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JENSEN'S INEQUALITY FOR $\mathscr{M}_\Psi$–CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. This paper investigates novel refinements and reversals of Jensen's inequality within the framework of generalized convexity, particularly focusing on M_Ψ-convex functions. These functions extend classical convexity by incorporating nonlinear mean-type structures via a strictly monotonic function Ψ. We present sharper forms of both Jensen and Jensen-Mercer inequalities, providing double-sided bounds and reverse inequalities that significantly improve classical results.

Building upon recent advances, our contributions include enhanced inequalities adapted to k-harmonically and k-geometrically convex functions. These extensions are achieved by selecting specific transformation functions Ψ, such as $\Psi(t) = 1/(t-k)$ and $\Psi(t) = \log(t-k)$, which yield new insights into the structure of generalized convexity.

Furthermore, we establish Jensen-type inequalities in operator settings, leveraging harmonic convexity. Our operator inequalities yield refined spectral bounds and deepen the connection between convexity and functional analysis. In particular, a new operator-Jensen inequality and a McCarthy-type inequality are proved for the class of harmonic convex functions.

Altogether, this unified treatment of generalized convexity broadens the applicability of classical inequalities and offers powerful tools for future studies in analysis, optimization, information theory, and operator theory.

1. Introduction and preliminaries

A function $f : I \to \mathbb{R}$, defined on an interval $I \subseteq \mathbb{R}$, is said to be convex if for all $u, v \in I$ and $\kappa \in [0,1]$, the inequality

$$f(\kappa u + (1-\kappa)v) \leqslant \kappa f(u) + (1-\kappa) f(v)$$

holds. This definition captures the intuitive idea that the graph of a convex function lies below the straight line segment joining any two points on its graph.

Convex functions are fundamental objects in mathematical analysis due to their well-behaved structural properties. They are automatically continuous on the interior of their domain, possess subdifferentials, and obey powerful inequalities. In the field of optimization, convexity ensures that any local minimum is also a global one, which significantly simplifies both theoretical investigations and numerical methods. As such,

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convex optimization has become a cornerstone of applied mathematics, economics, data science, and control theory.

Convexity also underlies numerous classical inequalities. Among the most celebrated is Jensen's inequality, which asserts that for a convex function f and real numbers $u_1,\ldots,u_n \in I$ with positive weights $\kappa_1,\ldots,\kappa_n$ summing to 1, we have

$$f\left(\sum_{i=1}^{n}\kappa_i u_i\right) \leqslant \sum_{i=1}^{n}\kappa_i f(u_i). \tag{1}$$

This inequality has profound implications in fields ranging from probability theory and information theory to mathematical physics.

In entropy analysis, convexity plays a crucial role as well. The Shannon entropy function, $H(p) = -\sum_i p_i \log p_i$, is concave, and its key properties can be derived using Jensen's inequality. These ideas are fundamental in information theory, thermodynamics, and statistical mechanics, where entropy quantifies disorder or uncertainty.

For additional results highlighting the applicability of convexity across various domains such as matrix means, operator theory, and entropy the reader is referred to [6, 10, 12, 13, 16, 17, 18, 19, 20].

A significant and natural extension of Jensen's inequality is the Jensen-Mercer inequality, which incorporates symmetric structures and refinements of the classical formulation. Owing to its flexibility and broad applicability, it has garnered substantial interest in the mathematical literature.

One particularly notable version of this refined inequality is the following

$$f\left(H+h-\sum_{i=1}^{n}\kappa_i u_i\right) \leqslant f(H)+f(h)-\sum_{i=1}^{n}\kappa_i f(u_i), \tag{2}$$

where f is a convex function defined on an interval $[h,H]$, $\kappa_i > 0$ are weights with $\sum_{i=1}^{n}\kappa_i = 1$, and $u_1,\ldots,u_n \in [h,H]$. Inequality (2) reflects a delicate balance between symmetric means and function evaluations and plays a significant role in sharpening existing bounds.

Recently, in [14] a refinement and a reverse form of Jensen's inequality has been given. These results offer sharper bounds compared to the classical version, in the following form.

THEOREM 1. ([14]) *Let f be a convex function defined on an interval $[h,H]$. Then for every collection of points $u_1,\ldots,u_n \in [h,H]$, and every set of positive real numbers $\kappa_1,\ldots,\kappa_n$ such that $\sum_{i=1}^{n}\kappa_i = 1$, the following inequalities hold:*

$$\begin{aligned} f\left(\sum_{i=1}^{n}\kappa_i u_i\right) &\leqslant 2\sum_{i=1}^{n}\kappa_i f\left(\frac{\sum_{j=1}^{n}\kappa_j u_j + u_i}{2}\right) - f\left(\sum_{i=1}^{n}\kappa_i u_i\right) \\ &\leqslant \sum_{i=1}^{n}\kappa_i f(u_i). \end{aligned} \tag{3}$$

Moreover,

$$\sum_{i=1}^{n}\kappa_i f(u_i)-f\left(\sum_{i=1}^{n}\kappa_i u_i\right)\leqslant 2\left[\sum_{i=1}^{n}\kappa_i f(u_i)-\sum_{i=1}^{n}\kappa_i f\left(\frac{\sum_{j=1}^{n}\kappa_j u_j+u_i}{2}\right)\right]. \tag{4}$$

In the same work [14], the authors also derived a significant refinement and reverse form of the Jensen–Mercer inequality, which we state next.

THEOREM 2. ([14]) *Let f be a convex function defined on the interval $[h,H]$. Then for every $u_1,\dots,u_n\in[h,H]$ and every set of positive real numbers $\kappa_1,\dots,\kappa_n$ with $\sum_{i=1}^{n}\kappa_i=1$, the following inequality holds:*

$$\begin{aligned}f\left(H+h-\sum_{i=1}^{n}\kappa_i u_i\right)&\leqslant 2\sum_{i=1}^{n}\kappa_i f\left(H+h-\frac{\sum_{j=1}^{n}\kappa_j u_j+u_i}{2}\right)-f\left(H+h-\sum_{i=1}^{n}\kappa_i u_i\right)\\&\leqslant f(H)+f(h)-\sum_{i=1}^{n}\kappa_i f(u_i).\end{aligned} \tag{5}$$

Moreover,

$$\begin{aligned}&f(H)+f(h)-\sum_{i=1}^{n}\kappa_i f(u_i)-f\left(H+h-\sum_{i=1}^{n}\kappa_i u_i\right)\\&\leqslant 2\left[f(H)+f(h)-\sum_{i=1}^{n}\kappa_i f(u_i)-\sum_{i=1}^{n}\kappa_i f\left(H+h-\frac{\sum_{j=1}^{n}\kappa_j u_j+u_i}{2}\right)\right].\end{aligned} \tag{6}$$

For some recent works on the Jensen gap, which is also the main topic of this manuscript, the referee is kindly referred to the following paper [5].

