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SOME ASPECTS OF NEW SKEW GEOMETRIC CONSTANTS IN BANACH SPACES

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Abstract. In this paper, two new skew geometric constants are introduced. These constants are used to characterize Hilbert spaces. Some basic properties of these constants in Banach spaces are derived, and the values of the constants in specific spaces are calculated. On this basis, the relationships between the new geometric constants and other famous constants are studied. Finally, based on these identities, the relationship between the new geometric constants and the geometric properties in Banach spaces is discussed, such as uniform non-square and normal structure.

1. Introduction

It is well known to all that the geometric theory of Banach spaces plays an important role in functional analysis. This theory has been fully developed, and many scholars have defined some special constants to study some abstract properties of Banach spaces. For example, Clarkson introduced the concept of convexity modules to describe uniformly convex spaces [13], and the von Neumann constant to describe inner product spaces and uniform non-square spaces [4]. We also mention the von Neumann-Jordan constant $C_{NJ}(X)$, and the James constant $J(X)$ defined by Gao and Lau [8]. After the appearance of these constants, many others were introduced. For more papers on geometric constants, refer to [1, 7, 14, 15, 17, 18, 20]. These constants become a simple and intuitive tool to quantify the properties of a given Banach space.

In inner product spaces, there are various concepts of orthogonality that result to be equivalent to the traditional orthogonality relation. For example, James [10] introduced isosceles orthogonality stating that $x \perp_I y$ if and only if $\|x+y\| = \|x-y\|$. Roberts [16] introduced his concept of orthogonality: $x \perp_R y$ if and only if $\forall t \in \mathbb{R}, \|x+ty\| = \|x-ty\|$.

In recent years, the orthogonal geometric constants have been extended by several scholars. Inspired by the above two orthogonality relationships, we propose new geometric constants $A_{\alpha-\beta}(X)$ and $A'_{\alpha-\beta}(X)$ by incorporating a special skew orthogonality relationship. They are an important tool for studying Banach spaces. These new constants aim to more accurately characterize the spatial structure from various perspectives.

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In Section 2, as a preparation, lists some classical studies and their conclusions.

In Section 3, we introduce the constant $A_{\alpha-\beta}(X)$. Simultaneously, we discuss some basic properties it possesses, such as its range, how to characterize Hilbert spaces, and comparisons with other classic constants. Furthermore, we presented some examples in specific spaces and provided results related to uniform non-square and normal structure.

In Section 4, we introduce the constant $A'_{\alpha-\beta}(X)$ by examining the influence of isosceles orthogonality on the previously defined constant $A_{\alpha-\beta}(X)$. We also discuss its range, how to characterize Hilbert spaces, and compare it with the James constant. Finally, we demonstrate through a counterexample that the constants $A'_{\alpha-\beta}(X)$ and $A_{\alpha-\beta}(X)$ are generally different.

2. Preliminaries

Throughout the article, X will denote a real Banach space and X^* will denote the dual of X; $B_X = \{x \in X : \|x\| \leqslant 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$ will denote the unit ball and the unit sphere of X, respectively. We will assume that the dimension of X is at least 2. Now, we recall the notion of some well-known constants in Banach spaces.

The James non-square constant of a Banach space X was introduced by Gao can be used to characterize uniformly non-square spaces.

DEFINITION 1. [8] Let X be a Banach space, the James constant is defined as

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

We have collected some common conclusions as follows that
(1) $\sqrt{2} \leqslant J(X) \leqslant 2$. If X is a Hilbert space, then $J(X) = \sqrt{2}$.
(2) $J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in B_X\}$.
(3) X is uniformly non-square if and only if $J(X) < 2$.
(4) $J(X) = \sup\{J(Y) : Y$ is a subspace of X, $\dim Y = 2\}$.

We need to point out that a Banach space satisfies $J(X) = \sqrt{2}$ if and only if X is a Hilbert space, only holds in the Banach space of $\dim X \geqslant 3$ [12].

The von Neumann-Jordan constant $C_{NJ}(X)$ was defined in 1937 by Clarkson as

DEFINITION 2. [4] Let X be a Banach space, the von-Neumann constant and the modified von-Neumann constant are defined as

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, (x,y) \neq (0,0)\right\},$$

and

$$C'_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X\right\}.$$

It has the following well-known conclusions that

(1) $1 \leqslant C_{NJ}(X) \leqslant 2$ for all Banach spaces X, X is a Hilbert space if and only if $C_{NJ}(X) = 1$.

(2) X is uniformly non-square if and only if $C_{NJ}(X) < 2$.

(3) If $X < \frac{1+\sqrt{3}}{2}$, then X has normal structure.

The modulus of convexity introduced by Clarkson as

DEFINITION 3. [5] Let X be a Banach space, the modulus of convexity is defined as

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x,y \in S_X, \|x-y\| = \varepsilon\right\}, \quad 0 \leqslant \varepsilon \leqslant 2.$$

In addition, M. Baronti, E. Casini and P.L. Papini [2] introduced the constant $A_2(X)$.

