

CONSTRUCTING RIESZ–FISCHER SEQUENCES FROM A MINIMAL SEQUENCE IN A HILBERT SPACE \mathcal{H}

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Abstract. In this paper we prove that if $U = \{u_n\}$ is a minimal sequence in a separable Hilbert space \mathcal{H} , then multiplying each vector u_n by an appropriate constant c_n , yields a family of functions $P := \{c_n \cdot u_n\}$, such that P is a Riesz-Fischer sequence in \mathcal{H} .

If U is a minimal and complete sequence in \mathcal{H} , therefore having a unique biorthogonal sequence $V = \{v_n\}$ in \mathcal{H} , then P is a complete Riesz-Fischer sequence in \mathcal{H} and its unique biorthogonal family $\{v_n/\overline{c_n}\}$ is a Bessel sequence in \mathcal{H} .

1. Introduction and result

Let \mathcal{H} be a separable Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\|\cdot\|$. Let $U := \{u_n\}_{n \in J}$ be a countable family of vectors in \mathcal{H} with $J \subset \mathbb{Z}$. We say that

(i) U is *complete* if the closed span of U in \mathcal{H} is equal to \mathcal{H} .

(ii) U is *minimal* if each u_n does not belong to the closed span of the remaining vectors of U in \mathcal{H} . That is, denoting the *distance* of u_n from $\overline{\text{span}}(U \setminus u_n)$ in \mathcal{H} by

$$D_n := \inf_{g \in \overline{\text{span}}(U \setminus u_n)} \|u_n - g\|, \quad (1)$$

then $D_n > 0$ for all $n \in J$.

REMARK 1. It is well known that $\{u_n\}_{n \in J}$ is a minimal sequence in \mathcal{H} if and only if it has a biorthogonal sequence $\{v_n\}_{n \in J}$ in \mathcal{H} , that is

$$\langle v_n, u_m \rangle = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

An exact sequence in \mathcal{H} , that is a sequence which is both complete and minimal, has a unique biorthogonal sequence in \mathcal{H} .

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(iii) U is uniformly minimal if

$$\inf_{n \in J} \frac{D_n}{\|u_n\|} > 0. \quad (2)$$

(iv) U is a Bessel sequence if there is a positive constant B so that the following upper frame condition holds:

$$\sum_{n \in J} |\langle f, u_n \rangle|^2 \leq B \cdot \|f\|^2 \quad \forall f \in \mathcal{H}.$$

REMARK 2. One may consult the books of Young [11], Christensen, [3], and Heil [4] for further reading on the above topics.

We also say that $U = \{u_n\}_{n \in J}$ is a Riesz-Fischer sequence (see [11, Chapter 4, Section 2]) if the moment problem

$$\langle f, u_n \rangle = a_n \quad n \in J$$

has a solution $f \in \mathcal{H}$ for every sequence $\{a_n\}_{n \in J}$ in the space $l^2(J)$ where

$$l^2(J) := \left\{ \{a_n\}_{n \in J} : \sum_{n \in J} |a_n|^2 < \infty \right\}.$$

We point out that (see [11, Chapter 4, Section 2, Theorem 3]) U is a Riesz-Fischer sequence in \mathcal{H} if and only if there exists a positive number A so that for any finite scalar sequence $\{\beta_n\}$ we have

$$A \sum |\beta_n|^2 \leq \left\| \sum \beta_n u_n \right\|^2,$$

a result attributed to Nina Bari.

REMARK 3. It follows from the above inequality and (1) that a Riesz-Fischer sequence is also a minimal sequence.

Our goal in this article is to show that we can always construct a Riesz-Fischer sequence from a given minimal sequence U in \mathcal{H} , if we multiply each vector u_n by an appropriate constant c_n . We will prove the following result.