The development of Jensen–Mercer-type inequalities has led to a broad spectrum of applications, ranging from numerical integration and operator theory to information theory and entropy analysis. These inequalities not only generalize the classical Jensen inequality but also establish deeper connections between convexity and functional estimates. In particular, they have been instrumental in studying generalized convexity classes, such as $\mathscr{M}_\Psi$-convex functions, where the usual convex combination is replaced by more generalized mean-type structures.

The quasi-arithmetic mean is defined, for a continuous and strictly monotonic function Ψ, as

$$\mathscr{M}_\Psi(\kappa,u,v):=\Psi^{-1}\left(\kappa\Psi(u)+(1-\kappa)\Psi(v)\right), \tag{7}$$

where $u,v\in I\subset\mathbb{R}$ and $\kappa\in[0,1]$. This general form recovers several classical means for specific choices of Ψ: the arithmetic mean for $\Psi(x)=x$, the geometric mean for $\Psi(x)=\ln x$, and the harmonic mean for $\Psi(x)=1/x$. Such means extend linear interpolation to nonlinear contexts and play a central role in generalized convexity and functional inequalities.

DEFINITION 1. ([11, 23]) A set $I\subseteq\mathbb{R}$ is called $\mathscr{M}_\Psi$-convex if

$$\mathscr{M}_\Psi(\kappa,u,v)\in I\quad\text{for all } u,v\in I \text{ and } \kappa\in[0,1]. \tag{8}$$

This generalization broadens the class of admissible sets, making it applicable to nonlinear analysis, generalized inequalities, and related fields. Among these generalizations, one particularly rich and flexible class is that of $\mathscr{M}_\Psi$-convex functions, which encompasses many known convexity concepts as special cases. The key idea is to replace the linear interpolation in standard convexity with interpolation via a nonlinear mean governed by the function Ψ.

For the remainder of this paper, we consider the interval $I \subset \mathbb{R}$ and $[h,H] \subset \mathbb{R}$ to be an M_Ψ-convex subset of $\mathbb{R}$.

DEFINITION 2. ([11, 22, 23]) A function $f : I \to \mathbb{R}$ is said to be $\mathscr{M}_\Psi$-convex if, for every $u,v \in I$ and $\kappa \in [0,1]$, the following inequality holds:

$$f(\mathscr{M}_\Psi(\kappa,u,v)) \leqslant \kappa f(u) + (1-\kappa)f(v). \tag{9}$$

If the inequality (9) is reversed, the function f is said to be $\mathscr{M}_\Psi$-concave.

REMARK 1. As a direct consequence of this definition, we have the following observations.

1. By setting $\Psi(t) = t$ in our definition, we directly recover the concept of convexity.
2. Furthermore, by setting $\Psi(t) = \frac{1}{t-k}$ for $t > k$, we obtain the notion of k-harmonically convexity as introduced in [21].
3. Additionally, when $\Psi(t) = \log(t-k)$ for $t > k$, we arrive at the notion of k-geometric convexity introduced in [3].

The classical Jensen's inequality has been extended to a broader framework involving the concept of $\mathscr{M}_\Psi$-convexity, as described below.

THEOREM 3. ([22]) *Let f be an $\mathscr{M}_\Psi$-convex function defined on the interval I. Let $u_1,\dots,u_n \in I$, and let $\kappa_1,\dots,\kappa_n$ be positive weights such that $\sum_{i=1}^n \kappa_i = 1$. Then*

$$f\left(\Psi^{-1}\left(\sum_{i=1}^n \kappa_i \Psi(u_i)\right)\right) \leqslant \sum_{i=1}^n \kappa_i f(u_i). \tag{10}$$

If the function f is $\mathscr{M}_\Psi$-concave instead of $\mathscr{M}_\Psi$-convex, then the direction of the inequality in (10) is reversed.

This inequality generalizes numerous well-known Jensen-type inequalities found in the literature.

Very recently Luo et al. [11] established a Jensen-Mercer type inequality for $\mathscr{M}_\Psi$-convex functions, as follows.

THEOREM 4. *Let f be an $\mathscr{M}_\Psi$-convex function defined on the interval $[h,H]$. For any collection of points $u_1,\dots,u_n \in [h,H]$ and any set of positive weights $\kappa_1,\dots,\kappa_n$ satisfying $\sum_{i=1}^n \kappa_i = 1$, the following holds:*

$$f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\sum_{i=1}^n \kappa_i \Psi(u_i)\right)\right) \leqslant f(H)+f(h)-\sum_{i=1}^n \kappa_i f(u_i). \tag{11}$$

This framework provides a unified approach to study a variety of convexity-related inequalities, and will be particularly useful in the context of Jensen-type and Jensen–Mercer-type inequalities explored in this work.

This paper is devoted to a comprehensive and systematic study of Jensen and Jensen–Mercer type inequalities, with particular emphasis on their generalized forms for the class of $\mathscr{M}_\Psi$-convex functions. Our primary objective is to extend Theorems 1 and 2 to this broader context, thereby providing a variety of new extensions, refinements, and reverse inequalities that build upon those introduced earlier in this work. These generalizations not only enrich the theoretical landscape of convexity-related inequalities but also yield meaningful implications in the field of mathematical analysis, particularly in relation to mean-type inequalities. Furthermore, we establish and refine a Jensen-type inequality tailored for operators, specifically within the framework of harmonically convex functions. This result serves as a significant advancement, bridging classical Jensen theory with operator analysis through the lens of harmonic convexity.

After that, several operator inequalities are proved for harmonic convex functions, extending some known forms for convex functions, when dealing with Hilbert space operators. For this particular interest, which is a main goal of this work, we refer the reader to Section 3 below.

