)<1$, we obtain

$$\|Mf\|_{\mathscr{M}_{\Phi,L}^{p_0}}\leqslant\frac{C_0}{1-C_0\Phi^{-1}(\delta)}\min\left\{\|f\|_{\mathscr{M}_1^{p_0}},\frac{1}{\delta}\left\|M^{\sharp}f\right\|_{\mathscr{M}_\Phi^{p_0}}\right\}.\tag{77}$$

Estimates (62) and (77) give

$$\|Mf\|_{\mathscr{M}_\Phi^{p_0}}\lesssim\|f\|_{\mathscr{M}_1^{p_0}}+\left\|M^{\sharp}f\right\|_{\mathscr{M}_\Phi^{p_0}}.$$

By Lemma 4.12, we obtain Theorem 3.11. □

REMARK 5.1. By $\lim_{\delta \to 0+0} \Phi^{-1}(\delta) = 0$, we can choose $\delta > 0$ so that $C_0 \Phi^{-1}(\delta) < 1$.

Proof of Theorem 1.4. Since $|[b, I_\alpha]f(x)| \leqslant M([b, I_\alpha]f)(x)$ a.e. $x \in \mathbb{R}^n$,

$$\|[b, I_\alpha]f\|_{\mathscr{M}_\Psi^{q_0}} \leqslant \|M([b, I_\alpha]f)\|_{\mathscr{M}_\Psi^{q_0}}.$$

Note that we assume that $\Psi \in \triangle_2 \cap \nabla'$. Then, Theorems 3.1, 3.11 and Proposition 2.8 show that

$$\begin{aligned}\|M([b, I_\alpha]f)\|_{\mathscr{M}_\Psi^{q_0}} &\lesssim \left\|M^\sharp([b, I_\alpha]f)\right\|_{\mathscr{M}_\Psi^{q_0}} \lesssim \|b\|_{BMO} \left(\|I_\alpha f\|_{\mathscr{M}_\Psi^{q_0}} + \|M_{B,\alpha} f\|_{\mathscr{M}_\Psi^{q_0}}\right) \\ &\lesssim \|b\|_{BMO} \|M_{B,\alpha} f\|_{\mathscr{M}_\Psi^{q_0}},\end{aligned}$$

here, $B(t) = t \log(e + t)$. Theorem 3.3 gives the desired result. □

Acknowledgement. The author is grateful to Professor Marjan Praljak referee having checked the manuscript carefully. Furthermore, the author would like to thank Professor Y. Sawano for carefully reviewing the revised paper and providing valuable feedback and suggestions that have improved the quality of this paper.

REFERENCES

[1] D. ADAMS, *A note on Riesz potentials*, Duke Math. J., 42, 1975, 765–778.

[2] E. BURTSEVA, L. MALIGRANDA AND K. MATSUOKA, *Boundedness of the Riesz potential in central Morrey-Orlicz spaces*, Positivity, 26, 1, 22, 2022, 26 pp.

[3] D. CRUZ-URIBE, SFO AND A. FIORENZA, *Endpoint estimates and weighted norm inequalities for commutators of fractional integrals*, Publ. Mat. 47, 2003, 103–131.

[4] D. CRUZ-URIBE, SFO AND K. MOEN, *A fractional Muckenhoupt-Wheeden theorem and its consequences*, Integral Equations Operator Theory, 76, 2013, 3, 421–446.

[5] D. CRUZ-URIBE, SFO, JOSÉ MARIA MARTELL AND C. PÉREZ, *Weights, extrapolation and the theory of Rubio de Francia. Operator Theory: Advances and Applications*, 215. Birkhäuser/Springer Basel AG, Basel, 2011. xiv+280 pp.

[6] D. CRUZ-URIBE, SFO AND S. RODNEY, *Bounded weak solutions to elliptic PDE with data in Orlicz spaces*, J. Diff. Eq., 297, 2021, 409–432.

[7] F. DERINGOZ, V. S. GULIYEV AND S. G. HASANOV, *Maximal operator and its commutators on generalized weighted Orlicz-Morrey spaces*, Tokyo J. Math., 41, 2, 2018, pp. 347–369, doi:10.3836/tjm/1502179260.

[8] G. DI FAZIO AND M. A. RAGUSA, *Commutators and Morrey spaces*, Boll. Un. Mat. Ital., A 7, 5, 1991, 3, 323–332.

[9] C. FEFFERMAN AND E. M. STEIN, *H^p spaces of several variables*, Acta Math., 129, 1972, 137–193.