THEOREM 1. Let $U = \{u_n\}_{n \in J}$ be a minimal sequence in \mathcal{H} and let $\{D_n\}_{n \in J}$ be the Distances as in (1). Choose numbers $\{c_n\}_{n \in J}$ so that

$$\sum_{n \in J} \frac{1}{D_n \cdot |c_n|} < \infty. \quad (3)$$

Then the family

$$P := \{p_n : p_n = c_n \cdot u_n\}_{n \in J}$$

is a Riesz-Fischer sequence in the closed span of P in \mathcal{H} . Moreover, there is some $A > 0$ so that the following lower frame condition holds:

$$A \cdot \|f\|^2 \leq \sum_{n \in J} |\langle f, p_n \rangle|^2 \quad \forall f \in \overline{\text{span}}(P) \text{ in } \mathcal{H}. \quad (4)$$

The proof of Theorem 1 is given in Section 3 followed by some corollaries in case U is an exact sequence or uniformly minimal. The article ends with two examples in Section 4.

2. Connecting Riesz-Fischer sequences with Bessel sequences

In Casazza et al. [2], we find the following nice connection between Bessel sequences and Riesz-Fischer sequences.

PROPOSITION A. [2, Proposition 2.3, (ii)] *The Riesz-Fischer sequences in \mathcal{H} are precisely the families for which a biorthogonal Bessel sequence exists.*

Combining Proposition A with [3, Proposition 3.5.4] gives the following sufficient condition so that two biorthogonal families $\{v_n\}_{n \in J}$ and $\{u_n\}_{n \in J}$ are Bessel and Riesz-Fischer sequences respectively. We point out that Lemma 1 plays a crucial role for proving Theorem 1.

LEMMA 1. *Consider two biorthogonal families $\{u_n\}_{n \in J}$ and $\{v_n\}_{n \in J}$ in \mathcal{H} and suppose there is some $M > 0$ so that*

$$\sum_{n \in J} |\langle v_n, v_m \rangle| < M \quad \text{for all } m \in J.$$

Then $\{v_n\}_{n \in J}$ is a Bessel sequence in \mathcal{H} and $\{u_n\}_{n \in J}$ is a Riesz-Fischer sequence in \mathcal{H} .

We also note that the lower frame bound (4) follows from Casazza et al. [2, Theorem 3.2], restated below.

THEOREM A. *Suppose that a family $U = \{u_n\}_{n \in J}$ in \mathcal{H} is a complete Riesz-Fischer sequence in \mathcal{H} . It then satisfies the following lower frame condition: there is some $A > 0$ so that*

$$A \cdot \|f\|^2 \leq \sum_{n \in J} |\langle f, u_n \rangle|^2 \quad \forall f \in \mathcal{H}.$$

REMARK 4. Some other interesting results on general sequences in \mathcal{H} , including Bessel sequences and Riesz-Fischer sequences, classifying them by frame-related operators are given in [1]. We point out that a sequence which is both Bessel and Riesz-Fischer is called a Riesz sequence (see Seip [8, p. 138]). It is well known, that a complete Riesz sequence in \mathcal{H} is a Riesz basis for \mathcal{H} . A nice characterization of such bases was given recently by Stoeva [9].

3. Proof of Theorem 1 and some Corollaries

3.1. Proof of Theorem 1

First we establish the known result that a minimal sequence has a biorthogonal sequence.

Consider a minimal family U in \mathcal{H} and let $\{D_n\}_{n \in J}$ be the Distances as in (1). Then $\overline{\text{span}}(U \setminus u_n)$ in \mathcal{H} is a proper closed subspace of \mathcal{H} , hence there exists a unique element in $\overline{\text{span}}(U \setminus u_n)$ in \mathcal{H} that we denote by q_n , so that

$$D_n = \|u_n - q_n\|.$$

The function $u_n - q_n$ is orthogonal to all the elements of the closed span of $U \setminus u_n$ in \mathcal{H} , hence to q_n itself. Therefore

