

# WEIGHTED WEAK ESTIMATE FOR COMMUTATORS OF FRACTIONAL TYPE PARAMETRIC MARCINKIEWICZ INTEGRALS OVER NON-HOMOGENEOUS METRIC SPACES

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*Abstract.* Let  $(\mathcal{X}, d)$  be a metric space satisfying the geometrically doubling condition, and  $\mu$  be a Borel measure satisfying the upper doubling condition. In this paper, the authors prove the weak type weighted  $L^p(\omega)$  boundedness of the commutators  $\mathcal{T}_{\beta, \rho, q}^b$  generated by the  $RBMO(\mu)$  function  $b$  and the fractional type parametric Marcinkiewicz integral operator  $\mathcal{T}_{\beta, \rho, q}$ , which is defined over the non-homogeneous metric space  $(\mathcal{X}, d, \mu)$ .

## 1. Introduction

As it's well-known that Stein [14] first introduced the classical Marcinkiewicz integral over Euclidean space  $\mathbb{R}^n (n \geq 2)$ , and then Hörmander [9] introduced the parametric Marcinkiewicz integral, and the fractional Marcinkiewicz integral is also considered by many researchers, see Lin-Lin-Tao-Yu [13] for example among others. These Marcinkiewicz integral operators can be uniformly written as fractional type parametric Marcinkiewicz integral operator in the following form,

$$\mu_{\Omega, \alpha}^{\rho}(f)(x) = \left\{ \int_0^{\infty} \left| \frac{1}{t^{\alpha+\rho}} \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y) dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}, \quad x \in \mathbb{R}^n, \quad (1)$$

where  $\rho > 0$  and  $\alpha \geq 0$ , and that  $\Omega$  is homogeneous of degree zero in  $\mathbb{R}^n$ , integrable and has mean value zero on the unit sphere  $\mathbb{S}^{n-1}$ . If let  $\rho = 1, \alpha = 0$  in (1) then it is just the Marcinkiewicz integral  $\mu_{\Omega}$ . If  $\alpha = 0$  in (1) then it is the parametric Marcinkiewicz integral  $\mu_{\Omega}^{\rho}$ . Hörmander [9] proved the  $L^p (1 < p < \infty)$  boundedness for  $\mu_{\Omega}^{\rho}$  whenever  $\Omega$  is Lipschitz continuous. In 1990, Torchinsky and Wang [16] first studied the  $L^p$  boundedness for the commutator  $\mu_{\Omega, b}$  generated by the Marcinkiewicz integral  $\mu_{\Omega}$  and a BMO function  $b$ . In 2009, Lin-Lin-Tao-Yu [13] showed that, if  $\Omega$  satisfies a class of Dini condition, then the fractional Marcinkiewicz integral  $\mu_{\Omega, \alpha}$ , i.e., the case

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$\rho = 1$  in (1), is bounded from  $L^p(1 < p < 2)$  to  $L^q$  with  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2n}$ , and bounded from  $H^p$  to  $H^q$  for  $\frac{2n}{2n+\alpha} \leq p \leq 1$  and  $0 < \alpha < 1$ .

In this paper, we will consider commutators of some fractional type parametric Marcinkiewicz integral on the non-homogeneous metric space, and discuss the weak weighted boundedness of these generalized Marcinkiewicz integral operator over metric space. To this end, let's recall some necessary concepts by starting with the Hytönen's non-homogeneous metric space.

Hytönen [10] introduced the non-homogeneous space  $(\mathcal{X}, d, \mu)$ , which is the metric space satisfying the following geometric doubling condition and the upper doubling condition.

DEFINITION 1. [3, 2] A metric space  $(\mathcal{X}, d)$  is said to satisfy the geometrically doubling condition if there exist some  $N_0 \in \mathbb{N}$  such that, for all balls  $B(x, r) \subset \mathcal{X}$ , there exists a finite ball covering  $\{B(x_i, \frac{r}{2})\}_i$  of  $B(x, r)$  such that the cardinality of this covering is at most  $N_0$ .

DEFINITION 2. [10] A metric measure space  $(\mathcal{X}, d, \mu)$  is said to satisfy the upper doubling condition if  $\mu$  is a Borel measure on  $\mathcal{X}$  and there exists a dominating function  $\lambda : \mathcal{X} \times (0, \infty) \rightarrow (0, \infty)$  and a positive constant  $C_\lambda$  such that, for each  $x \in \mathcal{X}$ ,  $r \rightarrow \lambda(x, r)$  is non-decreasing and, for any  $x \in \mathcal{X}$ ,  $r > 0$ ,

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, r/2). \tag{2}$$

Furthermore, in [11], it shows that there exists a dominating function  $\bar{\lambda}$  such that  $\bar{\lambda} \leq \lambda$ ,  $C_{\bar{\lambda}} \leq C_\lambda$  and for any  $x, y \in \mathcal{X}$ ,  $d(x, y) \leq r$ ,

$$\bar{\lambda}(x, r) \leq C_{\bar{\lambda}} \bar{\lambda}(y, r). \tag{3}$$

Hence we can assume that the dominating function  $\lambda$  satisfies both (2) and (3).

Suppose that  $\alpha, \beta \in (1, \infty)$ , a ball  $B \subset \mathcal{X}$  is called  $(\alpha, \beta)$ -doubling if  $\mu(\alpha B) \leq \beta \mu(B)$ . It is proved in [10] that, if  $(\mathcal{X}, d, \mu)$  satisfies the upper doubling condition and  $\beta > c_\lambda^{\log_2 \alpha} = \alpha^\nu$ , then for any ball  $B$ , there exist some  $j \in \mathbb{N} \cup \{0\}$  so that  $\alpha^j B$  is  $(\alpha, \beta)$ -doubling. Furthermore, if  $(\mathcal{X}, d, \mu)$  satisfies the geometrically doubling condition and  $\beta > \alpha^n$  with  $n = \log_2 N_0$ , Hytönen [10] also proved that there exist  $(\alpha, \beta)$ -doubling balls centered at  $x$  and the doubling ball can be arbitrary small. More than that, for any preassigned  $r > 0$ , their radius can be chosen to be the form  $\alpha^{-j}$  for  $j \in \mathbb{N}$ . For any  $\alpha \in (1, \infty)$  and ball  $B$ ,  $\bar{B}^\alpha$  denote the smallest  $(\alpha, \beta_\alpha)$ -doubling ball of the form  $\alpha^j B$  with  $j \in \mathbb{N}$ , where

$$\beta_\alpha = \max\{\alpha^{3n}, \alpha^{3\nu}\} + 30^n + 30^\nu.$$

In the following discussion, if there is no special statement, for any  $\nu \in (1, \infty)$  and  $B \subset \mathcal{X}$ ,  $\bar{B}$  always denote the smallest  $(30\nu, \beta_{30\nu})$ -doubling ball which have the form of  $(30\nu)^j B$ ,  $j \in \mathbb{N}$ .