DEFINITION 4. Let X be a Banach space, $A_2(X)$ is defined as

$$A_2(X) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} : x,y \in S_X\right\}.$$

The geometric constant $A_2(X)$ in Banach spaces satisfies:

(1) $1 \leqslant A_2(X) \leqslant 2$.

(2) If X is a Hilbert space, then $A_2(X) = \sqrt{2}$.

(3) X is uniformly non-square if and only if $A_2(X) < 2$.

The above famous constants help to understand some of the geometric properties of Banach spaces such as uniformly non-square and uniform normal structure. We recall now some important properties of Banach spaces.

DEFINITION 5. [11] A Banach space X is called uniformly non-square if there exists $\delta \in (0,1)$ such that for any $x,y \in S_X$, then

$$\frac{\|x+y\|}{2} \leqslant 1-\delta \text{ or } \frac{\|x-y\|}{2} \leqslant 1-\delta.$$

DEFINITION 6. [3] A Banach space X is said to have a (weak) normal structure, if for every (weakly compact) closed bounded convex subset K of X containing more than one point, there exists a point $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < \sup\{\|x-y\| : x,y \in K\}.$$

Moreover, a Banach space X is said to have uniform normal structure, if there exists $0 < c < 1$ such that for any closed bounded convex subset K of X containing more than one point, there exists a point $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < c\sup\{\|x-y\| : x,y \in K\}.$$

For a reflexive Banach spaces X, normal structure and weak normal structure coincide.

3. The constant $A_{\alpha-\beta}(X)$

In this section, inspired by Roberts orthogonality, we consider the following constant: let $\alpha,\beta>0$,

$$A_{\alpha-\beta}(X)=\sup\left\{\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}:x,y\in S_X\right\}.$$

To better understand the geometric picture, see Figure 1.

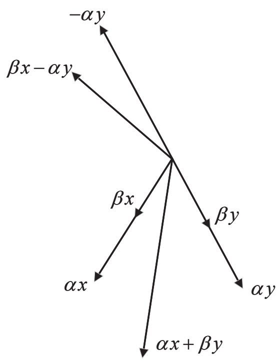

Figure 1: *x,y vector diagram of fixed parameters α and β.*

Next, we compute the value of the constant $A_{\alpha-\beta}(X)$ in a Hilbert space, and based on this, we give the upper and lower bounds of the constant $A_{\alpha-\beta}(X)$.

PROPOSITION 1. *If X is a Hilbert space, then for all $\alpha,\beta>0$, we have $A_{\alpha-\beta}(X)=\sqrt{\alpha^2+\beta^2}$.*

Proof. Since X is a Hilbert space, then for any $x,y\in S_X$ and $\alpha,\beta>0$, we have

$$\begin{aligned}&\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}\\ &=\frac{\sqrt{\|\alpha x+\beta y\|^2}+\sqrt{\|\beta x-\alpha y\|^2}}{2}\\ &=\frac{\sqrt{\alpha^2\|x\|^2+\beta^2\|y\|^2+2\alpha\beta\langle x,y\rangle}+\sqrt{\beta^2\|x\|^2+\alpha^2\|y\|^2-2\alpha\beta\langle x,y\rangle}}{2}\end{aligned}$$

where the supremum is attained for $x\perp y$. Therefore, by taking $x_0,y_0\in S_X$ such that $x_0\perp y_0$, we get $A_{\alpha-\beta}(X)=\sqrt{\alpha^2+\beta^2}$. $\square$

PROPOSITION 2. *Let X be an infinite dimensional Banach space. Then*

$$\sqrt{\alpha^2+\beta^2}\leqslant A_{\alpha-\beta}(X)\leqslant\alpha+\beta.$$

Proof. First, according to Dvoretzki's theorem [9] (see Theorem 10.43 of the reference), given $\varepsilon > 0$, if the dimension of X is large enough (in particular, if $\dim(X) = \infty$), then there exists a subspace Y of X, with $\dim(Y) = 2$, such that

$$\left|A_{\alpha-\beta}(Y) - \sqrt{\alpha^2+\beta^2}\right| < \varepsilon,$$

which implies that

$$A_{\alpha-\beta}(Y) \geqslant \sqrt{\alpha^2+\beta^2}.$$

On the other hand,

$$\frac{\|\alpha x+\beta y\| + \|\beta x-\alpha y\|}{2} \leqslant \alpha+\beta,$$

as desired. □

Now, we will compute the values of the constant $A_{\alpha-\beta}(X)$ for some specific spaces.