$$\langle u_n - q_n, u_n - q_n \rangle = \langle u_n - q_n, u_n \rangle.$$

Hence

$$(D_n)^2 = \langle u_n - q_n, u_n \rangle.$$

Define now

$$v_n(t) := \frac{u_n(t) - q_n(t)}{(D_n)^2}.$$

It then follows that $\langle v_n, u_n \rangle = 1$ and v_n is orthogonal to all the elements of the system $U \setminus u_n$. Thus, the family $\{v_n : n \in J\}$ is biorthogonal to the family U in \mathcal{H} . Since $q_n \in \overline{\text{span}}(U \setminus u_n)$ in \mathcal{H} , then $v_n \in \overline{\text{span}}(U)$ in \mathcal{H} . One also has

$$\|v_n\| = \frac{1}{D_n}.$$

Next, for every $n \in J$ define

$$r_n(t) := \frac{v_n(t)}{c_n}.$$

Then clearly the family $\{r_n : n \in J\}$ is biorthogonal to the family $P = \{c_n \cdot u_n : n \in J\}$ in \mathcal{H} . Also, since $v_n \in \overline{\text{span}}(U)$ in \mathcal{H} , then $r_n \in \overline{\text{span}}(P)$ in \mathcal{H} as well. Moreover, one has

$$\|r_n\| = \frac{1}{D_n \cdot |c_n|}.$$

Hence

$$|\langle r_n, r_m \rangle| \leq \frac{1}{D_n \cdot |c_n|} \cdot \frac{1}{D_m \cdot |c_m|} \quad \forall n, m \in J.$$

Now, for every fixed $n \in J$ consider the series

$$\sum_{m \in J} |\langle r_n, r_m \rangle|.$$

We then get

$$\sum_{m \in J} |\langle r_n, r_m \rangle| < \frac{1}{D_n \cdot |c_n|} \cdot \sum_{m \in J} \frac{1}{D_m \cdot |c_m|}.$$

Condition (3) implies the existence of a positive number M so that

$$\sum_{m \in J} |\langle r_n, r_m \rangle| < M \quad \text{for all } n \in J.$$

Since $\overline{\text{span}}(P)$ in \mathcal{H} is itself a Hilbert space and each r_n belongs to $\overline{\text{span}}(P)$ in \mathcal{H} , it then follows by Lemma 1 that $\{r_n\}$ and $\{p_n\}$ are Bessel and Riesz-Fischer sequences respectively in $\overline{\text{span}}(P)$ in \mathcal{H} .

Moreover, since the family P is complete in $\overline{\text{span}}(P)$ in \mathcal{H} , then the lower frame condition (4) follows from Theorem A. The proof of Theorem 1 is now complete.

3.2. Riesz-Fischer sequences from exact sequences or from uniformly minimal sequences

Next we state two corollaries of Theorem 1 in case $U = \{u_n\}_{n \in J}$ is not just a minimal sequence in \mathcal{H} , but it is either exact or uniformly minimal.

Firstly, from Theorem 1, Proposition A, and the definition of a Bessel sequence, we get the following result.

COROLLARY 1. *Let $U = \{u_n\}_{n \in J}$ be an exact sequence in \mathcal{H} , therefore it has a unique biorthogonal sequence $V = \{v_n\}_{n \in J}$. Let the sequence $\{c_n\}_{n \in J}$ satisfy (3). Then the family $\{c_n \cdot u_n\}_{n \in J}$ is an exact Riesz-Fischer sequence in \mathcal{H} and its unique biorthogonal family $\{\frac{v_n}{c_n}\}_{n \in J}$ is a Bessel sequence in \mathcal{H} . Thus, there are some positive constants A and B so that the following lower frame and upper frame conditions hold:*

$$\sum_{n \in J} |\langle f, c_n \cdot u_n \rangle|^2 \geq A \cdot \|f\|^2 \quad \forall f \in \mathcal{H},$$

and

$$\sum_{n \in J} |\langle f, \frac{v_n}{c_n} \rangle|^2 \leq B \cdot \|f\|^2 \quad \forall f \in \mathcal{H}.$$

Secondly, if U is a uniformly minimal sequence, combining Theorem 1 with (2) gives the following.