In [8], Hu et al. introduced the following  $A_p^\nu$  weight.

DEFINITION 3. [8] For  $v \in [1, \infty)$ ,  $p \in (1, \infty)$  and  $p' = p/(p - 1)$ , a nonnegative  $\mu$ -measurable function  $\omega$  is said to belong to  $A_p^v$ , if there exists a positive constant  $C$  so that, for every ball  $B \subset \mathcal{X}$ ,

$$\left[ \frac{1}{\mu(vB)} \int_B \omega(x) d\mu(x) \right] \left\{ \frac{1}{\mu(vB)} \int_B [\omega(x)]^{1-p'} d\mu(x) \right\}^{p-1} \leq C, \tag{4}$$

and  $\omega$  is said to belong to  $A_1^v$ , if there exists a positive constant  $C$  so that, for every ball  $B \subset \mathcal{X}$ ,

$$\frac{1}{\mu(vB)} \int_B \omega(x) d\mu(x) \leq C \inf_{y \in B} \omega(y),$$

and let  $A_\infty^v(\mu) := \bigcup_{p=1}^\infty A_p^v(\mu)$ .

In this paper, we will use some notations introduced by Bui and Duong [1]. For two balls  $B$  and  $R$  in  $\mathcal{X}$  such that  $B \subset R$ , let

$$K_{B,R} = 1 + \sum_{i=1}^{N_{B,R}} \frac{\mu(6^i B)}{\lambda(c_B, 6^i r_B)}, \tag{5}$$

where  $N_{B,R}$  denotes the smallest integer satisfying  $6^{N_{B,R}} r_B \geq r_R$ .

For any  $v \in (0, \infty)$  and any two balls  $B \subset R \subset \mathcal{X}$ , let

$$\tilde{K}_{B,R}^{(v)} := 1 + \sum_{k=-\lfloor \log_v 2 \rfloor}^{N_{B,R}^{(v)}} \frac{\mu(v^k B)}{\lambda(c_B, v^k r_B)},$$

where  $N_{B,R}^{(v)}$  is the smallest integer satisfying  $v^{N_{B,R}^{(v)}} r_B \geq r_R$  and for any  $a \in \mathbb{R}$ ,  $\lfloor a \rfloor$  is the largest integer which is less than or equal to  $a$ . It is easy to deduce that

$$\tilde{K}_{B,R}^{(v)} \sim 1 + \sum_{k=1}^{N_{B,R}^{(v)} + \lfloor \log_v 2 \rfloor + 1} \frac{\mu(v^k B)}{\lambda(c_B, v^k r_B)}.$$

Now we recall the definition of  $RBMO(\mu)$  introduced in [10].

DEFINITION 4. [10] For  $\rho \in (1, \infty)$ , a function  $f \in L_{loc}^1(\mu)$  is said to belong to the space  $RBMO(\mu)$  if, for any ball  $B \subset \mathcal{X}$ , there exists a positive constant  $C$  and a number  $f_B$  such that

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B| d\mu(x) \leq C, \tag{6}$$

and for any two balls  $B \subset R \subset \mathcal{X}$ ,

$$|f_B - f_R| \leq C \delta_{B,R}, \tag{7}$$

where

$$\delta(B, R) = 1 + \int_{2R \setminus B} \frac{d\mu(x)}{\lambda(c_B, d(x, c_B))}.$$

The  $RBMO(\mu)$  norm  $\|f\|_{RBMO(\mu)}$  of  $f$  is the infimum of the positive constant  $C$  in (6) and (7). It's worthy to point out that the norm  $\|f\|_{RBMO(\mu)}$  does not depend on  $\rho$ , see [10].

LEMMA 1. [11] For  $\rho \in (1, \infty)$  and  $f \in L^1_{loc}(\mu)$ ,  $f \in RBMO(\mu)$  if and only if for any doubling balls  $B \subset R$ , there exists a positive constant  $C$  such that

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - m_{\bar{B}}f| d\mu(x) \leq C,$$

and

$$|m_B f - m_R f| \leq C \delta_{B,R},$$

where we denote by  $m_B f = \frac{1}{\mu(B)} \int_B f(x) \mu(x)$ .

LEMMA 2. [6] Let  $\rho \in (1, \infty)$  and  $r \in [1, \infty)$ , If  $f \in RBMO(\mu)$ , then there exists a positive constant  $C$  such that for any ball  $B$ ,

$$\left\{ \frac{1}{\mu(\rho B)} \int_B |f(x) - m_{\bar{B}}f|^r d\mu(x) \right\}^{1/r} \leq C \|f\|_{RBMO(\mu)}.$$

We now give the definition of fractional type parametric Marcinkiewicz integral over the non-homogeneous metric space  $(\mathcal{X}, d, \mu)$ . Let  $K(x, y)$  be a locally integrable function in  $(\mathcal{X} \times \mathcal{X}) \setminus \{(x, x) : x \in \mathcal{X}\}$  satisfying, for any  $x, y \in \mathcal{X}$  with  $x \neq y$ ,

$$|K(x, y)| \leq C \frac{[d(x, y)]^{1+\beta}}{\lambda(x, d(x, y))}, \tag{8}$$

and if  $d(x, y) \geq 2d(x, x')$ ,

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \frac{[d(x, x')]^{\delta+\beta+1}}{[d(x, y)]^\delta \lambda(x, d(x, y))}, \tag{9}$$

for some  $\beta \in [0, \infty)$ ,  $\delta \in (0, 1]$  and the positive constant  $C$ .