[10] S. GALA, Y. SAWANO AND H. TANAKA, *A remark on two generalized Orlicz-Morrey spaces*, J. Approx. Theory, 198 2015, 1–9.

[11] L. GRAFAKOS, *Classical Fourier analysis*, Third Edition, Graduate Texts in Math., no. 249, Springer, New York, 2014. xviii+638 pp.

[12] L. GRAFAKOS, *Modern Fourier analysis*, Third Edition, Graduate Texts in Math., no. 250, Springer, New York, 2014. xvi+624pp.

[13] V. S. GULIYEV, S. G. HASANOV, Y. SAWANO AND T. NOI, *Non-smooth atomic decompositions for generalized Orlicz-Morrey spaces of the third kind*, Acta Appl. Math., 145, 2016, 137–174, doi:10.1007/s10440-016-0052-7.

[14] N. HATANO, R. KAWASUMI, Y. SAWANO, *Sparse non-smooth atomic decomposition of quasi-Banach lattices*, J. Fourier Anal. Appl. 28, 2022, 4, no. 61, 21 pp.

[15] T. IIDA, *Orlicz-fractional maximal operators in Morrey and Orlicz-Morrey Spaces*, Positivity, 25, 2021, 243–272, doi:10.1007/s11117-020-00762-w.

[16] T. IIDA AND Y. SAWANO, *Orlicz-fractional maximal operators on weighted L^p spaces*, J. Math. Inequal., 13, 2019, 2, 369–413.

[17] Y. KOMORI AND T. MIZUHARA, *Notes on commutators and Morrey spaces*, Hokkaido Math. J. 32, 2003, 2, 345–353.

[18] L. MALIGRANDA AND K. MATSUOKA, *Maximal function in Beurling-Orlicz and central Morrey-Orlicz spaces*, Colloq. Math. 138, 2, 2015, 165–181.

[19] L. MALIGRANDA AND K. MATSUOKA, *Calderón-Zygmund singular integrals in central Morrey spaces*, Tohoku. Math. J., 2, 72, 2, 2020, 235–259.

[20] C. B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. 43, 1938, 1, 126–166.

[21] E. NAKAI, *Generalized fractional integrals on Orlicz-Morrey spaces*, Banach and function spaces, Yokohama publ., Yokohama, 2004, 323–333.

[22] S. NAKAMURA AND Y. SAWANO, *The singular integral operator and its commutator on weighted Morrey spaces*, Collect. Math., 68, 2017, 2, 145–174, doi:10.1007/s13348-017-0193-7.

[23] W. ORLICZ, *Über eine gewisse Klasse von Räumen vom Typus B*, Bull. Acad. Polonaise. A (1932), 207–220; reprinted in his Collected Papers, PWN, Warszawa 1988, 217–230 (German).

[24] W. ORLICZ, *Über Räume(L^M)*, Bull. Acad. Polonaise A (1936), 93–107; reprinted in his Collected Papers, PWN, Warszawa (1988), 345–359 (German).

[25] C. PÉREZ, *On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p-spaces with different weights*, Proc. London Math. Soc. 3, 71, 1, 1995, 135–157.

[26] M. M. RAO AND Z. D. REN, *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 146, Marcel Dekker Inc., New York, 1991. xii+449 pp.

[27] Y. SAWANO AND D. I. HAKIM, *Complex interpolation and commutators acting on Morrey spaces*, Rom. J. Math. Comput. Sci. 11, 2021, 1, 10–24.

[28] Y. SAWANO AND H. TANAKA, *Sharp maximal inequalities and commutators on Morrey spaces with non-doubling measures*, Taiwanese J. Math, 11, 4, 2007, 1091–1112.

[29] Y. SAWANO, G. DI. FAZIO AND D. I. HAKIM, *Morrey spaces-introduction and applications to integral operators and PDE's*, Vol. I. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2020. xxi+479 pp.

[30] Y. SAWANO, G. DI. FAZIO AND D. I. HAKIM, *Morrey spaces-introduction and applications to integral operators and PDE's*, Vol. II. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2020. xxi+409 pp.

[31] Y. SAWANO, S. SUGANO AND H. TANAKA, *Orlicz-Morrey spaces and fractional operators*, Potential Analysis, 36, 4, 2012, 517–556.

(Received August 2, 2022)

Takeshi Iida
Department of General Education
National Institute of Technology, Fukushima College
Nagao 30, Kamiarakawa-aza, Taira, Iwaki city,
Fukushima-Ken, 970-8034, Japan
e-mail: tiida@fukushima-nct.ac.jp