COROLLARY 2. *Let $U = \{u_n\}_{n \in J}$ be a uniformly minimal sequence in \mathcal{H} and choose numbers $\{c_n\}_{n \in J}$ so that*

$$\sum_{n \in J} \frac{1}{\|u_n\| \cdot |c_n|} < \infty.$$

Then the family $P = \{c_n \cdot u_n\}_{n \in J}$ is a Riesz-Fischer sequence in $\overline{\text{span}}(P)$ in \mathcal{H} .

If in addition $\inf_{n \in J} \|u_n\| > 0$, then the family P is a Riesz-Fischer sequence in $\overline{\text{span}}(P)$ in \mathcal{H} for any sequence $\{c_n\}_{n \in J}$ such that $\{1/c_n\}_{n \in J}$ belongs to the space $l^1(J)$ where

$$l^1(J) := \left\{ \{a_n\}_{n \in J} : \sum_{n \in J} |a_n| < \infty \right\}.$$

4. Examples

We end this paper with some examples involving either exponential systems $\{e^{i\lambda_n t}\}$ or $\{e^{\lambda_n t}\}$, $\lambda_n \in \mathbb{R}$, in the classical $L^2(a, b)$ spaces.

4.1. Example 1

Consider the exponential system $\{e^{i\lambda_n t}\}_{-\infty}^{\infty}$ where

$$\lambda_n = \begin{cases} n + \frac{1}{4} & n > 0 \\ 0 & n = 0 \\ n - \frac{1}{4} & n < 0 \end{cases}.$$

The system $\{e^{i\lambda_n t}\}_{n \in \mathbb{Z}}$ is a uniformly minimal sequence in $L^2(-\pi, \pi)$ (see [7, Theorem 5]) and it is also complete in $L^2(-\pi, \pi)$ (see [11, Chapter 3, Section 2, Theorem 4]). Hence it is exact in $L^2(-\pi, \pi)$.

REMARK 5. We note that by [10] the unique biorthogonal family to an exact exponential system $\{e^{i\lambda_n t}\}_{n \in \mathbb{Z}}$ in $L^2(-\pi, \pi)$ is itself exact.

We also point out that the system $\{e^{i\lambda_n t}\}_{n \in \mathbb{Z}}$ is not a Riesz-Fischer sequence: if it were, it would also be a Bessel sequence (see [5, Proposition 1]). Combined with its completeness would mean that the system is a Riesz basis for $L^2(-\pi, \pi)$ but this is not the case (see [7, Theorem 4]).

On the other hand, it follows by Corollary 2 that the family

$$\{c_n \cdot e^{i\lambda_n t}\}_{n \in \mathbb{Z}} \quad \text{where} \quad \{1/c_n\}_{n \in \mathbb{Z}} \in l^1(\mathbb{Z})$$

is an exact Riesz-Fischer sequence in $L^2(-\pi, \pi)$. In other words, for every sequence $\{a_n\}_{-\infty}^{\infty}$ in the space $l^2(\mathbb{Z})$, there exists a function $f \in L^2(-\pi, \pi)$ so that

$$\int_{-\pi}^{\pi} f(t) \cdot c_n \cdot e^{i\lambda_n t} dt = a_n \quad n \in \mathbb{Z}.$$

4.2. Example 2

Let $\{\lambda_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive real numbers, diverging to infinity, satisfying the following two conditions:

- (I) $\sum_{n=1}^{\infty} 1/\lambda_n < \infty$.
- (II) There is some $c > 0$ so that $\lambda_{n+1} - \lambda_n > c$ for all $n \in \mathbb{N}$.

Assuming these, and inspired by the celebrated Müntz-Szász theorem, Luxemburg and Korevaar [6], studied the properties of the exponential system $\{e^{\lambda_n t}\}_{n=1}^{\infty}$ in the spaces $L^p(a, b)$ for $p \geq 1$ and $-\infty < a < b < \infty$. They proved (see [6, relation (1.9)]) that the distance D_n of the function $e^{\lambda_n t}$ from the closed span of the remaining exponential functions in $L^2(a, b)$, satisfies the following lower bound: for every $\varepsilon > 0$, there is a positive constant m_ε which does not depend on $n \in \mathbb{N}$, so that

$$D_n \geq m_\varepsilon \cdot e^{(b-\varepsilon)\lambda_n}. \tag{5}$$

As a result, they characterized (see [6, Theorem 8.2]) the closed span of the system $\{e^{\lambda_n t}\}_{n=1}^\infty$ in the $L^p(a, b)$ spaces as follows.

