

CHARACTERIZATIONS OF w_ρ -BIRKHOFF—JAMES ORTHOGONALITY AND w_ρ -PARALLELISM

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Abstract. We study the concepts of Birkhoff–James orthogonality and parallelism in Hilbert space operators, induced by the operator radius norm $w_\rho(\cdot)$. In particular, we completely characterize Birkhoff–James orthogonality and parallelism with respect to $w_\rho(\cdot)$. As an application of the results presented, we obtain a well-known characterization due to R. Bhatia and P. Šemrl for the classical Birkhoff–James orthogonality of Hilbert space operators. Some other related results are also discussed.

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