EXAMPLE 1. Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, then $A_{\alpha-\beta}(X) = \alpha+\beta$.
Let $x = (1,1)$, $y = (-1,1)$ satisfying $x,y \in S_X$. We have

$$\|\alpha x+\beta y\|_\infty = \|\beta x-\alpha y\|_\infty = \alpha+\beta.$$

Thus,

$$\frac{\|\alpha x+\beta y\|_\infty + \|\beta x-\alpha y\|_\infty}{2} = \alpha+\beta.$$

EXAMPLE 2. Let $X = \mathbb{R}^2$, $\alpha \geqslant \beta$, and assign the following $l_\infty - l_1$ norm

$$\|x\| = \|(x_1,x_2)\| = \begin{cases} \|x\|_1, & x_1x_2 \leqslant 0, \\ \|x\|_\infty, & x_1x_2 \geqslant 0. \end{cases}$$

Thanks to Krein-Milman theorem, we only need to consider extremal points. We may assume without loss of generality that $x = (a,b)$, $y = (c,d) \in \mathrm{ext}(B_X)$. We consider $a,c \geqslant 0$, $b,d \leqslant 0$. (In the other cases, the discussion is similar). Since $x,y \in S_X$, we have $a-b=1$ and $c-d=1$. Then $\|\alpha x+\beta y\| = \alpha+\beta$, $\|\beta x-\alpha y\| = \|(\beta a-\alpha c, \beta a-\alpha c-\beta+\alpha)\|$.

Case 1: $\beta a-\alpha c \geqslant 0$. We have

$$\frac{\|\alpha x+\beta y\| + \|\beta x-\alpha y\|}{2} \leqslant \alpha+\frac{\beta}{2},$$

Case 2: $\beta a-\alpha c-\beta+\alpha \leqslant 0$. We have

$$\begin{aligned}\frac{\|\alpha x+\beta y\| + \|\beta x-\alpha y\|}{2} &= \frac{\alpha+\beta+\alpha c-\beta a}{2} \\ &\leqslant \alpha+\frac{\beta}{2}.\end{aligned}$$

Case 3: $\beta a-\alpha c<0,\ \beta a-\alpha c-\beta+\alpha \geqslant 0$, we have

$$\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}=\alpha.$$

To sum up, for any $x,y\in \operatorname{ext}(B_X)$, we have

$$\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}\leqslant \alpha+\frac{\beta}{2}.$$

On the other hand, for $x=(1,1),\ y=(-1,0)$, we have

$$\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}=\alpha+\frac{\beta}{2}.$$

Thus,

$$A_{\alpha-\beta}(X)=\alpha+\frac{\beta}{2}.$$

REMARK 1. In Example 2, we chose the $l_\infty-l_1$ norm to demonstrate that the constants $A_{\alpha-\beta}(X)$ and $A'_{\alpha-\beta}(X)$ (see Section 4) are generally not equal.

After having obtained upper and lower bounds for $A_{\alpha-\beta}(X)$ and having computed the constant $A_{\alpha-\beta}(X)$ in some specific space, we further analyse the relation between the constant $A_{\alpha-\beta}(X)$ and other famous constants such as $J(X)$, $C_{NJ}(X)$, $C'_{NJ}(X)$, $A_2(X)$, $\delta_X(\varepsilon)$.

THEOREM 1. *Let X be a Banach space. Then*

$$\max\{\alpha,\beta\}J(X)-|\alpha-\beta|\leqslant A_{\alpha-\beta}(X)\leqslant \frac{1}{2}J(X)+1+|1-\alpha|+|1-\beta|.$$

Proof. For $x,y\in S_X$ and $\alpha,\beta>0$, we have

$$\begin{aligned}\alpha\min\{\|x+y\|,\|x-y\|\}&=\min\{\|\alpha x+\alpha y\|,\|\alpha x-\alpha y\|\}\\&\leqslant \min\{\|\alpha x+\beta y\|+|\alpha-\beta|,\|\beta x-\alpha y\|+|\alpha-\beta|\}\\&=\min\{\|\alpha x+\beta y\|,\|\beta x-\alpha y\|\}+|\alpha-\beta|\\&\leqslant \frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}+|\alpha-\beta|,\end{aligned}$$

and

$$\begin{aligned}\beta\min\{\|x+y\|,\|x-y\|\}&=\min\{\|\beta x+\beta y\|,\|\beta x-\beta y\|\}\\&\leqslant \min\{\|\alpha x+\beta y\|+|\alpha-\beta|,\|\beta x-\alpha y\|+|\alpha-\beta|\}\\&=\min\{\|\alpha x+\beta y\|,\|\beta x-\alpha y\|\}+|\alpha-\beta|\\&\leqslant \frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}+|\alpha-\beta|.\end{aligned}$$

This shows that $\max\{\alpha,\beta\}J(X)-|\alpha-\beta|\leqslant A_{\alpha-\beta}(X)$.

On the other hand, we have

$$\begin{aligned}\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2} &\leqslant \frac{\|x+y\|+\|x-y\|}{2}+|1-\alpha|+|1-\beta| \\ &= \frac{J(X)}{2}+1+|1-\alpha|+|1-\beta|,\end{aligned}$$

as desired. $\square$

In the following Remark 2, we only consider Banach spaces X with $\dim X\geqslant 3$.