THEOREM B. *Let $\{\lambda_n\}_{n=1}^\infty$ satisfy conditions (I) and (II). Let f be in the space $L^p(a, b)$. Then f belongs to the closed span of the system $\{e^{\lambda_n t}\}_{n=1}^\infty$ in $L^p(a, b)$ if and only if $f(x) = g(x)$ almost everywhere on (a, b) , where g is an analytic function in the half plane $\Re z < b$, admitting the Dirichlet series representation*

$$g(z) = \sum_{n=1}^\infty a_n e^{\lambda_n z} \quad a_n \in \mathbb{C}, \quad \forall z \in \Re z < b,$$

with the series converging uniformly on compact subsets of the half plane $\Re z < b$.

Combining the above with Theorem 1 yields the following result.

THEOREM 2. *Let $\{\lambda_n\}_{n=1}^\infty$ be a strictly sequence of positive real numbers diverging to infinity satisfying conditions (I) and (II). Consider the space $L^2(a, b)$ and choose non-zero constants c_n for $n = 1, 2, \dots$ such that,*

$$\frac{1}{|c_n|} = O(e^{\alpha \lambda_n}) \quad \text{where} \quad \alpha < b. \tag{6}$$

Then, the system $\{c_n \cdot e^{\lambda_n t}\}_{n=1}^\infty$ is a Riesz-Fischer sequence in $\overline{\text{span}}(\{e^{\lambda_n t}\}_{n=1}^\infty)$ in $L^2(a, b)$. In fact, for every sequence $A = \{a_n\}_{n=1}^\infty$ in the space $l^2(\mathbb{N})$, there exists an analytic function f_A in the half-plane $\Re z < b$, admitting a Dirichlet series representation of the form

$$f_A(z) = \sum_{n=1}^\infty d_{A,n} e^{\lambda_n z}, \quad d_{A,n} \in \mathbb{C},$$

converging uniformly on compact subsets of the half-plane $\Re z < b$, such that $f_A \in L^2(a, b)$ and

$$\int_a^b f_A(t) \cdot c_n \cdot e^{\lambda_n t} dt = a_n \quad n \in \mathbb{N}. \tag{7}$$

Proof. Choose $\varepsilon = (b - \alpha)/4$ where $\alpha < b$ as in (6). Then, combining the lower bound (5) with (6), shows that condition (3) holds. Hence, it follows from Theorem 1 that the system $\{c_n \cdot e^{\lambda_n t}\}_{n=1}^\infty$ is a Riesz-Fischer sequence in $\overline{\text{span}}\{e^{\lambda_n t}\}_{n=1}^\infty$ in $L^2(a, b)$. Thus, for every sequence $A = \{a_n\}_{n=1}^\infty$ in the space $l^2(\mathbb{N})$, there is a function f_A in $\overline{\text{span}}\{e^{\lambda_n t}\}_{n=1}^\infty$ in $L^2(a, b)$ so that (7) is valid. By Theorem B, any function in this closure extends analytically in the half-plane $\Re z < b$, as a Dirichlet series. \square

For example, for a fixed real number $\alpha < b$ and every sequence $A = \{a_n\}_{n=1}^\infty$ in the space $l^2(\mathbb{N})$, there exists a Dirichlet series

$$f_A(z) = \sum_{n=1}^\infty d_{A,n} e^{n^3 z}, \quad d_{A,n} \in \mathbb{C},$$

analytic in the half-plane $\Re z < b$, with $f_A \in L^2(a, b)$, so that

$$\int_a^b f_A(t) \cdot e^{n^3 t} dt = a_n \cdot e^{\alpha \cdot n^3} \quad n \in \mathbb{N}.$$

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