For  $\rho \in (0, \infty)$ ,  $q \in (1, \infty)$ , the fractional type parametric Marcinkiewicz integral operator  $\mathcal{T}_{\beta, \rho, q}$  with the kernel  $K(x, y)$  is then defined by

$$\mathcal{T}_{\beta, \rho, q}(f)(x) := \left\{ \int_0^\infty \left| \frac{1}{t^{\beta+\rho}} \int_{d(x, y) < t} \frac{K(x, y)}{[d(x, y)]^{1-\rho}} f(y) d\mu(y) \right|^q \frac{dt}{t} \right\}^{\frac{1}{q}}. \tag{10}$$

Let  $b \in RBMO(\mu)$ , we consider the commutators generated by  $b$  and the fractional type parametric Marcinkiewicz integral operator  $\mathcal{T}_{\beta,\rho,q}$ , which is defined by

$$\begin{aligned} &\mathcal{T}_{\beta,\rho,q}^b(f)(x) \\ &:= \left\{ \int_0^\infty \left| \frac{1}{t^{\beta+\rho}} \int_{d(x,y)<t} \frac{K(x,y)}{[d(x,y)]^{1-\rho}} [b(x) - b(y)] f(y) d\mu(y) \right|^q \frac{dt}{t} \right\}^{\frac{1}{q}}. \end{aligned} \tag{11}$$

In [15], the authors proved the boundedness in  $L^p(\mu)$  of the commutators  $\mathcal{T}_{0,\rho,2}^b$  if the kernel  $K(x,y)$  satisfies some Log-Dini condition and that  $\mathcal{T}_{0,\rho,2}$  is bounded on  $L^2(\mu)$ . Zhou [4] proved some similar results under the following Hörmander type condition for the kernel  $K(x,y)$  satisfying

$$\begin{aligned} &\sup_{\substack{r>0 \\ d(y,y') \leq r}} \sum_{i=1}^\infty i \int_{6^i r < d(x,y) \leq 6^{i+1} r} [|K(x,y) - K(x,y')| \\ &+ |K(y,x) - K(y',x)|] \frac{1}{d(x,y)} d\mu(x) \leq C. \end{aligned}$$

In this paper, we devote to give the weak type weighted boundedness of the commutator  $\mathcal{T}_{\beta,\rho,q}^b$  as follows.

**THEOREM 1.** *Let  $v \in [1, \infty)$  and  $\omega \in A_p^v$ ,  $1 < p < \infty$ , and let  $K(x,y)$  satisfy (8) and (9). Suppose that  $\mathcal{T}_{\beta,\rho,q}$  is bounded on  $L^2(\mu)$ , then for the commutator  $\mathcal{T}_{\beta,\rho,q}^b$ , we have*

$$\sup_{t>0} t \left[ \omega \left( \left\{ x \in \mathcal{X} : \mathcal{T}_{\beta,\rho,q}^b(f)(x) > t \right\} \right) \right]^{\frac{1}{p}} \lesssim \|f\|_{L^p(\omega)}.$$

### 2. Some lemmas

For a  $\mu$ -measurable real function  $f$  and a ball  $B$  with  $\mu(B) \neq 0$ , we let  $m_f(B)$  be a real number such that  $\inf_{\xi \in \mathbb{R}} m_B(|f - \xi|)$  is attained. Moreover,  $m_f(B)$  satisfies

$$\mu(\{x \in B : f(x) > m_f(B)\}) \leq \mu(B)/2$$

and

$$\mu(\{x \in B : f(x) < m_f(B)\}) \leq \mu(B)/2.$$

In the case that  $\mu(B) = 0$ , set  $m_f(B) \equiv 0$ . For a complex-valued  $f$ , we then take  $m_f(B) \equiv m_{Re f}(B) + i m_{Im f}(B)$ , where  $i$  is imaginary unit. It is known that, for a complex number  $z$ ,  $Re(z)$  and  $Im(z)$  denote the real part and the imaginary part, respectively.

For any  $\mu$ -measurable function  $f$  and ball  $B$ , when  $\mu(B) > 0$ , define  $m_{0,s;B}^{\sigma,v}(f)$  by setting

$$m_{0,s;B}^{\sigma,v}(f) := \inf\{t > 0 : \mu(\{y \in B : |f(y)| > t\}) < s\mu(\sigma v B)\}$$

where  $s \in (0, 1)$ ,  $\sigma \in [1, \infty)$  and  $v \in [1, \infty)$ , while  $\mu(B) = 0$ , set  $m_{0,s;B}^{\sigma,v}(f) = 0$ . By the definition of  $m_{0,s;B}^{\sigma,v}(f)$ , we give the following John-Strömberg-type maximal operator, which is defined as

$$M_{0,s}^{\sigma,v}(f)(x) := \sup_{B \ni x, B(30v, \beta_{30v})\text{-doubling}} m_{0,s;B}^{\sigma,v}(f),$$

and the John-Strömberg-type sharp maximal operator corresponding to  $M_{0,s}^{\sigma,v}(f)$  is defined as

$$M_{0,s}^{\sigma,v;\sharp}(f)(x) := \sup_{B \ni x} m_{0,s;B}^{\sigma,v} \left( f - m_f(\tilde{B}) \right) + \sup_{\substack{x \in B \subset R \\ B, R(30v, \beta_{30v})\text{-doubling}}} \frac{|m_f(B) - m_f(R)|}{\tilde{K}_{B,R}^{(v)}}.$$

For any  $f \in L^1_{loc}(\mu)$  and  $x \in \mathcal{X}$ , maximal operator  $M_{r,\eta}$  is denoted by

$$M_{r,\eta}f(x) = \sup_{B \ni x} \left\{ \frac{1}{\mu(\eta B)} \int_B |f(y)|^r d\mu(y) \right\}^{\frac{1}{r}},$$

where  $v \in [1, \infty)$ ,  $\eta, \sigma \in (1, \infty)$  and  $r \in (0, \infty)$ . Moreover, sharp maximal operator  $M^{v,\sigma;\sharp}$  is given by

$$M^{v,\sigma;\sharp}(f)(x) := \sup_{B \ni x} \left( \frac{1}{\mu(\sigma v B)} \int_B |f(y) - m_f(\tilde{B})| d\mu(y) \right) + \sup_{\substack{x \in B \subset R \\ B, R(30v, \beta_{30v})\text{-doubling}}} \frac{|m_f(B) - m_f(R)|}{\tilde{K}_{B,R}^{(v)}}.$$

Similar to Lemma 2.6 in [8], we can easily get the following lemma, where we omit the details.

LEMMA 3. For  $v \in [1, \infty)$ ,  $\eta \in [5v, \infty)$ ,  $p \in (1, \infty)$  and  $\omega \in A_p^v(\mu)$ ,  $M_{r,\eta}$  is bounded from  $L^p(\omega)$  to  $L^p(\omega)$ .