REMARK 2. In Proposition 1, we stated that for Hilbert spaces, it holds $A_{\alpha-\beta}(X) = \sqrt{\alpha^2+\beta^2}$. Let us assume now that $A_{\alpha-\beta}(X)=\sqrt{\alpha^2+\beta^2}$. According to Theorem 1, we have

$$J(X)\leqslant\frac{\sqrt{\alpha^2+\beta^2}+|\alpha-\beta|}{\max\{\alpha,\beta\}}.$$

We know that when $J(X)=\sqrt{2}$, the space X is a Hilbert space [12]. Thus, if

$$\frac{\sqrt{\alpha^2+\beta^2}+|\alpha-\beta|}{\max\{\alpha,\beta\}}\leqslant\sqrt{2},$$

we have $J(X)=\sqrt{2}$. In this way, we might deduce that X is a Hilbert space by means of some lower bound for the constant $A_{\alpha-\beta}(X)$ and some constraints on α and β. In fact, set

$$Z(\alpha,\beta)=\frac{\sqrt{\alpha^2+\beta^2}+|\alpha-\beta|}{\max\{\alpha,\beta\}}-\sqrt{2}.$$

Assuming $\alpha\geqslant\beta$ without losing generality, we will simply obtain $\frac{\sqrt{\alpha^2+\beta^2}+|\alpha-\beta|}{\max\{\alpha,\beta\}}\geqslant\frac{\sqrt{2\alpha\beta}+|\alpha-\beta|}{\alpha}$. The function $Z(\alpha,\beta)$ attains its minimum value of 0 only when $\alpha=\beta$.

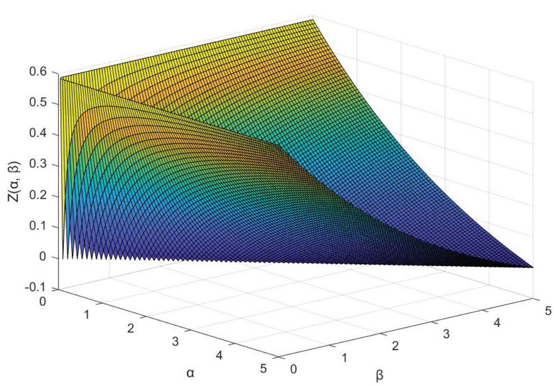

Figure 2: *Plots of* $\frac{\sqrt{\alpha^2+\beta^2}+|\alpha-\beta|}{\max\{\alpha,\beta\}}\geqslant\sqrt{2}$.

Figure 2 provides a more intuitive geometric image, we plot Z as a function of α and β.

According to the figure, it is also not difficult to see that when the parameters α,β are equal, we have that $Z(\alpha,\beta)=0$, then $J(X)=\sqrt{2}$, and we infer that X is a Hilbert space; if $\alpha\neq\beta$, then $Z(\alpha,\beta)>0$, we cannot infer that X is a Hilbert space.

Now, from Theorem 1 and the fact that X is uniformly non-square if and only if $J(X)<2$, we get a simple corollary.

COROLLARY 1. *Let X be a Banach space. Then the following three conditions are equivalent:*

(1) X is uniformly non-square.

(2) For all $\alpha,\beta\geqslant 1$, we have $A_{\alpha-\beta}(X)<\alpha+\beta$.

(3) There exist $\alpha,\beta\geqslant 1$ such that $A_{\alpha-\beta}(X)<\alpha+\beta$.

S. Dhompongsa, A. Kaewkhao and S. Tasena [6] introduced the following lemma. We will use this lemma to discuss the relationship between $A_{\alpha-\beta}(X)$ and the uniform normal structure.

LEMMA 1. *Let X be a Banach space. If $J(X)<\frac{1+\sqrt{5}}{2}$, then X has uniform normal structure.*

COROLLARY 2. *For any non-trivial Banach space X and for any $\alpha,\beta>0$, if*

$$A_{\alpha-\beta}(X)<\frac{(1+\sqrt{5})}{2}\max\{\alpha,\beta\}-|\alpha-\beta|,$$

then X has uniform normal structure.

Proof. The proof is obtained combining Theorem 1 and Lemma 1. □

Now, we will compare the constant $A_{\alpha-\beta}(X)$ with other well-known constants such as $C_{NJ}(X), C'_{NJ}(X), A_2(X), \delta_X(\varepsilon)$ in turn. First, we introduce the following lemma to compare the constant $A_{\alpha-\beta}(X)$ and $C_{NJ}(X)$.

LEMMA 2. [19] *Let X be a Banach space. Then*

$$C_{NJ}(X)\leqslant 1+\frac{J(X)^2}{4}.$$

PROPOSITION 3. *Let X be a Banach space. Then*

$$C_{NJ}(X)\leqslant 1+\left(\frac{A_{\alpha-\beta}(X)+|\alpha-\beta|}{2\max\{\alpha,\beta\}}\right)^2.$$

Proof. Apply Lemma 2 and Theorem 1, it's easy to get this result. □

THEOREM 2. *Let X be a Banach space. Then*

$$\begin{aligned}&\max\{\alpha,\beta\}C'_{NJ}(X)-|\alpha-\beta|\\&\leqslant A_{\alpha-\beta}(X)\\&\leqslant\sqrt{2\max\{\alpha^2,\beta^2\}C'_{NJ}(X)+2\sqrt{2}\max\{\alpha,\beta\}|\alpha-\beta|\sqrt{C'_{NJ}(X)}+|\alpha-\beta|^2}.\end{aligned}$$