For example, we show that if f is a harmonic convex function defined on the interval $[0,+\infty)$, $\mathscr{T}$ is a positive invertible operator on complex separable Hilbert space, and if $x \in \mathscr{H}$ is a unit vector, the following holds

$$\begin{aligned} f(\langle \mathscr{T}^{-1}x,x\rangle^{-1}) &\leqslant 2\left\langle f\left(\frac{\mathscr{T}^{-1}+\langle \mathscr{T}^{-1}x,x\rangle \cdot I_{\mathscr{H}}}{2}\right)^{-1} x,x\right\rangle - f(\langle \mathscr{T}^{-1}x,x\rangle^{-1}) \\ &\leqslant \langle f(\mathscr{T})x,x\rangle. \end{aligned}$$

This provides a significant extension of the well-known operator Jensen inequality that is valid for convex functions. Many other operator versions are discussed with striking applications that generalize some celebrated results.

2. On Jensen and Jensen–Mercer type inequalities for $\mathscr{M}_\Psi$-convex functions

In this section, we present refinements and reverses of the Jensen and Jensen–Mercer type inequalities for the class of $\mathscr{M}_\Psi$-convex functions. These results are developed in accordance with Theorems 1 and 2, providing significant improvements over the classical forms. Our contributions extend the theoretical framework of these inequalities by offering more general and precise versions tailored to $\mathscr{M}_\Psi$-convexity.

This marks a notable advancement in the analysis of convex-type inequalities and broadens their applicability within mathematical analysis.

THEOREM 5. *Let f be an $\mathscr{M}_{\Psi}$-convex function defined on the interval $[h,H]$, let $u_1,\dots,u_n \in [h,H]$, and let $\kappa_1,\dots,\kappa_n$ be positive weights satisfying $\sum_{i=1}^{n}\kappa_i = 1$. Then*

$$\begin{aligned}
&f\left(\Psi^{-1}\left(\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right)\\
&\leqslant 2\sum_{i=1}^{n}\kappa_i f\left(\Psi^{-1}\left(\frac{\sum_{j=1}^{n}\kappa_j\Psi(u_j)+\Psi(u_i)}{2}\right)\right)-f\left(\Psi^{-1}\left(\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right)\\
&\leqslant \sum_{i=1}^{n}\kappa_i f(u_i),
\end{aligned} \tag{12}$$

and likewise,

$$\begin{aligned}
&\sum_{i=1}^{n}\kappa_i f(u_i)+2\left[\sum_{i=1}^{n}\kappa_i f(u_i)-\sum_{i=1}^{n}\kappa_i f\left(\Psi^{-1}\left(\frac{\sum_{j=1}^{n}\kappa_j\Psi(u_j)+\Psi(u_i)}{2}\right)\right)\right]\\
&\leqslant f\left(\Psi^{-1}\left(\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right).
\end{aligned} \tag{13}$$

The inequalities (12) and (13) are satisfied in the reverse direction when f is M_{Ψ}-concave.

Proof. To establish the first inequality in (12), observe the following:

$$\begin{aligned}
&2\sum_{i=1}^{n}\kappa_i f\left(\Psi^{-1}\left(\frac{\sum_{j=1}^{n}\kappa_j\Psi(u_j)+\Psi(u_i)}{2}\right)\right)-f\left(\Psi^{-1}\left(\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right)\\
&\geqslant 2f\left(\sum_{i=1}^{n}\kappa_i\cdot\frac{\sum_{j=1}^{n}\kappa_j\Psi(u_j)+\Psi(u_i)}{2}\right)-f\left(\Psi^{-1}\left(\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right)\\
&\qquad\text{(by Jensen's inequality (10))}\\
&=2f\left(\Psi^{-1}\left(\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right)-f\left(\Psi^{-1}\left(\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right)\\
&=f\left(\Psi^{-1}\left(\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&2\sum_{i=1}^{n}\kappa_i f\left(\Psi^{-1}\left(\frac{\sum_{j=1}^{n}\kappa_j\Psi(u_j)+\Psi(u_i)}{2}\right)\right)-f\left(\Psi^{-1}\left(\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right)\\
&\leqslant 2\sum_{i=1}^{n}\kappa_i\cdot\frac{f\left(\Psi^{-1}\left(\sum_{j=1}^{n}\kappa_j\Psi(u_j)\right)\right)+f(u_i)}{2}-f\left(\Psi^{-1}\left(\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right)\\
&\quad\text{(by } \mathscr{M}_\Psi\text{-convexity)}\\
&=f\left(\Psi^{-1}\left(\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right)+\sum_{i=1}^{n}\kappa_i f(u_i)-f\left(\Psi^{-1}\left(\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right)\\
&=\sum_{i=1}^{n}\kappa_i f(u_i),
\end{aligned}
$$

completing the proof of inequality (12).

Finally, inequality (13) is a direct consequence of the bound:

$$
\begin{aligned}
&\sum_{i=1}^{n}\kappa_i f\left(\Psi^{-1}\left(\frac{\sum_{j=1}^{n}\kappa_j\Psi(u_j)+\Psi(u_i)}{2}\right)\right)\\
&\leqslant\frac{1}{2}\left(f\left(\Psi^{-1}\left(\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right)+\sum_{i=1}^{n}\kappa_i f(u_i)\right),
\end{aligned}
\tag{14}
$$

which follows from the M_Ψ-convexity of f. □

By setting $\Psi(t)=\frac{1}{t-k}$ for $t>k$ in Theorem 5, we obtain a new refinement and reverse form of the Jensen-type inequality adapted to the class of k-harmonically convex functions. This result, as established in [21], provides a significant extension of the classical inequality by incorporating the structure of k-harmonic convexity, thereby enriching the theory with broader applicability and sharper bounds.