LEMMA 4. By the definition of  $m_{0,s;B}^{\sigma,v}(f)$ , it is clear to see that for any constant  $C$ ,

$$m_{0,s;B}^{\sigma,v}(f - C) \leq s^{-1} \left( \frac{1}{\mu(\sigma v B)} \int_B |f(y) - C| d\mu(y) \right).$$

Particularly, we take  $C = m_f(\tilde{B})$  then we obtain

$$m_{0,s;B}^{\sigma,v}(f - m_f(\tilde{B})) \leq s^{-1} \left( \frac{1}{\mu(\sigma v B)} \int_B |f(y) - m_f(\tilde{B})| d\mu(y) \right) \tag{12}$$

and from (12), it follows that

$$M_{0,s}^{\sigma,v;\sharp}f(x) \leq s^{-1} M^{v,\sigma;\sharp}f(x). \tag{13}$$

LEMMA 5. *The following properties about  $K_{B,R}$  are useful when we make estimates over non-homogeneous metric spaces, which were proved in [6, 1].*

(1) *For any  $\rho \in [1, \infty)$ , there exists a positive constant  $c_\rho$ , relying on  $\rho$ , such that, for all balls  $B \subset R$  with  $r_s \leq \rho r_B, K_{B,R} \leq c_\rho$ .*

(2) *There exists a positive constant  $c$ , such that, for all balls  $B, K_{B,\tilde{B}} \leq c$ .*

*Using the properties of  $K_{B,R}$ , from [5] we can easily find that  $\tilde{K}_{B,R}^{(v)} \sim K_{B,R}$ .*

LEMMA 6. [8] *For  $v, p \in [1, \infty), \eta \in [5v, \infty)$  and  $\omega \in A_p^v$ , there exist constants  $C$ , so that, for every ball  $B$  and  $\mu$ -measurable set  $A \subset B$ ,*

$$\frac{\omega(A)}{\omega(B)} \geq C^{-1} \left[ \frac{\omega(A)}{\omega(\eta B)} \right]^p.$$

LEMMA 7. [7] *Let  $f \in RBMO(\mu), q \in (0, \infty)$  and for all  $x \in \mathcal{X}$ ,*

$$f_q(x) := \begin{cases} f(x), & \text{if } |f(x)| \leq q, \\ q \frac{f(x)}{|f(x)|}, & \text{if } |f(x)| > q. \end{cases}$$

*Then  $f_q \in RBMO(\mu)$  and there exists a positive constant  $C$ , which is independent of  $f$ , such that  $\|f_q\|_{RBMO(\mu)} \leq C \|f\|_{RBMO(\mu)}$ .*

LEMMA 8. [8] *Let  $v \in [1, \infty), \sigma \in [1, 30], s_1 \in (0, \beta_{30v}^{-1}/4), p \in (0, \infty)$  and  $\omega \in A_{\infty}^v(\mu)$ . In the case that  $\mu(\mathcal{X}) = \infty, f \in L^{p_0, \infty}(\mu)$  for some  $p_0 \in (0, \infty)$ , and for all  $R \in (0, \infty)$ ,*

$$\sup_{t \in (0, R)} t^p \omega(\{x \in \mathcal{X} : |f(x)| > t\}) < \infty.$$

*Then there exists a constant  $C_0 \in (0, 1)$  which depend on  $s_1$  and  $\omega$ , and a positive constant  $C$  such that for any  $s_2 \in (0, C_0 s_1)$ ,*

$$\left\| M_{0, s_1}^{\sigma, v}(f) \right\|_{L^{p, \infty}(\omega)} \leq C \left\| M_{0, s_2}^{\sigma, v; \#}(f) \right\|_{L^{p, \infty}(\omega)}.$$

LEMMA 9. [8] *Let  $v, p \in [1, \infty), \sigma \in [1, 30]$  and  $s \in (0, \beta_{30v}^{-1})$ , then for any  $\mu$ -measurable functions  $f$  and  $t \in (0, \infty)$ ,*

(1)  $\{x \in \mathcal{X} : |f(x)| > t\} \subset \left\{x \in \mathcal{X} : M_{0, s}^{\sigma, v}(f)(x) \geq t\right\} \cup E$  with  $\mu(E) = 0$ .

(2) *For  $\omega \in A_p^v(\mu)$ , there exists a positive constant  $C$  which is independent of  $f$  and  $t$ , such that*

$$\omega\left(\left\{x \in \mathcal{X} : M_{0, s}^{\sigma, v}(f)(x) > t\right\}\right) \leq C s^{-p} \omega(\{x \in \mathcal{X} : |f(x)| > t\}).$$

We need the following boundedness of  $\mathcal{T}_{\beta, \rho, q}$  in [12].

LEMMA 10. [12] For  $\rho \in (0, \infty)$ ,  $\beta \in [0, \infty)$  and  $q \in [1, \infty)$ , let  $K(x, y)$  satisfy (8) and (9), and  $\mathcal{T}_{\beta, \rho, q}$  is defined as (10). Suppose that  $\mathcal{T}_{\beta, \rho, q}$  is bounded on  $L^2(\mu)$ , then for any  $p \in (1, \infty)$ , it is bounded on  $L^p$  and also bounded from  $L^1(\mu)$  into  $L^{1, \infty}(\mu)$ .

To prove Theorem 1, we should first establish the following pointwise estimate.

LEMMA 11. Let  $\beta \in [0, \infty)$ ,  $\rho \in (0, \infty)$ ,  $q \in (1, \infty)$  and  $K(x, y)$  satisfy (8) and (9). Suppose that  $\mathcal{T}_{\beta, \rho, q}$  is bounded on  $L^2(\mu)$ , then for any  $v \in [1, \infty)$ ,  $\sigma \in (5, 30]$ , there exists a constant  $C$ , such that for any function  $f \in L^\infty(\mu)$  and  $x \in \mathcal{X}$ ,

$$M^{v, \sigma, \sharp} \left( \mathcal{T}_{\beta, \rho, q}^b(f) \right) (x) \leq CM_{r, \frac{\sigma}{5}v}(f)(x).$$

*Proof.* Let

$$h_B := m_B \left[ \mathcal{T}_{\beta, \rho, q} \left( (b - m_{\bar{B}}(b)) f \chi_{(5B)^c} \right) \right].$$

