Mathematical
Inequalities
& Applications
Volume 28, Number 2 (2025), 173–187 doi:10.7153/mia-2025-28-12

CHARACTERIZATIONS OF w_ρ–BIRKHOFF—JAMES ORTHOGONALITY AND w_ρ–PARALLELISM

FUAD KITTANEH* AND ALI ZAMANI

(*Communicated by M. S. Moslehian*)

Abstract. We study the concepts of Birkhoff–James orthogonality and parallelism in Hilbert space operators, induced by the operator radius norm $w_\rho(\cdot)$. In particular, we completely characterize Birkhoff–James orthogonality and parallelism with respect to $w_\rho(\cdot)$. As an application of the results presented, we obtain a well-known characterization due to R. Bhatia and P. Šemrl for the classical Birkhoff–James orthogonality of Hilbert space operators. Some other related results are also discussed.

1. Introduction

Let $(\mathscr{H},\langle\cdot,\cdot\rangle)$ be a complex Hilbert space equipped with the norm $\|\cdot\|$, and let $\mathbf{S}_{\mathscr{H}}$ denote the unit ball of $\mathscr{H}$, i.e., $\mathbf{S}_{\mathscr{H}}=\{x\in\mathscr{H}:\|x\|\leqslant 1\}$. Let $\mathbb{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on $\mathscr{H}$. For $\rho>0$ an operator $A\in\mathbb{B}(\mathscr{H})$ is called a ρ-contraction (see [16]) if there is a Hilbert space $\mathscr{K}\,(\supseteq\mathscr{H})$ and a unitary operator U on $\mathscr{K}$ such that $A^nx=\rho PU^nx$ for all $x\in\mathscr{H}, n=1,2,\cdots$, where P is the orthogonal projection from $\mathscr{K}$ to $\mathscr{H}$. Holbrook [9] and Williams [22] defined the operator radii $w_\rho(\cdot)$ as the generalized Minkowski distance functionals on $\mathbb{B}(\mathscr{H})$, i.e.,

$$w_\rho(A)=\inf\{t>0:t^{-1}A \text{ is a } \rho\text{-contraction}\}.$$

The operator radius $w_\rho(\cdot)$, usually referred to in the literature as the ρ-radius, plays a very important role in the study of unitary ρ-dilations (see, e.g., [17]). It is well known that $w_\rho(A^*)=w_\rho(A)$ and $w_\rho(U^*AU)=w_\rho(A)$ for all A and all unitary $U\in\mathbb{B}(\mathscr{H})$ i.e., $w_\rho(\cdot)$ is, respectively, self-adjoint and weakly unitarily invariant. Moreover, the operator radii $w_\rho(\cdot)$ have the properties:

$$w_1(A)=\|A\|,$$

where $\|\cdot\|$ is the Hilbert space operator norm, that is, $\|A\|=\sup\{\|Ax\|:x\in\mathbf{S}_{\mathscr{H}}\}$ and

$$w_2(A)=w(A),$$

Mathematics subject classification (2020): 46B20, 47A12, 47A20, 47A30, 47L05.
Keywords and phrases: Operator radius, operator norm, numerical radius, Birkhoff–James orthogonality, parallelism.
* Corresponding author.

Paper MIA-28-12

where $w(\cdot)$ is the numerical radius, that is, $w(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathbf{S}_{\mathscr{H}}\}$. For every $A \in \mathbb{B}(\mathscr{H})$, we also have

$$\frac{1}{\rho}\|A\| \leqslant w_\rho(A) \leqslant \frac{1+|1-\rho|}{\rho}\|A\|. \tag{1}$$

If A is normal (i.e., $A^*A = AA^*$), then $w_\rho(A) = \frac{1+|1-\rho|}{\rho}\|A\|$ and if A is 2-nilpotent (i.e., $A^2 = 0$), then $w_\rho(A) = \frac{1}{\rho}\|A\|$. Notice that there is a major difference between the case when $0 < \rho \leqslant 2$ and $2 < \rho < \infty$. It is known that for $\rho \in (0,2]$, $w_\rho(\cdot)$ is a norm on $\mathbb{B}(\mathscr{H})$ but for $\rho \in (2,\infty)$ is only a quasi-norm. For proofs and more facts about the operator radii, we refer the reader to [2, 5, 9, 10, 17, 22].

Now, let $\rho \in (0,2]$ and $A, B \in \mathbb{B}(\mathscr{H})$. We say that A is w_ρ-Birkhoff–James orthogonal to B (see [6, 11]), in short $A \perp_{w_\rho} B$, if

$$w_\rho(A+\gamma B) \geqslant w_\rho(A) \quad \text{for all } \gamma \in \mathbb{C},$$

or equivalently,

$$\min_{\gamma\in\mathbb{C}} w_\rho(A+\gamma B) = w_\rho(A).$$

In particular, when $\rho = 1$ and $\rho = 2$, we obtain, respectively, the definitions of the classical Birkhoff–James orthogonality (written $A \perp B$) and the numerical radius orthogonality (written $A \perp_w B$) in $\mathbb{B}(\mathscr{H})$. The Birkhoff–James orthogonality plays a very crucial role in the geometry of Hilbert space operators, see [4, 18, 19, 21] and the references therein. Characterizations of the Birkhoff–James orthogonality for operators were given in [8, 13, 15, 23, 26].

Furthermore, we say that A is w_ρ-parallel to B (see [20]), and we write $A \parallel_{w_\rho} B$, if

$$w_\rho(A+\lambda B) = w_\rho(A) + w_\rho(B) \quad \text{for some } \lambda \in \mathbb{T},$$

or equivalently,

$$\max_{\lambda\in\mathbb{T}} w_\rho(A+\lambda B) = w_\rho(A) + w_\rho(B).$$

Here, as usual, $\mathbb{T}$ is the unit circle of the complex plane. For $\rho = 1$ and $\rho = 2$, we also obtain, respectively, the definitions of the the operator norm parallelism (written $A \parallel B$) and the numerical radius parallelism (written $A \parallel_w B$) in $\mathbb{B}(\mathscr{H})$. The concept of parallelism plays a significant role in the study of the geometric and the analytic properties of Banach space, for instance, see [7, 20, 25]. Some other authors studied different aspects of parallelism of bounded linear operators, see [14, 24] and the references therein.