Proof. For any $x,y\in S_X$, we have

$$\begin{aligned}&\left(\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}\right)^2\\&\leqslant\frac{\|\alpha x+\beta y\|^2+\|\beta x-\alpha y\|^2}{2}\\&\leqslant\frac{(\alpha\|x+y\|+|\alpha-\beta|\|y\|)^2+(\alpha\|x-y\|+|\alpha-\beta|\|x\|)^2}{2}\\&\leqslant 2\alpha^2C'_{NJ}(X)+2\sqrt{2}\alpha|\alpha-\beta|\sqrt{C'_{NJ}(X)}+|\alpha-\beta|^2.\end{aligned}$$

Similarly, the following inequality holds:

$$\begin{aligned}&\left(\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}\right)^2\\&\leqslant 2\beta^2C'_{NJ}(X)+2\sqrt{2}\beta|\alpha-\beta|\sqrt{C'_{NJ}(X)}+|\alpha-\beta|^2.\end{aligned}$$

On the other hand, since $C'_{NJ}(X)\leqslant J(X)$ and combining Theorem 1, we can obtain that

$$C'_{NJ}(X)\leqslant\frac{A_{\alpha-\beta}(X)+|\alpha-\beta|}{\max\{\alpha,\beta\}},$$

as desired. □

Next, we will compare the constant $A_{\alpha-\beta}(X)$ with the constant $A_2(X)$. We first prove the following lemma.

LEMMA 3. *(1) The function $f(t)=\|x+ty\|+\|tx-y\|$ is a convex function of t on $\mathbb{R}$.*

(2) The function $g(t)=\|tx+y\|+\|x-ty\|$ is a convex function of t on $\mathbb{R}$.

Proof. (1) Let $t_1,t_2\in\mathbb{R}$, $\lambda\in(0,1)$, we have

$$\begin{aligned}&\|x+(\lambda t_1+(1-\lambda)t_2)y\|+\|(\lambda t_1+(1-\lambda)t_2)x-y\|\\&\leqslant\lambda\left(\|x+t_1y\|+\|t_1x-y\|\right)+(1-\lambda)\left(\|x+t_2y\|+\|t_2x-y\|\right)\end{aligned}$$

which implies that

$$f(\lambda t_1+(1-\lambda)t_2)\leqslant\lambda f(t_1)+(1-\lambda)f(t_2).$$

(2) Obviously. □

PROPOSITION 4. *Let X be a Banach space. Then*

$$\frac{A_{\alpha-\beta}(X)}{\min\{\alpha,\beta\}} \geqslant A_2(X) \geqslant \frac{A_{\alpha-\beta}(X)}{\max\{\alpha,\beta\}}.$$

Proof. Consider the function $f:\mathbb{R}\to\mathbb{R}$ defined as

$$f(t) = \|x+ty\| + \|tx-y\|.$$

Then for any $x,y \in S_X$, we have

$$f(0) = \|x\| + \|y\| = 2$$

and

$$f(1) = f(-1) = \|x+y\| + \|x-y\| \geqslant 2.$$

Combined with Lemma 3, $f(t)$ is a convex function, then $f(\frac{\beta}{\alpha}) \geqslant f(1)$ for $\frac{\beta}{\alpha} \geqslant 1$ and $f(\frac{\beta}{\alpha}) \leqslant f(1)$ for $\frac{\beta}{\alpha} \leqslant 1$. Using the same technique, we also have $g(\frac{\alpha}{\beta}) \geqslant g(1)$ for $\frac{\alpha}{\beta} \geqslant 1$ and $g(\frac{\alpha}{\beta}) \leqslant g(1)$ for $\frac{\alpha}{\beta} \leqslant 1$.

Hence, for any $x,y \in S_X$ and $\beta \geqslant \alpha$, we obtain that

$$\frac{\alpha(\|x+\frac{\beta}{\alpha}y\| + \|\frac{\beta}{\alpha}x-y\|)}{2\alpha} \geqslant \frac{\|x+y\|+\|x-y\|}{2}.$$

In addition, we can also obtain that

$$\frac{\beta(\|\frac{\alpha}{\beta}x+y\| + \|x-\frac{\alpha}{\beta}y\|)}{2\beta} \leqslant \frac{\|x+y\|+\|x-y\|}{2}.$$

Thus

$$\frac{A_{\alpha-\beta}(X)}{\alpha} \geqslant A_2(X) \geqslant \frac{A_{\alpha-\beta}(X)}{\beta}.$$

Similarly, for any $x,y \in S_X$ and $\beta \leqslant \alpha$, we have

$$\frac{A_{\alpha-\beta}(X)}{\beta} \geqslant A_2(X) \geqslant \frac{A_{\alpha-\beta}(X)}{\alpha}.$$

In summary, we have

$$\frac{A_{\alpha-\beta}(X)}{\min\{\alpha,\beta\}} \geqslant A_2(X) \geqslant \frac{A_{\alpha-\beta}(X)}{\max\{\alpha,\beta\}}. \qquad \square$$

Finally, we compare the constant $A_{\alpha-\beta}(X)$ with the modulus of convexity.