To prove Lemma 11, it is sufficient to prove for any ball  $B$ ,  $x \in B$ ,

$$\frac{1}{\mu(\sigma v B)} \int_B \left| \mathcal{T}_{\beta, \rho, q}^b(f)(y) - h_B \right| d\mu(y) \lesssim M_{r, \frac{\sigma}{5}v}(f)(x) \quad (14)$$

and for any two balls  $B \subset R$ ,  $R$  is a doubling ball,

$$|h_B - h_R| \lesssim K_{B, R} M_{r, \frac{\sigma}{5}v}(f)(x). \quad (15)$$

First, we estimate (14). It follows that

$$\begin{aligned} & \frac{1}{\mu(\sigma v B)} \int_B \left| \mathcal{T}_{\beta, \rho, q}^b(f)(y) - h_B \right| d\mu(y) \\ & \leq \frac{1}{\mu(\sigma v B)} \int_B |b(y) - m_{\bar{B}}(b)| \mathcal{T}_{\beta, \rho, q}(f)(y) d\mu(y) \\ & \quad + \frac{1}{\mu(\sigma v B)} \int_B \mathcal{T}_{\beta, \rho, q} \left( (b - m_{\bar{B}}(b)) f_1 \right) (y) d\mu(y) \\ & \quad + \frac{1}{\mu(\sigma v B)} \int_B \left| \mathcal{T}_{\beta, \rho, q} \left( (b - m_{\bar{B}}(b)) f_2 \right) (y) - h_B \right| d\mu(y) \\ & := A_1 + A_2 + A_3, \end{aligned}$$

where,  $f_1 = f \chi_{(5B)}$ ,  $f_2 = f \chi_{(5B)^c}$ . By Hölder's inequality, Lemma 2 and Lemma 10, we see that

$$\begin{aligned} A_1 & \leq \frac{1}{\mu(\sigma v B)^{\frac{1}{r} + \frac{1}{r'}}} \left[ \int_B |b(y) - m_{\bar{B}}(b)|^{r'} d\mu(y) \right]^{1/r'} \left[ \int_B (\mathcal{T}_{\beta, \rho, q}(f))^r(y) \right]^{1/r} \\ & \leq \|b\|_{RBMO(\mu)} \left[ \frac{1}{\mu(\sigma v B)} \int_{5B} (\mathcal{T}_{\beta, \rho, q}(f))^r(y) \right]^{1/r} \\ & \lesssim M_{r, \frac{\sigma}{5}v}(f)(x). \end{aligned}$$

To estimate  $A_2$ , by Hölder’s inequality, the  $L^2$ -boundedness of  $\mathcal{T}_{\beta,\rho,q}(f)$ , Lemma 2 and Lemma 5, we conclude that

$$\begin{aligned} A_2 &\leq \frac{\mu(B)^{1/2}}{\mu(\sigma\nu B)} \left[ \int_B |\mathcal{T}_{\beta,\rho,q} [(b - m_{\bar{B}}(b)) f_1](y)|^2 d\mu(y) \right]^{1/2} \\ &\lesssim \left[ \frac{1}{\mu(\sigma\nu B)} \int_B |(b(y) - m_{\widetilde{5B}}(b)) f_1(y)|^2 d\mu(y) \right]^{1/2} \\ &\quad + \left[ \frac{1}{\mu(\sigma\nu B)} \int_B |(m_{\widetilde{5B}}(b) - m_{\bar{B}}(b)) f_1(y)|^2 d\mu(y) \right]^{1/2} \\ &\lesssim \|f\|_{L^\infty(\mu)} \left[ \frac{1}{\mu(\sigma\nu B)} \int_{5B} |b(y) - m_{\widetilde{5B}}(b)|^2 d\mu(y) \right]^{1/2} \\ &\quad + \|f\|_{L^\infty(\mu)} \left[ \frac{1}{\mu(\sigma\nu B)} \int_B |m_{\widetilde{5B}}(b) - m_{\bar{B}}(b)|^2 d\mu(y) \right]^{1/2} \\ &\lesssim \|f\|_{L^\infty(\mu)} \left( \|b\|_{RBMO(\mu)} + \left[ \frac{\mu(B)}{\mu(\sigma\nu B)} \right]^{\frac{1}{2}} \right) \\ &\lesssim \|f\|_{L^\infty(\mu)}, \end{aligned}$$

where we utilize the fact that  $|m_{\widetilde{5B}}(b) - m_{\bar{B}}(b)| \leq c (K_{B,\bar{B}} + K_{5B,\widetilde{5B}} + K_{B,5B}) \leq c$ . To estimate  $A_3$ , we observe that

$$\begin{aligned} &|\mathcal{T}_{\beta,\rho,q} ((b - m_{\bar{B}}(b)) f_2)(y) - h_B| \\ &= |\mathcal{T}_{\beta,\rho,q} ((b - m_{\bar{B}}(b)) f_2)(y) - m_B [\mathcal{T}_{\beta,\rho,q} ((b - m_{\bar{B}}(b)) f_2)]| \\ &\leq \left| \frac{1}{\mu(B)} \int_B |\mathcal{T}_{\beta,\rho,q} ((b - m_{\bar{B}}(b)) f_2)(y) \right. \\ &\quad \left. - \mathcal{T}_{\beta,\rho,q} ((b - m_{\bar{B}}(b)) f_2)(x) | d\mu(x) \right|. \end{aligned}$$

For  $x, y \in B$ , we obtain

$$\begin{aligned} &|\mathcal{T}_{\beta,\rho,q} ((b - m_{\bar{B}}(b)) f_2)(y) - \mathcal{T}_{\beta,\rho,q} ((b - m_{\bar{B}}(b)) f_2)(x)| \\ &\leq \left( \int_0^\infty \left| \int_{d(y,z)<t} \frac{K(y,z)}{|d(y,z)|^{1-\rho}} [b(z) - m_{\bar{B}}(b)] f_2(z) d\mu(z) \right. \right. \\ &\quad \left. \left. - \int_{d(x,z)<t} \frac{K(x,z)}{|d(x,z)|^{1-\rho}} [b(z) - m_{\bar{B}}(b)] f_2(z) d\mu(z) \right|^q \frac{dt}{t^{q(\beta+\rho)+1}} \right)^{\frac{1}{q}} \\ &\leq \left( \int_0^\infty \left| \int_{d(y,z) \leq t \leq d(x,z)} \frac{K(y,z)}{|d(y,z)|^{1-\rho}} [b(z) - m_{\bar{B}}(b)] \right. \right. \\ &\quad \left. \left. \times f_2(z) d\mu(z) \right|^q \frac{dt}{t^{q(\beta+\rho)+1}} \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
 & + \left( \int_0^\infty \left| \int_{d(x,z) \leq t \leq d(y,z)} \frac{K(x,z)}{|d(x,z)|^{1-\rho}} [b(z) - m_{\bar{B}}(b)] \right. \right. \\
 & \quad \left. \left. \times f_2(z) d\mu(z) \right|^q \frac{dt}{t^{q(\beta+\rho)+1}} \right)^{\frac{1}{q}} \\
 & + \left( \int_0^\infty \left| \int_{t > \max\{d(y,z), d(x,z)\}} \left( \frac{K(y,z)}{|d(y,z)|^{1-\rho}} - \frac{K(x,z)}{|d(x,z)|^{1-\rho}} \right) \right. \right. \\
 & \quad \left. \left. \times [b(z) - m_{\bar{B}}(b)] f_2(z) d\mu(z) \right|^q \frac{dt}{t^{q(\beta+\rho)+1}} \right)^{\frac{1}{q}} \\
 & := B_1 + B_2 + B_3.
 \end{aligned}$$