Motivated by these, in this paper we explore the w_ρ-Birkhoff–James orthogonality and the w_ρ-parallelism on $\mathbb{B}(\mathscr{H})$. In particular, we present characterizations of the w_ρ-Birkhoff–James orthogonality and the w_ρ-parallelism. Our results extend some previously known results which appeared in the literature [4, 13, 14, 24].

2. The w_ρ -Birkhoff–James orthogonality

We start our work with the following proposition, which contains some basic properties of the relation $\perp_{w_\rho}$.

PROPOSITION 1. *Let* $A,B \in \mathbb{B}(\mathscr{H})$ *and* $\rho \in (0,2]$. *The following conditions are mutually equivalent:*

(i) $A \perp_{w_\rho} B$,

(ii) $A^* \perp_{w_\rho} B^*$,

(iii) $\alpha A \perp_{w_\rho} \beta B$ *for all* $\alpha,\beta \in \mathbb{C}\setminus\{0\}$,

(iv) $U^*AU \perp_{w_\rho} U^*BU$ *for all unitary* $U \in \mathbb{B}(\mathscr{H})$.

Proof. The proof immediately follows from the definition of the relation $\perp_{w_\rho}$ and the properties of $w_\rho(\cdot)$. $\square$

REMARK 1. Let $A,B \in \mathbb{B}(\mathscr{H})$.

(i) If $\rho \in (0,2]$ and A is 2-nilpotent, then $w_\rho(A) = \frac{1}{\rho}\|A\|$, and so the condition $A \perp B$ implies $A \perp_{w_\rho} B$. Indeed, for every $\gamma \in \mathbb{C}$, by (1) it follows that

$$w_\rho(A+\gamma B) \geqslant \frac{1}{\rho}\|A+\gamma B\| \geqslant \frac{1}{\rho}\|A\| = w_\rho(A).$$

(ii) If $\rho \in [1,2]$ and A is normal, then $w_\rho(A) = \|A\|$, and hence the condition $A \perp_{w_\rho} B$ implies $A \perp B$. Indeed, for every $\gamma \in \mathbb{C}$, again by (1), we have

$$\|A+\gamma B\| \geqslant w_\rho(A+\gamma B) \geqslant w_\rho(A) = \|A\|.$$

In order to prove our desired characterization of the w_ρ -Birkhoff–James orthogonality, we need the following lemmas.

The first lemma has been proved recently in [12, Theorem 3.1].

LEMMA 1. *Let* $X \in \mathbb{B}(\mathscr{H})$ *and* $\rho \in (0,2]$. *Then*

$$w_\rho(X) = \frac{2}{\rho}\, w\left(\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}X \\ 0 & (1-\rho)X \end{bmatrix}\right).$$

The second lemma reads as follows. Our approach is similar to the one given in [26, Theorem 1].

LEMMA 2. *Let* $A,B \in \mathbb{B}(\mathscr{H})$ *and* $\rho \in (0,2]$. *The following conditions are equivalent:*

(i) $w_\rho(A+rB) \geqslant w_\rho(A)$ *for all* $r \geqslant 0$,

(ii) there exists a sequence $\left\{\begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\}$ *in* $\mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ *such that*

$$\lim_{n\to\infty}\left|\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\right| = w_\rho(A)$$

and

$$\lim_{n\to\infty}\mathrm{Re}\left(\left\langle\sqrt{\rho(2-\rho)}x_n+(1-\rho)y_n, Ay_n\right\rangle\left\langle By_n, \sqrt{\rho(2-\rho)}x_n+(1-\rho)y_n\right\rangle\right) \geqslant 0.$$

Proof. (i)$\Rightarrow$(ii) Let $w_\rho(A+rB) \geqslant w_\rho(A)$ for all $r \geqslant 0$. We may assume that $w_\rho(A) \neq 0$ otherwise (ii) trivially holds. So there exists $\varepsilon_0 \in (0,1)$ such that $w_\rho(A) \geqslant \varepsilon^2$ for all $\varepsilon \in (0,\varepsilon_0)$. Therefore,

$$w_\rho(A+\varepsilon B) \geqslant w_\rho(A) \geqslant w_\rho(A) - \varepsilon^2 \geqslant 0 \qquad \big(0<\varepsilon<\varepsilon_0\big). \tag{2}$$

Let $\varepsilon \in (0,\varepsilon_0)$. By Lemma 1 we have

$$w\left(\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}(A+\varepsilon B) \\ 0 & (1-\rho)(A+\varepsilon B)\end{bmatrix}\right) = \frac{\rho}{2}w_\rho(A+\varepsilon B),$$

and hence there exists a sequence $\left\{\begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\}$ in $\mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ such that

$$\lim_{n\to\infty}\left|\left\langle\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}(A+\varepsilon B) \\ 0 & (1-\rho)(A+\varepsilon B)\end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right| = \frac{\rho}{2}w_\rho(A+\varepsilon B). \tag{3}$$

Utilizing Lemma 1, (2) and (3), we have

$$\begin{aligned}
\frac{\rho}{2}w_\rho(A) + \varepsilon\frac{\rho}{2}w_\rho(B) &= w\left(\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}A \\ 0 & (1-\rho)A\end{bmatrix}\right) + \varepsilon w\left(\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}B \\ 0 & (1-\rho)B\end{bmatrix}\right) \\
&\geqslant \lim_{n\to\infty}\left|\left\langle\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}A \\ 0 & (1-\rho)A\end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right| \\
&\qquad + \varepsilon\lim_{n\to\infty}\left|\left\langle\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}B \\ 0 & (1-\rho)B\end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right| \\
&\geqslant \lim_{n\to\infty}\left|\left\langle\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}(A+\varepsilon B) \\ 0 & (1-\rho)(A+\varepsilon B)\end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right| \\
&= \frac{\rho}{2}w_\rho(A+\varepsilon B) \geqslant \frac{\rho}{2}w_\rho(A),
\end{aligned}$$

and so by letting $\varepsilon \to 0^+$ we obtain

$$\lim_{n\to\infty}\left|\left\langle\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}A \\ 0 & (1-\rho)A\end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right| = \frac{\rho}{2}w_\rho(A).$$