THEOREM 6. *Let f be a k-harmonically convex function defined on the interval I. Let $u_1,\ldots,u_n\in I$, and let $\kappa_1,\ldots,\kappa_n$ be positive weights such that $\sum_{i=1}^{n}\kappa_i=1$. Then*

$$
\begin{aligned}
&f\left(\frac{1}{\sum_{i=1}^{n}\frac{\kappa_i}{u_i-k}}+k\right)\\
&\leqslant 2\sum_{i=1}^{n}\kappa_i f\left(\frac{2}{\sum_{j=1}^{n}\frac{\kappa_j}{u_j-k}+\frac{1}{u_i-k}}+k\right)-f\left(\frac{1}{\sum_{i=1}^{n}\frac{\kappa_i}{u_i-k}}+k\right)\\
&\leqslant\sum_{i=1}^{n}\kappa_i f(u_i),
\end{aligned}
\tag{15}
$$

and likewise,

$$\sum_{i=1}^{n}\kappa_i f(u_i)+2\left[\sum_{i=1}^{n}\kappa_i f(u_i)-\sum_{i=1}^{n}\kappa_i f\left(\frac{2}{\sum_{j=1}^{n}\frac{\kappa_j}{u_j-k}+\frac{1}{u_i-k}}+k\right)\right]$$
$$\leqslant f\left(\frac{1}{\sum_{i=1}^{n}\frac{\kappa_i}{u_i-k}}+k\right). \tag{16}$$

By choosing $\Psi(t)=\log(t-k)$ for $t>k$ in Theorem 5, a novel refinement and reverse form of the Jensen-type inequality is derived, tailored to the class of k-geometrically convex functions. As shown in [3], this result significantly extends the classical inequality by integrating the framework of k-geometric convexity, thereby enhancing its scope and yielding more precise bounds.

THEOREM 7. *Let f be a k-geometrically convex function defined on the interval I. Let $u_1,\ldots,u_n\in I$, and let $\kappa_1,\ldots,\kappa_n$ be positive weights such that $\sum_{i=1}^{n}\kappa_i=1$. Then*

$$f\left(\prod_{i=1}^{n}(u_i-k)^{\kappa_i}+k\right)$$
$$\leqslant 2\sum_{i=1}^{n}\kappa_i f\left(\sqrt{\prod_{j=1}^{n}(u_j-k)^{\kappa_j}(u_i-k)}+k\right)-f\left(\prod_{i=1}^{n}(u_i-k)^{\kappa_i}+k\right)$$
$$\leqslant \sum_{i=1}^{n}\kappa_i f(u_i), \tag{17}$$

and likewise,

$$\sum_{i=1}^{n}\kappa_i f(u_i)+2\left[\sum_{i=1}^{n}\kappa_i f(u_i)-\sum_{i=1}^{n}\kappa_i f\left(\sqrt{\prod_{j=1}^{n}(u_j-k)^{\kappa_j}(u_i-k)}+k\right)\right]$$
$$\leqslant f\left(\prod_{i=1}^{n}(u_i-k)^{\kappa_i}+k\right). \tag{18}$$

In the next theorem, we establish a Jensen-Mercer type inequality for $\mathscr{M}_\Psi$-convex functions, which extends and generalizes Theorem 2. This result broadens the scope of the classical framework and demonstrates the flexibility of $\mathscr{M}_\Psi$-convexity in deriving refined inequalities.

THEOREM 8. *Let f be an $\mathscr{M}_\Psi$-convex function defined on the interval $[h,H]$. For any collection of points $u_1,\ldots,u_n\in[h,H]$ and any set of positive weights $\kappa_1,\ldots,\kappa_n$*

satisfying $\sum_{i=1}^{n}\kappa_i = 1$*, the following holds true*

$$\begin{aligned}
& f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right) \\
& \leqslant 2\sum_{i=1}^{n}\kappa_i f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\frac{\sum_{j=1}^{n}\kappa_j\Psi(u_j)+\Psi(u_i)}{2}\right)\right) \\
& \quad - f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right) \\
& \leqslant f(H)+f(h)-\sum_{i=1}^{n}\kappa_i f(u_i).
\end{aligned} \tag{19}$$

Moreover, the following estimate is also valid:

$$\begin{aligned}
& f(H)+f(h)-\sum_{i=1}^{n}\kappa_i f(u_i) - f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right) \\
& \leqslant 2\Bigg[f(H)+f(h)-\sum_{i=1}^{n}\kappa_i f(u_i) \\
& \quad -2\sum_{i=1}^{n}\kappa_i f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\frac{\sum_{j=1}^{n}\kappa_j\Psi(u_j)+\Psi(u_i)}{2}\right)\right)\Bigg]
\end{aligned} \tag{20}$$

Proof. Let us analyze the expression:

$$\begin{aligned}
\mathscr{J} &= 2\sum_{i=1}^{n}\kappa_i f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\frac{\sum_{j=1}^{n}\kappa_j\Psi(u_j)+\Psi(u_i)}{2}\right)\right) \\
& \quad - f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right).
\end{aligned}$$

By the $\mathscr{M}_\Psi$-convexity of f, Jensen's inequality gives:

$$\begin{aligned}
\mathscr{J} &\geqslant 2f\left(\Psi^{-1}\left[\sum_{i=1}^{n}\kappa_i\left(\Psi(H)+\Psi(h)-\frac{\sum_{j=1}^{n}\kappa_j\Psi(u_j)+\Psi(u_i)}{2}\right)\right]\right) \\
& \quad - f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right) \\
&= 2f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right) \\
& \quad - f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right) \\
&= f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right).
\end{aligned}$$

Hence,

$$
\begin{aligned}
& f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right) \\
& \leqslant 2\sum_{i=1}^{n}\kappa_i f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\frac{\sum_{j=1}^{n}\kappa_j\Psi(u_j)+\Psi(u_i)}{2}\right)\right) \\
& \quad - f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right).
\end{aligned}
$$

This proves the first inequality in (19).

To derive inequality (20), we invoke the bound:

$$
\begin{aligned}
& 2\sum_{i=1}^{n}\kappa_i f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\frac{\sum_{j=1}^{n}\kappa_j\Psi(u_j)+\Psi(u_i)}{2}\right)\right) \\
& \leqslant \frac{1}{2}\left[f\left(\Psi^{-1}\left(\Psi(H)+\Psi(h)-\sum_{i=1}^{n}\kappa_i\Psi(u_i)\right)\right)+f(H)+f(h)-\sum_{i=1}^{n}\kappa_i f(u_i)\right],
\end{aligned}
$$

and the result follows. $\square$

REMARK 2. As a result of the preceding theorem, we can make the following observations:

1. By setting $\Psi(t)=\frac{1}{t}$ in the above theorem, we obtain an improvement and a reverse of Theorem 2.4 from [2].

