

OSCILLATORY AND SPECTRAL PROPERTIES OF A CLASS OF FOURTH-ORDER DIFFERENTIAL OPERATORS VIA A NEW HARDY-TYPE INEQUALITY

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Abstract. In this paper, we study oscillatory properties of a fourth-order differential equation and spectral properties of a corresponding differential operator. These properties are established by first proving a new second-order Hardy-type inequality, where the weights are the coefficients of the equation and the operator. This new inequality, in its turn, is established for functions satisfying certain boundary conditions that depend on the boundary behavior of one of its weights at infinity and at zero.

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