This implies

$$\lim_{n\to\infty}\left|\left\langle \sqrt{\rho(2-\rho)}Ay_n, x_n\right\rangle + \left\langle (1-\rho)Ay_n, y_n\right\rangle\right| = \frac{\rho}{2}\, w_\rho(A),$$

or equivalently,

$$\lim_{n\to\infty}\left|\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\right| = w_\rho(A).$$

We also have

$$\begin{aligned}
&2\varepsilon \lim_{n\to\infty} \mathrm{Re}\left(\left\langle \sqrt{\rho(2-\rho)}x_n + (1-\rho)y_n, Ay_n\right\rangle \left\langle By_n, \sqrt{\rho(2-\rho)}x_n + (1-\rho)y_n\right\rangle\right)\\
&\qquad + \frac{\rho^2}{4}\, w_\rho^2(A) + \varepsilon^2 \frac{\rho^2}{4}\, w_\rho(B)\\
&= 2\varepsilon \lim_{n\to\infty} \mathrm{Re}\left(\left\langle \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}A \\ 0 & (1-\rho)A \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle \left\langle \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}B \\ 0 & (1-\rho)B \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right)\\
&\qquad + w^2\left(\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}A \\ 0 & (1-\rho)A \end{bmatrix}\right) + \varepsilon^2 w^2\left(\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}B \\ 0 & (1-\rho)B \end{bmatrix}\right)\\
&\geqslant 2\varepsilon \lim_{n\to\infty} \mathrm{Re}\left(\left\langle \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}A \\ 0 & (1-\rho)A \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle \left\langle \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}B \\ 0 & (1-\rho)B \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right)\\
&\quad + \lim_{n\to\infty}\left|\left\langle \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}A \\ 0 & (1-\rho)A \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right|^2\\
&\quad + \varepsilon^2 \lim_{n\to\infty}\left|\left\langle \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}B \\ 0 & (1-\rho)B \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right|^2\\
&= \lim_{n\to\infty}\left|\left\langle \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}(A+\varepsilon B) \\ 0 & (1-\rho)(A+\varepsilon B) \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right|^2\\
&= \frac{\rho^2}{4}\, w_\rho^2(A+\varepsilon B) \geqslant \frac{\rho^2}{4}\left(w_\rho(A) - \varepsilon^2\right)^2 = \frac{\rho^2}{4}\, w_\rho^2(A) - \varepsilon^2\,\frac{\rho^2}{2} w_\rho(A) + \varepsilon^4\,\frac{\rho^2}{4},
\end{aligned}$$

which yields

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{Re}\left(\left\langle \sqrt{\rho(2-\rho)}x_n + (1-\rho)y_n, Ay_n\right\rangle \left\langle By_n, \sqrt{\rho(2-\rho)}x_n + (1-\rho)y_n\right\rangle\right)\\
&\qquad \geqslant \varepsilon^3\,\frac{\rho^2}{8} - \varepsilon\,\frac{\rho^2}{4} w_\rho(A) - \varepsilon\frac{\rho^2}{8}\, w_\rho^2(B). \qquad (4)
\end{aligned}$$

Finally, by letting $\varepsilon \to 0^+$ in (4), we conclude that

$$\lim_{n\to\infty} \mathrm{Re}\left(\left\langle \sqrt{\rho(2-\rho)}x_n + (1-\rho)y_n, Ay_n\right\rangle \left\langle By_n, \sqrt{\rho(2-\rho)}x_n + (1-\rho)y_n\right\rangle\right) \geqslant 0.$$

(ii)$\Rightarrow$(i) Suppose (ii) holds. For any $r \geqslant 0$, by Lemma 1 we have

$$
\begin{aligned}
w_\rho^2(A+rB) &= \frac{4}{\rho^2} w^2\left(\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}(A+rB) \\ 0 & (1-\rho)(A+rB) \end{bmatrix}\right) \\
&\geqslant \frac{4}{\rho^2} \lim_{n\to\infty} \left| \left\langle \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}(A+rB) \\ 0 & (1-\rho)(A+rB) \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix} \right\rangle \right|^2 \\
&= \lim_{n\to\infty} \left| \left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}} x_n + \frac{2-2\rho}{\rho} y_n \right\rangle \right|^2 \\
&\quad + \frac{8r}{\rho^2} \lim_{n\to\infty} \mathrm{Re}\left(\left\langle \sqrt{\rho(2-\rho)} x_n + (1-\rho) y_n, Ay_n \right\rangle \left\langle By_n, \sqrt{\rho(2-\rho)} x_n + (1-\rho) y_n \right\rangle \right) \\
&\quad + r^2 \lim_{n\to\infty} \left| \left\langle By_n, \sqrt{\frac{8-4\rho}{\rho}} x_n + \frac{2-2\rho}{\rho} y_n \right\rangle \right|^2 \geqslant w_\rho^2(A),
\end{aligned}
$$

which implies $w_\rho(A+rB) \geqslant w_\rho(A)$ and the proof is completed. $\square$

We are now in a position to establish the main result of this section.