By setting $\Psi(t)=\frac{1}{t-k}$ for $t>k$ in Theorem 8, we derive a new refinement and reverse form of the Jensen Mercer-type inequality, specifically adapted to the class of k-harmonically convex functions. As established in [21], this result constitutes a meaningful extension of the classical inequality by incorporating the structure of k-harmonic convexity.

THEOREM 9. *Let f be a k-harmonically convex function defined on the interval $[h,H]$. Let $u_1,\ldots,u_n\in[h,H]$, and let $\kappa_1,\ldots,\kappa_n$ be positive weights such that $\sum_{i=1}^{n}\kappa_i=1$. Then*

$$
\begin{aligned}
& f\left(\frac{1}{\frac{1}{h-k}+\frac{1}{H-k}-\sum_{i=1}^{n}\frac{\kappa_i}{u_i-k}}+k\right) \\
& \leqslant 2\sum_{i=1}^{n}\kappa_i f\left(\frac{2}{\frac{2}{h-k}+\frac{2}{H-k}-\sum_{j=1}^{n}\frac{\kappa_j}{u_j-k}+\frac{1}{u_i-k}}+k\right)-f\left(\frac{1}{\frac{1}{h-k}+\frac{1}{H-k}-\sum_{i=1}^{n}\frac{\kappa_i}{u_i-k}}+k\right) \\
& \leqslant \sum_{i=1}^{n}\kappa_i f(u_i),
\end{aligned}
\tag{21}
$$

and similarly,

$$\sum_{i=1}^{n} \kappa_i f(u_i) + 2\left[\sum_{i=1}^{n} \kappa_i f(u_i) - f\left(\frac{2}{\frac{2}{h-k} + \frac{2}{H-k} - \sum_{j=1}^{n} \frac{\kappa_j}{u_j-k} + \frac{1}{u_i-k}} + k\right)\right]$$
$$\leqslant f\left(\frac{1}{\frac{1}{h-k} + \frac{1}{H-k} - \sum_{i=1}^{n} \frac{\kappa_i}{u_i-k}} + k\right). \tag{22}$$

By choosing $\Psi(t) = \log(t-k)$ for $t > k$ in Theorem 8, we obtain a refined and reversed form of the Jensen Mercer-type inequality, suited to the class of k-geometrically convex functions. As demonstrated in [3], this result offers a significant extension of the classical inequality by leveraging the structure of k-geometric convexity.

THEOREM 10. *Let f be a k-geometrically convex function defined on the interval $[h,H]$. Let $u_1,\ldots,u_n \in [h,H]$, and let $\kappa_1,\ldots,\kappa_n$ be positive weights such that $\sum_{i=1}^{n} \kappa_i = 1$. Then*

$$f\left(\frac{(h-k)(H-k)}{\prod_{i=1}^{n}(u_i-k)^{\kappa_i}} + k\right)$$
$$\leqslant 2\sum_{i=1}^{n} \kappa_i f\left(\frac{(h-k)(H-k)}{\sqrt{\prod_{j=1}^{n}(u_j-k)^{\kappa_j}(u_i-k)}} + k\right) - f\left(\frac{(h-k)(H-k)}{\prod_{i=1}^{n}(u_i-k)^{\kappa_i}} + k\right)$$
$$\leqslant f(h) + f(H) - \sum_{i=1}^{n} \kappa_i f(u_i), \tag{23}$$

and similarly,

$$f(h) + f(H) - \sum_{i=1}^{n} \kappa_i f(u_i)$$
$$+ 2\left[\sum_{i=1}^{n} \kappa_i f(u_i) - \sum_{i=1}^{n} \kappa_i f\left(\frac{(h-k)(H-k)}{\sqrt{\prod_{j=1}^{n}(u_j-k)^{\kappa_j}(u_i-k)}} + k\right)\right]$$
$$\leqslant f\left(\frac{(h-k)(H-k)}{\prod_{i=1}^{n}(u_i-k)^{\kappa_i}} + k\right). \tag{24}$$

REMARK 3. The obtained results present a significant generalization of classical inequalities, demonstrating the flexibility and applicability of the framework. By varying the function Ψ, we can retrieve well-known theorems from previous works, establishing connections between different convexity concepts. This reinforces the versatility of the approach in unifying diverse inequality results and provides a broader perspective on their interrelationships.

3. Inequalities for operators based on harmonic convexity

In this section, we establish several new Jensen-type inequalities for operators, specifically formulated for the class of harmonically convex functions. These results contribute novel insights to the field, highlighting how harmonic convexity can be effectively integrated into operator theory. The inequalities presented here enrich the existing literature with meaningful and applicable refinements.

Let $\mathscr{B}(\mathscr{H})$ denote the $\mathbb{C}^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ equipped with an inner product $\langle\cdot,\cdot\rangle$. An operator $\mathscr{T}\in\mathscr{B}(\mathscr{H})$ is called *positive*, denoted by $\mathscr{T}\geqslant 0$, if it satisfies $\langle\mathscr{T}x,x\rangle\geqslant 0$ for all $x\in\mathscr{H}$.

The set of all positive operators is denoted by $\mathscr{B}(\mathscr{H})^+$, while $\mathscr{B}(\mathscr{H})^{++}$ denotes the set of all positive invertible operators.

Jensen's inequality is a fundamental result in convex analysis, with powerful extensions to the setting of operator theory. In the context of Hilbert spaces, the inequality can be formulated for self-adjoint operators, offering a bridge between classical analysis and functional analysis.

THEOREM 11. *Let $\mathscr{T}\in\mathscr{B}(\mathscr{H})$ be a self-adjoint operator with spectrum contained in the interval $J\subset\mathbb{R}$. If $f:J\to\mathbb{R}$ is a convex function, then*

$$f(\langle\mathscr{T}x,x\rangle)\leqslant\langle f(\mathscr{T})x,x\rangle \tag{25}$$

for all $x\in\mathscr{H}$ with $\|x\|=1$. Furthermore, if f is concave instead of convex, then the inequality is reversed.

When $f(t)=t^r$, $|r|\geqslant 1$, (25) is reduced to the well-known McCarthy inequality [15]. These inequalities have played a vital role in the advancement of the field of operator inequalities, with many applications towards operator inequalities. We refer the reader to [7, 8, 9] for a sample of some work treating and applying these inequalities.

