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JENSEN'S INEQUALITY FOR $\mathscr{M}_\Psi$–CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. This paper investigates novel refinements and reversals of Jensen's inequality within the framework of generalized convexity, particularly focusing on M_Ψ-convex functions. These functions extend classical convexity by incorporating nonlinear mean-type structures via a strictly monotonic function Ψ. We present sharper forms of both Jensen and Jensen-Mercer inequalities, providing double-sided bounds and reverse inequalities that significantly improve classical results.

Building upon recent advances, our contributions include enhanced inequalities adapted to k-harmonically and k-geometrically convex functions. These extensions are achieved by selecting specific transformation functions Ψ, such as $\Psi(t) = 1/(t-k)$ and $\Psi(t) = \log(t-k)$, which yield new insights into the structure of generalized convexity.

Furthermore, we establish Jensen-type inequalities in operator settings, leveraging harmonic convexity. Our operator inequalities yield refined spectral bounds and deepen the connection between convexity and functional analysis. In particular, a new operator-Jensen inequality and a McCarthy-type inequality are proved for the class of harmonic convex functions.

Altogether, this unified treatment of generalized convexity broadens the applicability of classical inequalities and offers powerful tools for future studies in analysis, optimization, information theory, and operator theory.

Mathematics subject classification (2020): Primary 26A51, 47A30; Secondary 47A12, 47B15.
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