THEOREM 1. *Let* $A, B \in \mathbb{B}(\mathscr{H})$ *and* $\rho \in (0,2]$. *The following conditions are equivalent:*

(i) $A \perp_{w_\rho} B$,

(ii) for each $\theta \in [0, 2\pi)$, *there exists a sequence* $\left\{ \begin{bmatrix} x_n \\ y_n \end{bmatrix} \right\}$ *in* $\mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ *such that*

$$
\lim_{n\to\infty} \left| \left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}} x_n + \frac{2-2\rho}{\rho} y_n \right\rangle \right| = w_\rho(A)
$$

and

$$
\lim_{n\to\infty} \mathrm{Re}\left(e^{i\theta} \left\langle \sqrt{\rho(2-\rho)} x_n + (1-\rho) y_n, Ay_n \right\rangle \left\langle By_n, \sqrt{\rho(2-\rho)} x_n + (1-\rho) y_n \right\rangle \right) \geqslant 0.
$$

Proof. (i)$\Rightarrow$(ii) Let $A \perp_{w_\rho} B$ and let $\theta \in [0, 2\pi)$ be fixed. Thus, $w_\rho(A + re^{i\theta}B) \geqslant w_\rho(A)$ for all $r \geqslant 0$. By Lemma 2 there exists a sequence $\left\{ \begin{bmatrix} x_n \\ y_n \end{bmatrix} \right\}$ in $\mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ such that

$$
\lim_{n\to\infty} \left| \left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}} x_n + \frac{2-2\rho}{\rho} y_n \right\rangle \right| = w_\rho(A)
$$

and

$$
\lim_{n\to\infty} \mathrm{Re}\left(\left\langle \sqrt{\rho(2-\rho)} x_n + (1-\rho) y_n, Ay_n \right\rangle \left\langle e^{i\theta} By_n, \sqrt{\rho(2-\rho)} x_n + (1-\rho) y_n \right\rangle \right) \geqslant 0,
$$

and hence we deduce (ii).

(ii)$\Rightarrow$(i) Suppose (ii) holds. Let $\gamma \in \mathbb{C}$. Then $\gamma = e^{i\theta}|\gamma|$ for some $\theta \in [0,2\pi)$. So, there exists a sequence $\left\{ \begin{bmatrix} x_n \\ y_n \end{bmatrix} \right\}$ in $\mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ such that

$$\lim_{n\to\infty} \left| \left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}} x_n + \frac{2-2\rho}{\rho} y_n \right\rangle \right| = w_\rho(A)$$

and

$$\lim_{n\to\infty} \mathrm{Re}\left(e^{i\theta} \left\langle \sqrt{\rho(2-\rho)} x_n + (1-\rho) y_n, Ay_n \right\rangle \left\langle By_n, \sqrt{\rho(2-\rho)} x_n + (1-\rho) y_n \right\rangle \right) \geqslant 0.$$

Utilizing a similar argument as in Lemma 2, we get $w_\rho(A + |\gamma| e^{i\theta} B) \geqslant w_\rho(A)$. Thus, $w_\rho(A+\gamma B) \geqslant w_\rho(A)$, or equivalently, $A \perp_{w_\rho} B$. $\square$

When $\mathscr{H}$ is a finite dimensional Hilbert space, we have the following result.

COROLLARY 1. *Let $\mathscr{H}$ be a finite dimensional Hilbert space, and let $A, B \in \mathbb{B}(\mathscr{H})$ and $\rho \in (0,2]$. The following conditions are equivalent:*

(i) $A \perp_{w_\rho} B$,

(ii) for each $\theta \in [0,2\pi)$, there exists a vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ such that

$$\left| \left\langle Ay, \sqrt{\frac{8-4\rho}{\rho}} x + \frac{2-2\rho}{\rho} y \right\rangle \right| = w_\rho(A)$$

and

$$\mathrm{Re}\left(e^{i\theta} \left\langle \sqrt{\rho(2-\rho)} x + (1-\rho) y, Ay \right\rangle \left\langle By, \sqrt{\rho(2-\rho)} x + (1-\rho) y \right\rangle \right) \geqslant 0.$$

Proof. If $A \perp_{w_\rho} B$, then from Theorem 1 we obtain a sequence $\left\{ \begin{bmatrix} x_n \\ y_n \end{bmatrix} \right\}$ in $\mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ such that

$$\lim_{n\to\infty} \left| \left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}} x_n + \frac{2-2\rho}{\rho} y_n \right\rangle \right| = w_\rho(A)$$

and

$$\lim_{n\to\infty} \mathrm{Re}\left(e^{i\theta} \left\langle \sqrt{\rho(2-\rho)} x_n + (1-\rho) y_n, Ay_n \right\rangle \left\langle By_n, \sqrt{\rho(2-\rho)} x_n + (1-\rho) y_n \right\rangle \right) \geqslant 0.$$

Since $\left\{ \begin{bmatrix} x_n \\ y_n \end{bmatrix} \right\}$ is a bounded sequence, then it has a convergent subsequence converging to a vector $\begin{bmatrix} x \\ y \end{bmatrix}$. This $\begin{bmatrix} x \\ y \end{bmatrix}$ is the required vector. The proof in the other direction is straightforward. $\square$

As an immediate consequence of Theorem 1, we have the following result that can also be found in [13, Theorem 2.3].

COROLLARY 2. *Let $A,B \in \mathbb{B}(\mathscr{H})$. The following conditions are equivalent:*

(i) $A \perp_w B$,

(ii) for each $\theta \in [0,2\pi)$, *there exists a sequence* $\{y_n\}$ *in* $\mathbf{S}_{\mathscr{H}}$ *such that* $\lim_{n\to\infty} |\langle Ay_n, y_n\rangle| = w(A)$ *and* $\lim_{n\to\infty} \operatorname{Re}\left(e^{i\theta}\langle y_n, Ay_n\rangle\langle By_n, y_n\rangle\right) \geqslant 0$.

Proof. The proof follows immediately from Theorem 1 for $\rho = 2$. □

As another consequence of Theorem 1 we have the following result.

COROLLARY 3. *Let $A,B \in \mathbb{B}(\mathscr{H})$. The following conditions are mutually equivalent:*

(i) $A \perp B$,

(ii) for each $\theta \in [0,2\pi)$, *there exists a sequence* $\left\{\begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\}$ *in* $\mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ *such that* $\lim_{n\to\infty} |\langle Ay_n, x_n\rangle| = \frac{1}{2}\|A\|$ *and* $\lim_{n\to\infty} \operatorname{Re}\left(e^{i\theta}\langle x_n, Ay_n\rangle\langle By_n, x_n\rangle\right) \geqslant 0$,

(iii) there exists a sequence $\{z_n\}$ *in* $\mathbf{S}_{\mathscr{H}}$ *such that* $\lim_{n\to\infty} \|Az_n\| = \|A\|$ *and* $\lim_{n\to\infty} \langle Az_n, Bz_n\rangle = 0$.