This result expresses that, for convex functions, the functional calculus of an operator preserves the order of convexity when evaluated in the quadratic form $\langle\cdot x,x\rangle$. It plays a crucial role in operator inequalities and quantum information theory, among other applications.

To establish a Jensen-type inequality for harmonically convex functions, we require the following theorem, which presents a recently obtained result on Jensen's inequality for integrals involving harmonically convex functions.

THEOREM 12. ([4]) *Let μ be a probability measure on a space X and let I be an interval in $\mathbb{R}\setminus\{0\}$. If $f:X\to I$ is a measurable function, φ is a harmonically convex function on I, and $\varphi\circ f$ is μ-integrable, then*

$$\varphi\left(\left(\int_X\frac{1}{f}\,d\mu\right)^{-1}\right)\leqslant\int_X\varphi\circ f\,d\mu,$$

where $\varphi(0)=\lim_{t\to 0,t\in I}\varphi(t)$, if $\int_X\frac{1}{f}\,d\mu=\pm\infty$.

In the following theorem, we present a Jensen-type inequality for operators, specifically formulated for the class of harmonically convex functions.

THEOREM 13. *Let f be a harmonic convex function defined on the interval $[0,+\infty)$, and let $\mathscr{T} \in \mathscr{B}(\mathscr{H})^{++}$. Then, for every unit vector $x \in \mathscr{H}$, the following holds*

$$f(\langle \mathscr{T}^{-1}x,x\rangle^{-1}) \leqslant \langle f(\mathscr{T})x,x\rangle. \tag{26}$$

Proof. Let $\mathscr{T}$ be a positive and invertible operator on a Hilbert space $\mathscr{H}$. By the spectral theorem, we have

$$\mathscr{T} = \int_{\sigma(\mathscr{T})} \lambda \, dE_\lambda,$$

where $\{E_\lambda\}$ is the spectral family of $\mathscr{T}$. Moreover, for all $x \in \mathscr{H}$,

$$\langle \mathscr{T}x,x\rangle = \int_{\sigma(\mathscr{T})} \lambda \, d\langle E_\lambda x,x\rangle,$$

where E_λ is the spectral family associated with $\mathscr{T}$, and $\sigma(\mathscr{T})$ denotes the spectrum of $\mathscr{T}$. Using the functional calculus for self-adjoint operators and Theorem 12 we obtain the inequality:

$$\begin{aligned} f\left(\langle \mathscr{T}^{-1}x,x\rangle^{-1}\right) &= f\left(\left(\int_{\sigma(\mathscr{T})} \frac{1}{\lambda}\, d\langle E_\lambda x,x\rangle\right)^{-1}\right) \\ &\leqslant \int_{\sigma(\mathscr{T})} f(\lambda)\, d\langle E_\lambda x,x\rangle = \langle f(\mathscr{T})x,x\rangle. \quad \square \end{aligned}$$

The following result provides a multiple-operator extension of the preceding theorem.

THEOREM 14. *Let f be a harmonic convex function defined on the interval $[0,+\infty)$, and let $\mathscr{T}_i \in \mathscr{B}(\mathscr{H})^{++}$, for $i = 1,2,\ldots,n$. Let $x_1,x_2,\ldots,x_n \in \mathscr{H}$ be vectors such that*

$$\sum_{i=1}^{n} \|x_i\|^2 = 1.$$

Then

$$f\left(\left(\sum_{i=1}^{n} \langle \mathscr{T}_i^{-1}x_i,x_i\rangle\right)^{-1}\right) \leqslant \sum_{i=1}^{n} \langle f(\mathscr{T}_i)x_i,x_i\rangle. \tag{27}$$

Proof. If we put

$$\widetilde{\mathscr{T}} = \begin{pmatrix} \mathscr{T}_1 & 0 & \cdots & 0 \\ 0 & \mathscr{T}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathscr{T}_n \end{pmatrix}, \quad \widetilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

then we have $\mathrm{Sp}(\widetilde{\mathscr{T}}) \subset \mathscr{B}(\mathscr{H})^{++}$, $\|\widetilde{x}\| = 1$, and

$$\sum_{i=1}^{n} \langle \mathscr{T}_i^{-1} x_i, x_i \rangle = \langle \widetilde{\mathscr{T}}^{-1} \widetilde{x}, \widetilde{x} \rangle.$$

It follows from Theorem 13, that

$$f\left(\langle \widetilde{\mathscr{T}}^{-1} \widetilde{x}, \widetilde{x} \rangle^{-1}\right) \leqslant \langle f(\widetilde{\mathscr{T}}) \widetilde{x}, \widetilde{x} \rangle,$$

and hence we obtain the desired inequality. $\square$

The following theorem provides a significant refinement and reversal of Theorem 13. It sharpens the original inequality while also establishing a reversed form. This contributes to a deeper understanding of the operator version of Jensen's inequality.

THEOREM 15. *Let f be a harmonic convex function defined on the interval $[0, +\infty)$, and let $\mathscr{T} \in \mathscr{B}(\mathscr{H})^{++}$. Then, for every unit vector $x \in \mathscr{H}$, the following holds*

$$\begin{aligned} f(\langle \mathscr{T}^{-1} x, x \rangle^{-1}) &\leqslant 2 \left\langle f\left(\left(\frac{\mathscr{T}^{-1} + \langle \mathscr{T}^{-1} x, x \rangle \cdot I_{\mathscr{H}}}{2}\right)^{-1}\right) x, x \right\rangle - f(\langle \mathscr{T}^{-1} x, x \rangle^{-1}) \\ &\leqslant \langle f(\mathscr{T}) x, x \rangle. \end{aligned} \tag{28}$$

Furthermore,

$$\begin{aligned} &\langle f(\mathscr{T}) x, x \rangle - f\left(\langle \mathscr{T}^{-1} x, x \rangle^{-1}\right) \\ &\leqslant 2\left[\langle f(\mathscr{T}) x, x \rangle - \left\langle f\left(\left(\tfrac{\mathscr{T}^{-1} + \langle \mathscr{T}^{-1} x, x \rangle I_{\mathscr{H}}}{2}\right)^{-1}\right) x, x \right\rangle \right]. \end{aligned} \tag{29}$$

Both inequalities are reversed if f is harmonic concave.