As in the estimate for  $B_1$ , by Minkowski inequality, (8), and note that, for  $y \in B$  and  $z \in \mathcal{X} \setminus kB(k > 1)$ ,  $\lambda(y, d(y, z)) \sim \lambda(y, d(c_B, z)) \sim \lambda(c_B, d(c_B), z)$ . It follows that

$$\begin{aligned}
 B_1 & \lesssim \int_{\mathcal{X} \setminus 5B} \frac{|K(y, z)|}{d(y, z)^{1-\rho}} |b(z) - m_{\bar{B}}(b)| |f(z)| \\
 & \quad \times \left( \int_{d(y,z) \leq t \leq d(x,z)} \frac{dt}{t^{q(\beta+\rho)+1}} \right)^{\frac{1}{q}} d\mu(z) \\
 & \lesssim \sum_{k=1}^\infty \frac{1}{5^{kq}} \frac{1}{\lambda(c_B, 5^k r_B)} \int_{5^{k+1}B} |m_{5^{k+1}B}(b) - m_{\bar{B}}(b)| |f(z)| d\mu(z) \\
 & \quad + \sum_{k=1}^\infty \frac{1}{5^{kq}} \frac{1}{\lambda(c_B, 5^k r_B)} \int_{5^{k+1}B} |m_{5^{k+1}B}(b) - b(z)| |f(z)| d\mu(z) \\
 & \lesssim \sum_{k=1}^\infty \|f\|_{L^\infty(\mu)} \frac{k+1}{5^{kq}} \frac{\mu(5^{k+1}B)}{\lambda(c_B, 5^k r_B)} + \sum_{k=1}^\infty \|f\|_{L^\infty(\mu)} \|b\|_{RBMO(\mu)} \frac{1}{5^{kq}} \\
 & \lesssim \|f\|_{L^\infty(\mu)},
 \end{aligned}$$

where, for  $\rho > 1$ , if we take  $m = \lceil \log_2 5\rho \rceil$ , from (2), we know that

$$\frac{\mu(\rho 5^{k+1}B)}{\lambda(c_B, 5^k r_B)} \leq \frac{C_\lambda^m \lambda(c_B, \rho \frac{5^{k+1}}{2^m} r_B)}{\lambda(c_B, 5^k r_B)} \leq C_\lambda^m.$$

With the similar argument, we also have  $B_2 \lesssim \|f\|_{L^\infty(\mu)}$ . Next we estimate  $B_3$ , from Minkowski inequality, and note that for any  $x, y \in B$  and  $z \in (5B)^c$ ,  $d(x, z) \sim d(y, z) \sim d(c_B, z)$ , then it follows that

$$\begin{aligned}
 B_3 & \lesssim \int_{\mathcal{X} \setminus 5B} \left| \frac{K(y, z)}{d(y, z)^{1-\rho}} - \frac{K(x, z)}{d(x, z)^{1-\rho}} \right| |b(z) - m_{\bar{B}}(b)| \\
 & \quad \times |f(z)| \left( \int_{d(x,z)}^\infty \frac{dt}{t^{q(\beta+\rho)+1}} \right)^{\frac{1}{q}} d\mu(z) \\
 & \lesssim \int_{\mathcal{X} \setminus 5B} |K(y, z) - K(x, z)| |b(z) - m_{\bar{B}}(b)| \\
 & \quad \times |f(z)| \frac{1}{d(x, z)^{\beta+1}} d\mu(z)
 \end{aligned}$$

$$\begin{aligned}
 & + \int_{\mathcal{X} \setminus 5B} |K(x, z)| \left| \frac{1}{d(y, z)^{1-\rho}} - \frac{1}{d(x, z)^{1-\rho}} \right| |b(z) - m_{\tilde{B}}(b)| \\
 & \times |f(z)| \frac{1}{d(x, z)^{\beta+\rho}} d\mu(z) \\
 & := B_{31} + B_{32},
 \end{aligned}$$

Similar to the discussion with  $B_1$ , we conclude that  $B_{31} \lesssim \|f\|_{L^\infty(\mu)}$ . It is easy to show that

$$\begin{aligned}
 B_{32} & \lesssim \int_{\mathcal{X} \setminus 5B} \frac{1}{\lambda(c_B, d(c_B, z))} \frac{|d(x, z)^{1-\rho} - d(y, z)^{1-\rho}|}{d(x, z)^{1-\rho}} \\
 & \times |b(z) - m_{\tilde{B}}(b)| |f(z)| d\mu(z) \\
 & \lesssim \sum_{k=1}^{\infty} \frac{k+1}{5^k} \frac{1}{\lambda(c_B, 5^k r_B)} \int_{5^{k+1}} |f(z)| d\mu(z) \\
 & + \sum_{k=1}^{\infty} \frac{1}{5^k} \frac{1}{\lambda(c_B, 5^k r_B)} \int_{5^{k+1}} |m_{\widetilde{5^{k+1}B}}(b) - b(z)| |f(z)| d\mu(z) \\
 & \lesssim \sum_{k=1}^{\infty} \frac{k+1}{5^k} \frac{\mu(5^{k+1}B)}{\lambda(c_B, 5^k r_B)} \|f\|_{L^\infty} \\
 & + \sum_{k=1}^{\infty} \frac{1}{5^k} \|b\|_{RBMO(\mu)} \|f\|_{L^\infty(\mu)} \\
 & \lesssim \|f\|_{L^\infty(\mu)}.
 \end{aligned}$$

From  $B_{31}$  and  $B_{32}$ , we obtain  $B_3 \lesssim \|f\|_{L^\infty(\mu)}$ , which completes the estimate of (14).