Proof. (i)$\Rightarrow$(ii) This implication follows immediately from Theorem 1 for $\rho = 1$.

(ii)$\Rightarrow$(iii) Suppose (ii) holds. For $\theta = 0, \pi$, there exist, respectively, sequences $\left\{\begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} x'_n \\ y'_n \end{bmatrix}\right\}$ in $\mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ such that

$$\lim_{n\to\infty} |\langle Ay_n, x_n\rangle| = \lim_{n\to\infty} \left|\left\langle Ay'_n, x'_n\right\rangle\right| = \frac{1}{2}\|A\|$$

and

$$\lim_{n\to\infty} \operatorname{Re}\left(\left\langle x'_n, Ay'_n\right\rangle\left\langle By'_n, x'_n\right\rangle\right) \leqslant 0 \leqslant \lim_{n\to\infty} \operatorname{Re}\left(\langle x_n, Ay_n\rangle\langle By_n, x_n\rangle\right).$$

Therefore, we can find sequence $\left\{\begin{bmatrix} x''_n \\ y''_n \end{bmatrix}\right\}$ in $\mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ such that

$$\lim_{n\to\infty} \left|\left\langle Ay''_n, x''_n\right\rangle\right| = \frac{1}{2}\|A\| \quad \text{and} \quad \left\langle By''_n, x''_n\right\rangle = 0. \tag{5}$$

Put $z_n := 2\|x_n''\|y_n''$ and $e_n := \frac{x_n''}{\|x_n''\|}$. Hence, $\|e_n\| = 1$ and by the arithmetic-geometric mean inequality, we have $\|z_n\| = 2\|x_n''\|\|y_n''\| \leqslant \|x_n''\|^2 + \|y_n''\|^2 \leqslant 1$, and so $z_n \in \mathbf{S}_{\mathscr{H}}$. Rewrite (5) as

$$\lim_{n\to\infty} |\langle Az_n, e_n\rangle| = \|A\| \quad \text{and} \quad \langle e_n, Bz_n\rangle = 0. \tag{6}$$

Since $|\langle Az_n, e_n\rangle| \leqslant \|Az_n\| \leqslant \|A\|$ and $\left\|\frac{Az_n}{\|Az_n\|} - e_n\right\|^2 = 2 - \frac{2}{\|Az_n\|}\mathrm{Re}\left(\langle Az_n, e_n\rangle\right)$, by (6) it follows that $\lim\limits_{n\to\infty}\|Az_n\| = \|A\|$ and $\lim\limits_{n\to\infty}\left(\dfrac{Az_n}{\|Az_n\|} - e_n\right) = 0$. Therefore, the above facts imply that

$$\lim_{n\to\infty}\langle Az_n, Bz_n\rangle = \lim_{n\to\infty}\left(\|Az_n\|\left\langle \frac{Az_n}{\|Az_n\|} - e_n, Bz_n\right\rangle + \|Az_n\|\langle e_n, Bz_n\rangle\right) = 0.$$

(iii)$\Rightarrow$(i) Suppose that there exists a sequence $\{z_n\}$ in $\mathbf{S}_{\mathscr{H}}$ such that $\lim\limits_{n\to\infty}\|Az_n\| = \|A\|$ and $\lim\limits_{n\to\infty}\langle Az_n, Bz_n\rangle = 0$. We have

$$\|A+\gamma B\|^2 \geqslant \|(A+\gamma B)z_n\|^2 = \|Az_n\|^2 + 2\mathrm{Re}(\overline{\gamma}\langle Az_n, Bz_n\rangle) + |\gamma|^2\|Bz_n\|^2$$

for all $\gamma \in \mathbb{C}$ and $n \in \mathbb{N}$. Thus,

$$\|A+\gamma B\|^2 \geqslant \limsup_{n\to\infty}\|(A+\gamma B)z_n\|^2 \geqslant \|A\|^2$$

for all $\gamma \in \mathbb{C}$, and so $A \perp B$. $\square$

REMARK 2. The equivalence (i)$\Leftrightarrow$(iii) in Corollary 3 is due to R. Bhatia and P. Šemrl [4, Remark 3.1].

3. The w_ρ -parallelism

In this section, we explore the w_ρ -parallelism for Hilbert space operators. Here are some properties of the w_ρ -parallelism whose proofs are so easy that we omit them.

PROPOSITION 2. *Let $A, B \in \mathbb{B}(\mathscr{H})$ and $\rho \in (0,2]$. The following conditions are mutually equivalent:*

(i) $A \parallel_{w_\rho} B$,

(ii) $A^* \parallel_{w_\rho} B^*$,

(iii) $\zeta A \parallel_{w_\rho} \zeta B$ *for all* $\zeta \in \mathbb{C}\setminus\{0\}$,

(iii) $\alpha A \parallel_{w_\rho} \beta B$ *for all* $\alpha, \beta \in \mathbb{R}\setminus\{0\}$,

(iv) $U^*AU \parallel_{w_\rho} U^*BU$ *for all unitary* $U \in \mathbb{B}(\mathscr{H})$.

To achieve the following theorem, we mimic some ideas of [3, Theorem 2.1].