PROPOSITION 5. *Let X be a Banach space. Then*

$$A_{\alpha-\beta}(X) \leqslant \max\{\alpha,\beta\} + \sup\left\{\frac{\max\{\alpha,\beta\}(\varepsilon-2\delta_X(\varepsilon))}{2};\varepsilon\in[0,2]\right\} + |\alpha-\beta|.$$

Proof. Taking $x \in S_X$, if $y \in S_X$ and $\|x-y\| = \varepsilon$, then $\frac{\|x+y\|}{2} \leqslant 1- \delta_X(\varepsilon)$. So we obtain, for any $y \in S_X$,

$$\begin{aligned}
&\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2} \\
&\leqslant \frac{(\alpha\|x+y\|+|\alpha-\beta|\|y\|)+(\alpha\|x-y\|+|\alpha-\beta|\|x\|)}{2} \\
&\leqslant \frac{\alpha(\|x+y\|+\|x-y\|)}{2}+|\alpha-\beta| \\
&\leqslant \frac{\alpha(\varepsilon+2(1-\delta_X(\varepsilon)))}{2}+|\alpha-\beta|.
\end{aligned}$$

By taking the supremum for $x, y \in S_X$, we have

$$\begin{aligned}
A_{\alpha-\beta}(X) &\leqslant \sup\left\{\frac{\alpha(\varepsilon+2(1-\delta_X(\varepsilon)))}{2}; \varepsilon \in [0,2]\right\}+|\alpha-\beta| \\
&= \alpha+\sup\left\{\frac{\alpha(\varepsilon-2\delta_X(\varepsilon))}{2}; \varepsilon \in [0,2]\right\}+|\alpha-\beta|.
\end{aligned}$$

Similarly, the following inequality holds:

$$A_{\alpha-\beta}(X) \leqslant \beta+\sup\left\{\frac{\beta(\varepsilon-2\delta_X(\varepsilon))}{2}; \varepsilon \in [0,2]\right\}+|\alpha-\beta|.$$

Hence we have the thesis. □

Now, we shall give an estimate concerning the difference in the values of the constant $A_{\alpha-\beta}(X)$ on X and $A_{\alpha-\beta}(X^*)$ on X^*.

THEOREM 3. *Let X be a Banach space. Then*

$$2A_{\alpha-\beta}(X)-(\alpha+\beta) \leqslant A_{\alpha-\beta}(X^*) \leqslant 2A_{\alpha-\beta}(x)-|\alpha-\beta|.$$

Proof. First, according to the definition of $A_{\alpha-\beta}(X)$, for any $\varepsilon > 0$, there exist $x, y \in S_X$ such that

$$\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}+\varepsilon \geqslant A_{\alpha-\beta}(X).$$

In addition, according to Hahn-Banach theorem, there exist $f, g \in S_{X^*}$ such that

$$f(\alpha x+\beta y) = \|\alpha x+\beta y\|, \quad g(\beta x-\alpha y) = \|\beta x-\alpha y\|.$$

Then, we have

$$\begin{aligned}
A_{\alpha-\beta}(X^*) &\geqslant \min\{\|\alpha f+\beta g\|, \|\beta f-\alpha g\|\} \\
&= \|\alpha f+\beta g\|+\|\beta f-\alpha g\|-\max\{\|\alpha f+\beta g\|, \|\beta f-\alpha g\|\} \\
&\geqslant \|\alpha f+\beta g\|+\|\beta f-\alpha g\|-(\alpha+\beta) \\
&\geqslant (\alpha f+\beta g)(x)+(\beta f-\alpha g)(y)-(\alpha+\beta) \\
&= f(\alpha x+\beta y)+g(\beta x-\alpha y)-(\alpha+\beta) \\
&= \|\alpha x+\beta y\|+\|\beta x-\alpha y\|-(\alpha+\beta) \\
&\geqslant 2A_{\alpha-\beta}(X)-2\varepsilon-(\alpha+\beta).
\end{aligned}$$

Let $\varepsilon \to 0$, we get

$$A_{\alpha-\beta}(X^*) \geqslant 2A_{\alpha-\beta}(X)-(\alpha+\beta).$$

On the other hand, let $u,v \in S_{X^*}$, then for any $\varepsilon > 0$, there exist $x,y \in S_X$ such that

$$(\alpha u+\beta v)(x) > \|\alpha u+\beta v\|-\varepsilon, \quad (\beta u-\alpha v)(y) > \|\beta u-\alpha v\|-\varepsilon.$$

Thus,

$$\begin{aligned}
\frac{\|\alpha u+\beta v\|+\|\beta u-\alpha v\|}{2} &\leqslant \max\{\|\alpha u+\beta v\|,\|\beta u-\alpha v\|\} \\
&= \|\alpha u+\beta v\|+\|\beta u-\alpha v\|-\min\{\|\alpha u+\beta v\|,\|\beta u-\alpha v\|\} \\
&\leqslant \|\alpha u+\beta v\|+\|\beta u-\alpha v\|-|\alpha-\beta| \\
&< (\alpha u+\beta v)(x)+(\beta u-\alpha v)(y)+2\varepsilon-|\alpha-\beta| \\
&= u(\alpha x+\beta y)+v(\beta x-\alpha y)+2\varepsilon-|\alpha-\beta| \\
&\leqslant \|\alpha x+\beta y\|+\|\beta x-\alpha y\|+2\varepsilon-|\alpha-\beta| \\
&\leqslant 2A_{\alpha-\beta}(X)+2\varepsilon-|\alpha-\beta|.
\end{aligned}$$