Proof. The first inequality is a direct application of the convexity condition and follows from the observation

$$\begin{aligned} &2 \left\langle f\left(\left(\frac{\mathscr{T}^{-1} + \langle \mathscr{T}^{-1} x, x \rangle \cdot I_{\mathscr{H}}}{2}\right)^{-1}\right) x, x \right\rangle - f(\langle \mathscr{T}^{-1} x, x \rangle^{-1}) \\ &\geqslant 2 f \left\langle \left(\frac{\mathscr{T}^{-1} + \langle \mathscr{T}^{-1} x, x \rangle \cdot I_{\mathscr{H}}}{2}\right) x, x \right\rangle^{-1} - f(\langle \mathscr{T}^{-1} x, x \rangle^{-1}) \\ &= 2 f(\langle \mathscr{T}^{-1} x, x \rangle^{-1}) - f(\langle \mathscr{T}^{-1} x, x \rangle^{-1}) = f(\langle \mathscr{T}^{-1} x, x \rangle^{-1}). \end{aligned}$$

For the second inequality, we appeal to the standard harmonic convexity property: if $f : J \to \mathbb{R}$ is harmonic convex and $a, b \in J$, then

$$f\left(\left(\frac{a^{-1} + b}{2}\right)^{-1}\right) \leqslant \frac{f(a) + f(b^{-1})}{2}.$$

Using functional calculus for the operator $\mathscr{T}$, it follows that

$$f\left(\left(\frac{\mathscr{T}^{-1}+bI_{\mathscr{H}}}{2}\right)^{-1}\right) \leqslant \frac{f(\mathscr{T})+f(b^{-1})I_{\mathscr{H}}}{2},$$

and consequently, for any unit vector $x \in \mathscr{H}$,

$$\left\langle f\left(\left(\frac{\mathscr{T}^{-1}+bI_{\mathscr{H}}}{2}\right)^{-1}\right)x,x\right\rangle \leqslant \frac{\langle f(\mathscr{T})x,x\rangle+f(b^{-1})}{2}.$$

Taking $b=\langle \mathscr{T}^{-1}x,x\rangle$, we deduce the needed inequality. $\square$

As a direct consequence of the previous theorem, we obtain the following refinement and reverse of Theorem 14, which can be derived by replacing $\mathscr{T}$ with $\widetilde{\mathscr{T}}$.

COROLLARY 1. *Let f be a harmonic convex function defined on the interval $[0,+\infty)$, and let $\mathscr{T}_i \in \mathscr{B}(\mathscr{H})^{++}$, for $i=1,2,\ldots,n$. Let $x_1,x_2,\ldots,x_n \in \mathscr{H}$ be vectors such that*

$$\sum_{i=1}^{n}\|x_i\|^2=1.$$

Then

$$\begin{aligned}
&f\left(\sum_{i=1}^{n}\langle \mathscr{T}_i^{-1}x_i,x_i\rangle\right)^{-1}\\
&\leqslant 2\sum_{i=1}^{n}\left\langle f\left(\left(\frac{\mathscr{T}_i^{-1}+\sum_{i=1}^{n}\langle \mathscr{T}^{-1}x_i,x_i\rangle\cdot I_{\mathscr{H}}}{2}\right)^{-1}\right)x_i,x_i\right\rangle - f\left(\sum_{i=1}^{n}\langle \mathscr{T}_i^{-1}x_i,x_i\rangle\right)^{-1}\\
&\leqslant \sum_{i=1}^{n}\langle f(\mathscr{T}_i)x_i,x_i\rangle.
\end{aligned}\tag{30}$$

Moreover,

$$\begin{aligned}
&\sum_{i=1}^{n}\langle f(\mathscr{T}_i)x_i,x_i\rangle\\
&\leqslant f\left(\sum_{i=1}^{n}\langle \mathscr{T}_i^{-1}x_i,x_i\rangle\right)^{-1}\\
&\quad +2\left[\sum_{i=1}^{n}\langle f(\mathscr{T}_i)x_i,x_i\rangle-\sum_{i=1}^{n}\left\langle f\left(\left(\frac{\mathscr{T}_i^{-1}+\sum_{i=1}^{n}\langle \mathscr{T}^{-1}x_i,x_i\rangle\cdot I_{\mathscr{H}}}{2}\right)^{-1}\right)x_i,x_i\right\rangle\right].
\end{aligned}\tag{31}$$

Both inequalities are reversed if f is harmonic concave.

As a result of our findings, we obtain a significant refinement of the McCarthy-type inequality. This improvement specifically applies to the class of harmonically convex functions. The refinement strengthens the classical inequality under harmonic convexity assumptions. It highlights deeper structural properties within this function class.

COROLLARY 2. *Suppose $\mathscr{T} \in \mathscr{B}(\mathscr{H})^{++}$ and $x \in \mathscr{H}$ is a unit vector. Then, for any exponent $r \leqslant -1$,*

$$\begin{aligned}\langle \mathscr{T}^{-1}x,x\rangle^{-r} &\leqslant 2\left\langle \left(\frac{\mathscr{T}^{-1}+\langle \mathscr{T}^{-1}x,x\rangle \cdot I_{\mathscr{H}}}{2}\right)^{-r} x,x\right\rangle - \langle \mathscr{T}^{-1}x,x\rangle^{-r} \\ &\leqslant \langle \mathscr{T}^{r}x,x\rangle. \end{aligned} \tag{32}$$

And

$$\langle \mathscr{T}^{r}x,x\rangle - \langle \mathscr{T}^{-1}x,x\rangle^{-r} \leqslant 2\left[\langle \mathscr{T}^{r}x,x\rangle - \left\langle \left(\frac{\mathscr{T}^{-1}+\langle \mathscr{T}^{-1}x,x\rangle \cdot I_{\mathscr{H}}}{2}\right)^{-r} x,x\right\rangle\right]. \tag{33}$$