THEOREM 2. *Let $A, B \in \mathbb{B}(\mathscr{H})$ and $\rho \in (0,2]$. The following conditions are equivalent:*

(i) $A \parallel_{w_\rho} B$,

(ii) there exists a sequence $\left\{\begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\}$ *in* $\mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ *such that*

$$\lim_{n\to\infty}\left|\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\left\langle By_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\right| = w_\rho(A)w_\rho(B).$$

Proof. (i)⇒(ii) Let $A \parallel_{w_\rho} B$. Hence, there exists $\lambda \in \mathbb{T}$ such that $w_\rho(A+\lambda B) = w_\rho(A) + w_\rho(B)$. By Lemma 1 there exists a sequence $\left\{\begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\}$ in $\mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ such that

$$\lim_{n\to\infty}\left|\left\langle \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}(A+\lambda B) \\ 0 & (1-\rho)(A+\lambda B)\end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right| = \frac{\rho}{2}w_\rho(A+\lambda B),$$

or equivalently,

$$\lim_{n\to\infty}\left|\left\langle (A+\lambda B)y_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\right| = w_\rho(A+\lambda B). \tag{7}$$

We have

$$\begin{aligned}
&\left|\left\langle (A+\lambda B)y_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\right|^2 \\
&= \left|\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\right|^2 + \left|\left\langle By_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\right|^2 \\
&\quad + 2\mathrm{Re}\left(\overline{\lambda}\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\left\langle \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n, By_n\right\rangle\right) \\
&\leqslant \frac{4}{\rho^2}\left|\left\langle \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}A \\ 0 & (1-\rho)A\end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right|^2 + \frac{4}{\rho^2}\left|\left\langle \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}B \\ 0 & (1-\rho)B\end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right|^2 \\
&\quad + 2\left|\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\left\langle \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n, By_n\right\rangle\right| \\
&\leqslant w_\rho^2(A) + w_\rho^2(B) + 2\left|\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\left\langle By_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\right|
\end{aligned}$$

$$\leqslant w_\rho^2(A)+w_\rho^2(B) + \frac{8}{\rho^2}\left|\left\langle \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}A \\ 0 & (1-\rho)A \end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right| \left|\left\langle \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)}B \\ 0 & (1-\rho)B \end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}, \begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\rangle\right|$$
$$\leqslant w_\rho^2(A)+w_\rho^2(B)+2w_\rho(A)w_\rho(B),$$

and hence

$$\begin{aligned}&\left|\left\langle (A+\lambda B)y_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle\right|^2 \\ &\leqslant w_\rho^2(A)+w_\rho^2(B) \\ &\quad +2\left|\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle \left\langle By_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle\right| \\ &\leqslant w_\rho^2(A)+w_\rho^2(B)+2w_\rho(A)w_\rho(B)=w_\rho^2(A+\lambda B). \end{aligned} \tag{8}$$

By (7) and (8) we obtain

$$\lim_{n\to\infty}\left|\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle \left\langle By_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle\right| = w_\rho(A)w_\rho(B).$$

(ii)$\Rightarrow$(i) Suppose that there exists a sequence $\left\{\begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\}$ in $\mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ such that

$$\lim_{n\to\infty}\left|\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle \left\langle By_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle\right| = w_\rho(A)w_\rho(B). \tag{9}$$

Since

$$\begin{aligned}&\left|\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle \left\langle By_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle\right| \\ &\qquad \leqslant \left|\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle\right| w_\rho(B) \leqslant w_\rho(A)w_\rho(B),\end{aligned}$$

by (9) it follows that

$$\lim_{n\to\infty}\left|\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle\right| = w_\rho(A). \tag{10}$$

By using a similar argument, we also have

$$\lim_{n\to\infty}\left|\left\langle By_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle\right| = w_\rho(B). \tag{11}$$

Further, for every n, there exist $\lambda_n \in \mathbb{T}$ such that

$$\begin{aligned}&\left|\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\left\langle By_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\right|\\ &\quad = \lambda_n\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle\left\langle By_n, \sqrt{\frac{8-4\rho}{\rho}}x_n + \frac{2-2\rho}{\rho}y_n\right\rangle.\end{aligned}$$

From this and (9) it follows that

$$\lim_{n\to\infty}\lambda_n\left\langle Ay_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle\left\langle By_n, \sqrt{\frac{8-4\rho}{\rho}}x_n+\frac{2-2\rho}{\rho}y_n\right\rangle = w_\rho(A)w_\rho(B). \tag{12}$$

Since $\{\lambda_n\}$ is a bounded sequence in $\mathbb{T}$, there exists a subsequence $\{\lambda_{n_k}\}$ and a $\lambda \in \mathbb{C}$ such that $\lambda_{n_k} \to \overline{\lambda}$. We have

$$\big|1-|\lambda|\big| = \big||\lambda_{n_k}| - |\overline{\lambda}|\big| \leqslant \big|\lambda_{n_k} - \overline{\lambda}\big| \to 0,$$

and hence $|\lambda| = 1$. Thus, $\lambda \in \mathbb{T}$. Since

$$\begin{aligned}&\mathrm{Re}\left(\overline{\lambda}\left\langle Ay_{n_k}, \sqrt{\frac{8-4\rho}{\rho}}x_{n_k} + \frac{2-2\rho}{\rho}y_{n_k}\right\rangle\left\langle By_{n_k}, \sqrt{\frac{8-4\rho}{\rho}}x_{n_k} + \frac{2-2\rho}{\rho}y_{n_k}\right\rangle\right)\\ &= \mathrm{Re}\left((\overline{\lambda}-\lambda_{n_k})\left\langle Ay_{n_k}, \sqrt{\frac{8-4\rho}{\rho}}x_{n_k}+\frac{2-2\rho}{\rho}y_{n_k}\right\rangle\left\langle By_{n_k}, \sqrt{\frac{8-4\rho}{\rho}}x_{n_k}+\frac{2-2\rho}{\rho}y_{n_k}\right\rangle\right)\\ &\qquad +\mathrm{Re}\left(\lambda_{n_k}\left\langle Ay_{n_k}, \sqrt{\frac{8-4\rho}{\rho}}x_{n_k}+\frac{2-2\rho}{\rho}y_{n_k}\right\rangle\left\langle By_{n_k}, \sqrt{\frac{8-4\rho}{\rho}}x_{n_k}+\frac{2-2\rho}{\rho}y_{n_k}\right\rangle\right),\end{aligned}$$

by (12) we get

$$\begin{aligned}&\mathrm{Re}\left(\overline{\lambda}\left\langle Ay_{n_k}, \sqrt{\frac{8-4\rho}{\rho}}x_{n_k} + \frac{2-2\rho}{\rho}y_{n_k}\right\rangle\left\langle By_{n_k}, \sqrt{\frac{8-4\rho}{\rho}}x_{n_k} + \frac{2-2\rho}{\rho}y_{n_k}\right\rangle\right)\\ &\quad = w_\rho(A)w_\rho(B).\end{aligned} \tag{13}$$