Let $\varepsilon \to 0$, we get

$$A_{\alpha-\beta}(X^*) \leqslant 2A_{\alpha-\beta}(X)-|\alpha-\beta|. \quad \square$$

4. The constant $A'_{\alpha-\beta}(X)$

Taking into account the impact of orthogonality on the constant $A_{\alpha-\beta}(X)$, we impose a condition where x and y adhere to the principle of isosceles orthogonality. This leads us to introduce the following new constant: let $\alpha,\beta > 0$, $\dim X \geqslant 3$,

$$A'_{\alpha-\beta}(X) = \sup\left\{\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2} : x,y \in S_X, x \perp_I y\right\}.$$

REMARK 3. Since

$$\begin{aligned}
J(X) &= \sup\{\min\{\|x+y\|,\|x-y\|\} : x,y \in S_X, x \perp_I y\} \\
&= \sup\{\|x+y\| : x,y \in S_X, x \perp_I y\}, \\
&= \sup\{\min\{\|x+y\|,\|x-y\|\} : x,y \in S_X\}.
\end{aligned}$$

If $A'_{1-1}(X) = \sqrt{2}$, we can infer that

$$\begin{aligned}
&\sup\{\min\{\|x+y\|,\|x-y\|\} : x,y \in S_X, x \perp_I y\} \\
&\leqslant \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} : x,y \in S_X, x \perp_I y\right\} \\
&\leqslant \sqrt{2},
\end{aligned}$$

which implies $J(X) = \sqrt{2}$. Hence X is a Hilbert space (see page 328).

REMARK 4. It is easy to get that

$$A'_{\alpha-\beta}(X) = A'_{\beta-\alpha}(X).$$

Below, we compute the value of the constant $A'_{\alpha-\beta}(X)$ in a Hilbert space, and based on this, we give the upper and lower bounds of the constant $A'_{\alpha-\beta}(X)$.

PROPOSITION 6. *If X is a Hilbert space, then for all $\alpha,\beta > 0$, we have $A'_{\alpha-\beta}(X) = \sqrt{\alpha^2+\beta^2}$.*

Proof. The proof is obtained with the same technique as in Proposition 1. □

PROPOSITION 7. *Let X be an infinite dimensional Banach space. Then*

$$\sqrt{\alpha^2+\beta^2} \leqslant A'_{\alpha-\beta}(X) \leqslant \alpha+\beta.$$

Proof. The proof is obtained with the same technique as in Proposition 2. □

Next, we compute the relationship between the constant $A'_{\alpha-\beta}(X)$ and the James constant $J(X)$.

THEOREM 4. *Let X be a Banach space. Then*

$$\max\{\alpha,\beta\}J(X) - |\alpha-\beta| \leqslant A'_{\alpha-\beta}(X) \leqslant \frac{1}{2}J(X) + 1 + |1-\alpha| + |1-\beta|.$$

Proof. The proof is obtained with the same technique as in Theorem 1. □

Finally, we show with a counterexample that the constants $A'_{\alpha-\beta}(X)$ and $A_{\alpha-\beta}(X)$ are generally different. Isosceles orthogonality condition plays an important role in the constant structure.

EXAMPLE 3. Let $X = \mathbb{R}^2$, $\alpha \geqslant \beta$, and assign the following $l_\infty - l_1$ norm

$$\|x\| = \|(x_1,x_2)\| = \begin{cases} \|x\|_1, & x_1x_2 \leqslant 0, \\ \|x\|_\infty, & x_1x_2 \geqslant 0. \end{cases}$$

Then

$$A'_{\alpha-\beta}(X) = \begin{cases} \frac{3}{4}\alpha + \frac{3}{4}\beta & 0 < \beta \leqslant \alpha \leqslant 2\beta, \\ \alpha + \frac{1}{4}\beta & 2\beta < \alpha. \end{cases}$$

If $x = (y_1, 1+y_1)$, $y = (y_2, 1+y_2)$, where $-1 \leqslant y_1 \leqslant y_2 \leqslant 0$; $x = (y_1, y_1-1)$, $y = (y_2, y_2-1)$, where $0 \leqslant y_1 \leqslant y_2 \leqslant 1$. For both cases, by $x \perp_I y$, we can get $|y_1 - y_2| = 2$, which is contradictory. In order to estimate the constant value, only the following two cases need to be considered.

Case 1: Assuming that $x=(x_1,1)$, $y=(1,y_2)$, $0\leqslant x_1\leqslant y_2\leqslant 1$. Since $x\perp_I y$, we have

$$1+y_2=(1-x_1)+(1-y_2),$$

and hence $x_1+2y_2=1$, $y_2\in\left[\frac{1}{3},\frac{1}{2}\right]$.