Next, we consider (15). Note that

$$\begin{aligned}
 |h_B - h_R| & \leq \left| m_R \left[ \mathcal{T}_{\beta, \rho, q} \left( (b - m_{\tilde{B}}(b)) f \chi_{\mathcal{X} \setminus 5^N B} \right) \right] \right. \\
 & \quad \left. - m_B \left[ \mathcal{T}_{\beta, \rho, q} \left( (b - m_{\tilde{B}}(b)) f \chi_{\mathcal{X} \setminus 5^N B} \right) \right] \right| \\
 & + \left| m_R \left[ \mathcal{T}_{\beta, \rho, q} \left( (b - m_R(b)) f \chi_{\mathcal{X} \setminus 5^N B} \right) \right] \right. \\
 & \quad \left. - m_R \left[ \mathcal{T}_{\beta, \rho, q} \left( (b - m_{\tilde{B}}(b)) f \chi_{\mathcal{X} \setminus 5^N B} \right) \right] \right| \\
 & + \left| m_B \left[ \mathcal{T}_{\beta, \rho, q} \left( (b - m_{\tilde{B}}(b)) f \chi_{5^N B \setminus 5B} \right) \right] \right| \\
 & + \left| m_R \left[ \mathcal{T}_{\beta, \rho, q} \left( (b - m_{\tilde{B}}(b)) f \chi_{5^N B \setminus 5R} \right) \right] \right| \\
 & := C_1 + C_2 + C_3 + C_4,
 \end{aligned}$$

where,  $N = N_{B,R} + 1$ . Similar with  $A_3$ , we conclude that  $C_1 \lesssim \|f\|_{L^\infty(\mu)}$ . From Hölder's

inequality, Lemma 5 and Lemma 10, we see that

$$\begin{aligned}
C_2 &\lesssim m_R \left| (m_R(b) - m_{\widetilde{B}}(b)) \mathcal{T}_{\beta,\rho,q} \left( f \chi_{\mathcal{X} \setminus 5^N B} \right) \right| \\
&\lesssim \frac{K_{B,R} + K_{B,\widetilde{B}}}{\mu(R)} \int_R \mathcal{T}_{\beta,\rho,q} \left( f \chi_{\mathcal{X} \setminus 5^N B} \right) (y) d\mu(y) \\
&\lesssim \frac{K_{B,R}}{\mu(R)} \mu(R)^{1/r'} \left( \int_R \left| \mathcal{T}_{\beta,\rho,q} \left( f \chi_{\mathcal{X} \setminus 5^N B} \right) (y) \right|^r d\mu(y) \right)^{1/r} \\
&\lesssim K_{B,R} \left( \frac{1}{\mu(\sigma \cup R)} \int_{5R} \left| \mathcal{T}_{\beta,\rho,q} \left( f \chi_{\mathcal{X} \setminus 5^N B} \right) (y) \right|^r d\mu(y) \right)^{1/r} \\
&\lesssim K_{B,R} M_{r,\frac{q}{5}}(f)(x).
\end{aligned}$$

With the same argument of  $B_1$ , we get  $C_3 \lesssim \|f\|_{L^\infty(\mu)}$ . As for  $C_4$ , it follows that

$$\begin{aligned}
&\left| \mathcal{T}_{\beta,\rho,q} \left( (b - m_R(b)) f \chi_{5^N B \setminus 5R} \right) \right| \\
&\lesssim \|f\|_{L^\infty} \int_{6^N B} \frac{|b(z) - m_{\widetilde{6^2 R}}(b) + m_{\widetilde{6^2 R}}(b) - m_R(b)|}{\lambda(C_B, 6^N r_B)} \mu(z) \\
&\quad + \|f\|_{L^\infty} \frac{1}{\lambda(C_B, 6^N r_B)} \int_{6^N B} |m_R(b) - m_{\widetilde{6^2 R}}(b)| d\mu(z) \\
&\lesssim \|f\|_{L^\infty} \frac{\mu(5^N B)}{\lambda(C_B, 5^N r_B)} + \|f\|_{L^\infty} \frac{1}{\mu(6^2 R)} \int_{6^2 R} |b(z) - m_{\widetilde{6^2 R}}(b)| d\mu(z) \\
&\lesssim \|f\|_{L^\infty},
\end{aligned}$$

hence we obtain  $C_4 \lesssim \|f\|_{L^\infty}$ . Combining (14) and (15), we prove the Lemma 11.  $\square$

### 3. Proof of the Theorem 1

In order to prove Theorem 1, we first estimate that for  $R \in (0, \infty)$  and  $\omega \in A_p^V$ ,

$$\sup_{t \in (0, R)} t^p \omega(\{x \in \mathcal{X} : \mathcal{T}_{\beta,\rho,q}^b(f)(x) > t\}) < \infty. \quad (16)$$

Fix  $x_0 \in \mathcal{X}$ . Take  $l \in (0, \infty)$  be large enough such that  $\text{supp}(f_i)$  is contained in the ball  $B(x_0, l)$ , then we have

$$\sup_{t \in (0, R)} t^p \omega(\{x \in B(x_0, 2l) : \mathcal{T}_{\beta,\rho,q}^b(f)(x) > t\}) \leq R^p \omega(B(x_0, 2l)) < \infty. \quad (17)$$

From Lemma 7 and a standard limit argument, without loss of generality, we can suppose that  $b$  is bounded function. We note that, for any  $x \in \mathcal{X} \setminus B(x_0, 2l)$  and  $y \in B(x_0, l)$ ,  $d(x, x_0) \sim d(x, y)$ . From (8), Minkowski inequality and (3), for any

$x \in \mathcal{X} \setminus B(x_0, 2l)$ , we observe that,

$$\begin{aligned} \mathcal{T}_{\beta, \rho, q}^b(f)(x) &\leq \int_{\mathcal{X}} \frac{|f(x)|}{\lambda(x, d(x, y))} |b(x) - b(y)| d\mu(y) \\ &\lesssim \frac{\|f\|_{L^1(\mu)}}{\lambda(x_0, d(x, x_0))} \\ &= \frac{C_*}{\lambda(x_0, d(x, x_0))}, \end{aligned} \tag{18}$$

where  $C_*$  is a positive constant just depending on  $f$ . Note that  $\mu(\mathcal{X}) = \infty$ , thus for  $x_0 \in \mathcal{X}$ ,

$$\lim_{r \rightarrow \infty} \lambda(x_0, r) \geq \lim_{r \rightarrow \infty} \mu(B(x_0, r)) = \infty.$$