For every k, by Lemma 1 we also have

$$\begin{aligned}&w_\rho^2(A+\lambda B)\\ &\geqslant \left|\left\langle (A+\lambda B)y_{n_k}, \sqrt{\frac{8-4\rho}{\rho}}x_{n_k} + \frac{2-2\rho}{\rho}y_{n_k}\right\rangle\right|^2\\ &= \left|\left\langle Ay_{n_k}, \sqrt{\frac{8-4\rho}{\rho}}x_{n_k} + \frac{2-2\rho}{\rho}y_{n_k}\right\rangle\right|^2 + \left|\left\langle By_{n_k}, \sqrt{\frac{8-4\rho}{\rho}}x_{n_k} + \frac{2-2\rho}{\rho}y_{n_k}\right\rangle\right|^2\\ &\qquad +2\mathrm{Re}\left(\overline{\lambda}\left\langle Ay_{n_k}, \sqrt{\frac{8-4\rho}{\rho}}x_{n_k}+\frac{2-2\rho}{\rho}y_{n_k}\right\rangle\left\langle By_{n_k}, \sqrt{\frac{8-4\rho}{\rho}}x_{n_k}+\frac{2-2\rho}{\rho}y_{n_k}\right\rangle\right),\end{aligned}$$

and therefore by (10), (11) and (13) we conclude that

$$w_\rho^2(A+\lambda B) \geqslant w_\rho^2(A) + w_\rho^2(B) + 2w_\rho(A)w_\rho(B).$$

Hence, $w_\rho(A+\lambda B) = w_\rho(A) + w_\rho(B)$, that is, $A \parallel_{w_\rho} B$. $\square$

If $\mathscr{H}$ be finite dimensional, then we get a tractable characterization of the w_ρ-parallelism as follows.

COROLLARY 4. *Let $\mathscr{H}$ be a finite dimensional Hilbert space, and let $A, B \in \mathbb{B}(\mathscr{H})$ and $\rho \in (0,2]$. The following conditions are equivalent:*

(i) $A \parallel_{w_\rho} B$,

(ii) there exists a vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ such that

$$\left| \left\langle Ay, \sqrt{\frac{8-4\rho}{\rho}}x + \frac{2-2\rho}{\rho}y \right\rangle \left\langle By, \sqrt{\frac{8-4\rho}{\rho}}x + \frac{2-2\rho}{\rho}y \right\rangle \right| = w_\rho(A)w_\rho(B).$$

Proof. The proof is similar to the proof of Corollary 1, only we use Theorem 2 instead of Theorem 1. We omit the details. $\square$

The next result that was proved in [14, Theorem 2.2] (see also [1, Proposition 3.6]) follows immediately from Theorem 2 for $\rho = 2$.

COROLLARY 5. *Let $A, B \in \mathbb{B}(\mathscr{H})$. The following conditions are equivalent:*

(i) $A \parallel_w B$,

(ii) there exists a sequence $\{y_n\}$ in $\mathbf{S}_{\mathscr{H}}$ such that

$$\lim_{n\to\infty} |\langle Ay_n, y_n\rangle \langle By_n, y_n\rangle| = w(A)w(B).$$

We close this paper with the following result.

COROLLARY 6. *Let $A, B \in \mathbb{B}(\mathscr{H})$. The following conditions are mutually equivalent:*

(i) $A \parallel B$,

(ii) there exists a sequence $\left\{\begin{bmatrix} x_n \\ y_n \end{bmatrix}\right\}$ in $\mathbf{S}_{\mathscr{H}\oplus\mathscr{H}}$ such that $\lim_{n\to\infty} |\langle Ay_n, x_n\rangle \langle By_n, x_n\rangle| = \frac{1}{4}\|A\|\|B\|$,

(iii) there exists a sequence $\{z_n\}$ in $\mathbf{S}_{\mathscr{H}}$ such that $\lim_{n\to\infty} |\langle Az_n, Bz_n\rangle| = \|A\|\|B\|$.

Proof. (i)$\Rightarrow$(ii) This implication follows immediately from Theorem 2 for $\rho = 1$.

(ii)$\Rightarrow$(iii) Suppose (ii) holds. Put $z_n := 2\|x_n\|y_n$. Then, by the arithmetic-geometric mean inequality, we have $\|z_n\| = 2\|x_n\|\|y_n\| \leqslant \|x_n\|^2 + \|y_n\|^2 \leqslant 1$. Thus, $z_n \in \mathbf{S}_{\mathscr{H}}$. Also, for $x_n \neq 0$, by the Buzano inequality we have

$$\begin{aligned} 4\,|\langle Ay_n, x_n\rangle \langle By_n, x_n\rangle| &= \left|\left\langle A(2\|x_n\|y_n), \frac{x_n}{\|x_n\|}\right\rangle \left\langle \frac{x_n}{\|x_n\|}, B(2\|x_n\|y_n)\right\rangle\right| \\ &\leqslant \frac{\|Az_n\|\|Bz_n\| + |\langle Az_n, Bz_n\rangle|}{2} \\ &\leqslant \frac{\|A\|\|B\| + |\langle Az_n, Bz_n\rangle|}{2} \\ &\leqslant \frac{\|A\|\|B\| + \|Az_n\|\|Bz_n\|}{2} \leqslant \|A\|\|B\|, \end{aligned}$$

and so

$$4\,|\langle Ay_n, x_n\rangle \langle By_n, x_n\rangle| \leqslant \frac{\|A\|\|B\| + |\langle Az_n, Bz_n\rangle|}{2} \leqslant \|A\|\|B\|. \tag{14}$$

Clearly, (14) holds also when $x_n = 0$. Now, by letting $n \to \infty$ in (14), we obtain $\lim\limits_{n\to\infty} |\langle Az_n, Bz_n\rangle| = \|A\|\|B\|$.