Assuming that $\frac{1}{2}\leqslant\frac{\beta}{\alpha}$, we have

$$\begin{aligned}\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}&=\frac{2\alpha+(3\beta-\alpha)y_2}{2}\\&\leqslant\frac{3}{4}\alpha+\frac{3}{4}\beta.\end{aligned}$$

Taking the maximum at $x=(0,1)$, $y=\left(1,\frac{1}{2}\right)$ satisfies $\|x+y\|=\|x-y\|=\frac{3}{2}$.

Assuming that $\frac{1}{3}\leqslant\frac{\beta}{\alpha}\leqslant\frac{1}{2}$.

- When $\frac{1}{3}\leqslant y_2\leqslant\frac{\beta}{\alpha}$, $\|\beta x-\alpha y\|=\alpha+2\beta y_2-\alpha y_2$

$$\begin{aligned}\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}&=\frac{2\alpha+(3\beta-\alpha)y_2}{2}\\&<\frac{3}{4}\alpha+\frac{3}{4}\beta.\end{aligned}$$

- When $\frac{\beta}{\alpha}<y_2\leqslant\frac{1}{2}$, $\|\beta x-\alpha y\|=\alpha-\beta x_1$,

$$\begin{aligned}\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}&=\frac{2\alpha+\beta(3y_2-1)}{2}\\&\leqslant\alpha+\frac{1}{4}\beta.\end{aligned}$$

Taking the maximum at $x=(0,1)$, $y=\left(1,\frac{1}{2}\right)$ satisfies $\|x+y\|=\|x-y\|=\frac{3}{2}$.

Assuming that $\frac{1}{3}\geqslant\frac{\beta}{\alpha}$, we have

$$\begin{aligned}\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}&=\frac{2\alpha+\beta(3y_2-1)}{2}\\&\leqslant\alpha+\frac{1}{4}\beta.\end{aligned}$$

Case 2: Assuming that $x=(x_1,1)$, $y=(y_1,1+y_1)$ satisfies $-1\leqslant y_1\leqslant 0\leqslant x_1\leqslant 1$. Since $x\perp_I y$, we have $\|(x_1+y_1,2+y_1)\|=\|(x_1-y_1,-y_1)\|$.

If $-x_1\leqslant y_1$, then $2+y_1=x_1-y_1$ is true, hence $x_1-2y_1=2$, $y_1\in\left[-\frac{2}{3},-\frac{1}{2}\right]$.

Assuming that $\beta\leqslant\alpha\leqslant 2\beta$, we have

$$\begin{aligned}\frac{\|\alpha x+\beta y\|+\|\beta x-\alpha y\|}{2}&\leqslant\frac{\alpha+3\beta+(3\beta-\alpha)y_1}{2}\\&\leqslant\frac{3}{4}\alpha+\frac{3}{4}\beta.\end{aligned}$$

Taking the maximum at $x=(1,1)$, $y=\left(-\frac{1}{2},\frac{1}{2}\right)$ satisfies $\|x+y\|=\|x-y\|=\frac{3}{2}$.

Assuming that $2\beta < \alpha < 3\beta$.

- When $-\frac{2}{3} \leqslant y_1 \leqslant \frac{\beta-\alpha}{\alpha}$, $\|\beta x - \alpha y\| = \beta x_1 - \alpha y_1$

$$\begin{aligned}\frac{\|\alpha x+\beta y\| + \|\beta x - \alpha y\|}{2} &\leqslant \frac{\alpha + 3\beta + (3\beta - \alpha)y_1}{2} \\ &< \frac{3}{4}\alpha + \frac{3}{4}\beta.\end{aligned}$$

- When $\frac{\beta-\alpha}{\alpha} < y_1 \leqslant -\frac{1}{2}$, $\|\beta x - \alpha y\| = \alpha - \beta + \beta x_1$

$$\begin{aligned}\frac{\|\alpha x+\beta y\| + \|\beta x - \alpha y\|}{2} &\leqslant \frac{2\alpha + 2\beta + 3\beta y_1}{2} \\ &\leqslant \alpha + \frac{1}{4}\beta.\end{aligned}$$

Assuming that $\alpha \geqslant 3\beta$, we have

$$\begin{aligned}\frac{\|\alpha x+\beta y\| + \|\beta x - \alpha y\|}{2} &\leqslant \frac{2\alpha + 2\beta + 3\beta y_1}{2} \\ &\leqslant \alpha + \frac{1}{4}\beta.\end{aligned}$$

Similarly, if $y_1 \leqslant -x_1$, as discussed in Case 2, the proof is omitted.

Combining the above two cases, we have

$$A'_{\alpha-\beta}(X) = \begin{cases} \frac{3}{4}\alpha + \frac{3}{4}\beta & 0 < \beta \leqslant \alpha \leqslant 2\beta, \\ \alpha + \frac{1}{4}\beta & \alpha > 2\beta. \end{cases}$$

Combining Example 2, we get that for X in the space above, $A_{\alpha-\beta}(X) > A'_{\alpha-\beta}(X)$ for any $\alpha \geqslant \beta > 0$. Therefore, the introduction of the constant $A'_{\alpha-\beta}(X)$ in this section is valuable.

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