

## A LOGARITHMIC WEIGHTED ADAMS–TYPE INEQUALITY IN THE WHOLE OF $\mathbb{R}^N$ WITH AN APPLICATION

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**Abstract.** In this paper, we will establish a logarithmic weighted Adams inequality in a logarithmic weighted second order Sobolev space in the whole set of  $\mathbb{R}^N$ . Using this result, we delve into the analysis of a weighted fourth-order equation in  $\mathbb{R}^N$ . We assume that the nonlinearity of the equation exhibits either critical or subcritical exponential growth, consistent with the Adams-type inequalities previously established. By applying the Mountain Pass Theorem, we demonstrate the existence of a weak solution to this problem. The primary challenge lies in the lack of compactness in the energy caused by the critical exponential growth of the non-linear term  $f$ .

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