(iii)$\Rightarrow$(i) The proof is straightforward and is omitted. $\square$

REMARK 3. The equivalence (i)$\Leftrightarrow$(iii) of Corollary 6 already stated in [24, Theorem 3.3].

Acknowledgement. The authors are thankful to the referees for the useful comments and suggestions.

REFERENCES

[1] A. ABU-OMAR AND F. KITTANEH, *Notes on some spectral radius and numerical radius inequalities*, Studia Math. **227** (2015), no. 2, 97–109.

[2] T. ANDO AND C. K. LI, *Operator radii and unitary operators*, Oper. Matrices **4** (2) (2010), 273–281.

[3] M. BARRAA AND M. BOUMAZGOUR, *Inner derivations and norm equality*, Proc. Amer. Math. Soc. **130** (2002), no. 2, 471–476.

[4] R. BHATIA AND P. ŠEMRL, *Orthogonality of matrices and some distance problems*, Linear Algebra Appl. **287** (1–3) (1999), 77–85.

[5] P. BHUNIA, S. S. DRAGOMIR, M. S. MOSLEHIAN AND K. PAUL, *Lectures on numerical radius inequalities*, Infosys Science Foundation Series in Mathematical Sciences. Springer, 2022.

[6] G. BIRKHOFF, *Orthogonality in linear metric spaces*, Duke Math. J. **1** (1935), 169–172.

[7] T. BOTTAZZI, C. CONDE, M. S. MOSLEHIAN, P. WÓJCIK AND A. ZAMANI, *Orthogonality and parallelism of operators on various Banach spaces*, J. Aust. Math. Soc. **106** (2019), 160–183.

[8] S. M. ENDERAMI, M. ABTAHI AND A. ZAMANI, *An extension of Birkhoff–James orthogonality relations in semi-Hilbertian space operators*, Mediterr. J. Math. **19**, 234 (2022).

[9] J. A. R. HOLBROOK, *On the power-bounded operators of Sz.-Nagy and C. Foiaş*, Acta Sci. Math. (Szeged) **29** (1968), 299–310.

[10] J. A. R. HOLBROOK, *Inequalities governing the operator radii associated with unitary ρ-dilations*, Michigan Math. J. **18** (1971), 149–159.

[11] R. C. JAMES, *Orthogonality in normed linear spaces*, Duke Math. J. **12** (1945), 291–302.
[12] F. KITTANEH AND A. ZAMANI, *On the ρ -operator radii*, Linear Algebra Appl. **687** (2024), 132–156.
[13] A. MAL, K. PAUL AND J. SEN, *Birkhoff–James orthogonality and numerical radius inequalities of operator matrices*, Monatsh. Math. **197** (2022), 717–731.
[14] M. MEHRAZIN, M. AMYARI AND A. ZAMANI, *Numerical radius parallelism of Hilbert space operators*, Bull. Iran. Math. Soc. **46**, no. 3 (2020), 821–829.
[15] M. S. MOSLEHIAN AND A. ZAMANI, *Characterizations of operator Birkhoff–James orthogonality*, Canad. Math. Bull. **60** (2017), no. 4, 816–829.
[16] B. SZ.-NAGY AND C. FOIAŞ, *On certain classes of power-bounded operators in Hilbert space*, Acta Sci. Math. (Szeged) **27** (1966), 17–25.
[17] B. SZ.-NAGY, C. FOIAŞ, H. BERCOVICI AND L. KÉRCHY, *Harmonic Analysis of Operators on Hilbert Space*, second edition, Universitext, Springer, New York, 2010.
[18] S. PAL AND S. ROY, *Dilation and Birkhoff–James orthogonality*, J. Math. Anal. Appl. **537** (2024), no. 2, 128326.
[19] S. ROY, *The weak differentiability of norm and a generalized Bhatia–Šemrl theorem*, Linear Algebra Appl. **685** (2024), 46–65.
[20] A. SEDDIK, *Rank one operators and norm of elementary operators*, Linear Algebra Appl. **424** (2007), 177–183.
[21] A. TURNŠEK, *A remark on orthogonality and symmetry of operators in $B(H)$*, Linear Algebra Appl. **535** (2017), 141–150.
[22] J. P. WILLIAMS, *Schwarz norms for operators*, Pacific J. Math. **24** (1968), 181–188.
[23] A. ZAMANI, *Birkhoff–James orthogonality of operators in semi-Hilbertian spaces and its applications*, Ann. Funct. Anal. **10** (2019), no. 3, 433–445.
[24] A. ZAMANI AND M. S. MOSLEHIAN, *Exact and approximate operator parallelism*, Canad. Math. Bull. **58**(1) (2015), 207–224.
[25] A. ZAMANI, M. S. MOSLEHIAN, M. T. CHIEN AND H. NAKAZATO, *Norm-parallelism and the Davis–Wielandt radius of Hilbert space operators*, Linear Multilinear Algebra **67** (6) (2019), no. 11, 2147–2158.
[26] A. ZAMANI AND P. WÓJCIK, *Numerical radius orthogonality in C^* -algebras*, Ann. Funct. Anal. **11** (2020), no. 4, 1081–1092.

(Received July 16, 2024)

Fuad Kittaneh
Department of Mathematics
The University of Jordan
Amman, Jordan
and
Department of Mathematics
Korea University
Seoul 02841, South Korea e-mail: fkitt@ju.edu.jo

Ali Zamani
Department of Mathematics Education
Farhangian University
P. O. Box 14665-889, Tehran, Iran
and
School of Mathematics and Computer Sciences
Damghan University
P.O. Box 36715-364, Damghan, Iran
e-mail: zamani.ali85@yahoo.com