Therefore, we easily see that for any  $t \in (0, \infty)$ , there exists  $r_t \in (0, \infty)$  such that  $\frac{1}{t} \leq \lambda(x_0, r_t)$ . If there exists  $\bar{t} \in (0, \infty)$  such that for any  $r \in (0, \infty)$ ,  $\frac{C_*}{\bar{t}} \leq \lambda(x_0, r)$ , then for all  $t \in (\bar{t}, \infty)$  and  $r \in (0, \infty)$ ,  $\frac{C_*}{t} \leq \lambda(x_0, r)$ . Let

$$\hat{t} := \inf \left\{ \bar{t} \in (0, \infty) : \frac{C_*}{\bar{t}} \leq \lambda(x_0, r) \text{ holds for any } r \in (0, \infty) \right\}.$$

On the other hand, suppose that does not exist  $\bar{t} \in (0, \infty)$  such that for any  $r \in (0, \infty)$ ,  $\frac{C_*}{\bar{t}} \leq \lambda(x_0, r)$ , let  $\hat{t} = \infty$ . In the case that  $\bar{t} \in (0, \infty)$ , and for any  $t \in (\bar{t}, \infty)$  and  $x \in \mathcal{X}$  satisfying

$$t < \frac{C_*}{\lambda(x_0, d(x, x_0))}, \tag{19}$$

we get that  $d(x, x_0) = 0$  and hence  $x = x_0$ . Therefore, for any  $\bar{t} \in (0, \infty]$  and  $t \in (0, \bar{t})$ , there exists  $r_t \in (0, \infty)$  such that

$$\lambda(x_0, r_t) \geq \frac{C_*}{t} \quad \text{and} \quad \lambda(x_0, r_t/2) < \frac{C_*}{t}. \tag{20}$$

This indicates that for all  $x \in \mathcal{X}$  satisfying (19), we have  $d(x, x_0) < r_t$ . More than that, notice that for all  $x \in \mathcal{X} \setminus B(x_0, 2l)$ ,

$$\frac{1}{\lambda(x_0, d(x, x_0))} \leq \frac{1}{\lambda(x_0, l)}.$$

This indicates that for  $t > C_*/\lambda(x_0, l)$ , there is no point  $x \in \mathcal{X} \setminus B(x_0, 2l)$  such that  $\mathcal{T}_{\beta, \rho, q}^b(f)(x) > t$ .

Thus, by (18), (2), Lemma 6 with  $\omega \in A_p^v$  and (20), we conclude that if  $\bar{t} \leq$

$C_*/\lambda(x_0, l)$ , then

$$\begin{aligned} & \sup_{t \in (0, \infty)} t^p \omega \left( \left\{ x \in \mathcal{X} \setminus B(x_0, 2l) : \mathcal{T}_{\beta, \rho, q}^b(f)(x) > t \right\} \right) \\ &= \sup_{t \in (0, C_*/\lambda(x_0, l)]} t^p \omega \left( \left\{ x \in \mathcal{X} \setminus B(x_0, 2l) : \mathcal{T}_{\beta, \rho, q}^b(f)(x) > t \right\} \right) \\ &\leq \sup_{t \in (0, C_*/\lambda(x_0, l)]} t^p \omega \left( \left\{ x \in \mathcal{X} : \frac{C_*}{\lambda(x_0, d(x, x_0))} > t \right\} \right) \\ &\leq \sup_{t \in (0, \bar{t}]} t^p \omega(B(x_0, r_t)) + \sup_{t \in (\bar{t}, C_*/\lambda(x_0, l)]} t^p \omega(\{x_0\}) \\ &\lesssim 1 + \sup_{t \in (0, \bar{t}], r_t \in (0, l]} t^p \omega(B(x_0, r_t)) + \sup_{t \in (0, \bar{t}], r_t \in (l, \infty)} t^p \omega(B(x_0, r_t)) \\ &\lesssim 1 + \sup_{t \in (0, \bar{t}], r_t \in (l, \infty)} t^p \omega(B(x_0, l)) \left[ \frac{\mu(B(x_0, 5\nu r_t))}{\mu(B(x_0, l))} \right]^p \\ &\lesssim 1 + u(B(x_0, l)) \left[ \frac{1}{\mu(B(x_0, l))} \right]^p < \infty. \end{aligned}$$

Similarly, if  $\bar{t} > C_*/\lambda(x_0, l)$ , we have

$$\sup_{t \in (0, \infty)} t^p \omega \left( \left\{ x \in \mathcal{X} \setminus B(x_0, 2l) : \mathcal{T}_{\beta, \rho, q}^b(f)(x) > t \right\} \right) < \infty,$$

which, along with (17), implies (16).

From Lemma 9, Lemma 8, (13), Lemma 11 and Lemma 3, we have

$$\begin{aligned} & \sup_{t > 0} t \omega \left( \left\{ x \in \mathcal{X} : \mathcal{T}_{\beta, \rho, q}^b(f)(x) > t \right\} \right)^{\frac{1}{p}} \\ &\lesssim \sup_{t > 0} t \omega \left( \left\{ x \in \mathcal{X} : M_{0, s_1}^{\sigma, \nu} \left( \mathcal{T}_{\beta, \rho, q}^b(f) \right) (x) > t \right\} \right)^{\frac{1}{p}} \\ &\lesssim \sup_{t > 0} t \omega \left( \left\{ x \in \mathcal{X} : M_{0, s_2}^{\sigma, \nu, \#} \left( \mathcal{T}_{\beta, \rho, q}^b(f) \right) (x) > t \right\} \right)^{\frac{1}{p}} \\ &\lesssim \sup_{t > 0} t \omega \left( \left\{ x \in \mathcal{X} : M_{t, \frac{t}{5}}^{\sigma, \nu, \#} \left( \mathcal{T}_{\beta, \rho, q}^b(f) \right) (x) > t \right\} \right)^{\frac{1}{p}} \\ &\lesssim \sup_{t > 0} t \omega \left( \left\{ x \in \mathcal{X} : M_{t, \frac{t}{5}}(f)(x) > t \right\} \right)^{\frac{1}{p}} \\ &\lesssim \|f\|_{L^p(\omega)}, \end{aligned}$$

which completes the Theorem 1.

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