COROLLARY 3. *Suppose $\mathscr{T} \in \mathscr{B}(\mathscr{H})^{++}$ and $x \in \mathscr{H}$ is a unit vector. Then, for any exponent $r \leqslant -1$,*

$$\begin{aligned}&\left(\sum_{i=1}^{n}\langle \mathscr{T}_i^{-1}x_i,x_i\rangle\right)^{-r} \\ &\leqslant 2\sum_{i=1}^{n}\left\langle \left(\frac{\mathscr{T}_i^{-1}+\sum_{i=1}^{n}\langle \mathscr{T}^{-1}x_i,x_i\rangle \cdot I_{\mathscr{H}}}{2}\right)^{-1} x_i,x_i\right\rangle - \left(\sum_{i=1}^{n}\langle \mathscr{T}_i^{-r}x_i,x_i\rangle\right)^{-r} \\ &\leqslant \sum_{i=1}^{n}\langle \mathscr{T}_i^{-r}x_i,x_i\rangle. \end{aligned} \tag{34}$$

And

$$\begin{aligned}&\sum_{i=1}^{n}\langle \mathscr{T}_i^{-r}x_i,x_i\rangle \\ &\leqslant \left(\sum_{i=1}^{n}\langle \mathscr{T}_i^{-r}x_i,x_i\rangle\right)^{-1} \\ &\quad +2\left[\sum_{i=1}^{n}\langle \mathscr{T}_i^{-r}x_i,x_i\rangle - \sum_{i=1}^{n}\left\langle \left(\frac{\mathscr{T}_i^{-1}+\sum_{i=1}^{n}\langle \mathscr{T}^{-1}x_i,x_i\rangle \cdot I_{\mathscr{H}}}{2}\right)^{-r} x_i,x_i\right\rangle\right]. \end{aligned} \tag{35}$$

Both inequalities are reversed if f is harmonic concave.

The weighted arithmetic and harmonic means are central tools in matrix and operator analysis. For two positive invertible operators $\mathscr{T},\mathscr{S} \in \mathscr{B}(\mathscr{H})^{++}$, and $\kappa \in [0,1]$

their arithmetic mean is defined as $\mathscr{T}\nabla_\kappa\mathscr{S} := \kappa\mathscr{T} + (1-\kappa)\mathscr{S}$, while their harmonic mean is given by $\mathscr{T}!_\kappa\mathscr{S} := \left(\kappa\mathscr{T}^{-1} + (1-\kappa)\mathscr{S}^{-1}\right)^{-1}$. These operator means play a fundamental role in spectral theory and functional calculus, and they arise naturally in inequalities involving convex functions. Recent advancements, particularly those involving harmonic convexity, enable refined estimates of operator functions via Jensen-type inequalities. Such refinements, as developed in this paper, provide sharper norm bounds and reverse inequalities. In what follows, we apply these techniques to establish new norm inequalities for operator means. Specifically, we employ the results from Section 3 to investigate how harmonic convexity yields improved estimates involving the relationship between the arithmetic and harmonic means.

It is well known that for positive definite operators $\mathscr{T},\mathscr{S} \in \mathscr{B}^{++}(\mathscr{H})$, and for any $\kappa \in [0,1]$, the following norm inequality holds:

$$\|\mathscr{T}!_\kappa\mathscr{S}\| \leqslant \|\mathscr{T}\nabla_\kappa\mathscr{S}\|.$$

In the remainder of this paper, we derive additional norm inequalities comparing the harmonic and arithmetic means. These new results are obtained by applying our main findings presented in Theorems 13 and 14. Our approach yields refined estimates and extends known inequalities within the framework of operator means.

LEMMA 1. ([1]) *Let $f:[0,\infty)\to[0,\infty)$ be a convex function, and let $\mathscr{T},\mathscr{S} \in \mathscr{B}(\mathscr{H})$ be positive operators. If $0 \leqslant \kappa \leqslant 1$, then*

$$\|f(\kappa\mathscr{T} + (1-\kappa)\mathscr{S})\| \leqslant \|\kappa f(\mathscr{T}) + (1-\kappa)f(\mathscr{S})\|.$$

In the following theorem, we apply Theorem 13 to present new norm inequalities between the harmonic and arithmetic means.

THEOREM 16. *Let $f:[0,\infty)\to[0,\infty)$ be an increasing and convex function, and let $\mathscr{T},\mathscr{S} \in \mathscr{B}^{++}(\mathscr{H})$. If $0 \leqslant \kappa \leqslant 1$, then*

$$f\left(\|\mathscr{T}!_\kappa\mathscr{S}\|^{-1}\right) \leqslant \left\|f(\mathscr{T}^{-1})\nabla_\kappa f(\mathscr{S}^{-1})\right\|.$$

In particular, for $r \geqslant 1$:

$$\|\mathscr{T}!_\kappa\mathscr{S}\|^{-r} \leqslant \left\|\mathscr{T}^{-r}\nabla_\kappa\mathscr{S}^{-r}\right\|,$$

and for $r \leqslant -1$:

$$\|\mathscr{T}!_\kappa\mathscr{S}\|^{r} \leqslant \|\mathscr{T}^{r}\nabla_\kappa\mathscr{S}^{r}\|.$$

Proof. First, note that for $a,b>0$ and $\kappa \in [0,1]$,

$$\left(\kappa a^{-1} + (1-\kappa)b^{-1}\right)^{-1} \leqslant \kappa a + (1-\kappa)b.$$

Using the fact that f is an increasing and convex function, we obtain

$$f\left(\left(\kappa a^{-1} + (1-\kappa)b^{-1}\right)^{-1}\right) \leqslant f(\kappa a + (1-\kappa)b) \leqslant \kappa f(a) + (1-\kappa)f(b).$$

Therefore, f is a harmonically convex function. Now, for any unit vector x, we have

$$\begin{aligned} f\left(\langle(\mathscr{T}!_\kappa\mathscr{S})x,x\rangle^{-1}\right) &= f\left(\left\langle\left(\kappa\mathscr{T}^{-1}+(1-\kappa)\mathscr{S}^{-1}\right)^{-1}x,x\right\rangle^{-1}\right)\\ &\leqslant \left\langle f\left(\kappa\mathscr{T}^{-1}+(1-\kappa)\mathscr{S}^{-1}\right)x,x\right\rangle\\ &\leqslant \left\|f\left(\kappa\mathscr{T}^{-1}+(1-\kappa)\mathscr{S}^{-1}\right)\right\|\\ &\leqslant \left\|\kappa f(\mathscr{T}^{-1})+(1-\kappa)f(\mathscr{S}^{-1})\right\|\\ &= \left\|f(\mathscr{T}^{-1})\nabla_\kappa f(\mathscr{S}^{-1})\right\|. \end{aligned}$$

where the first inequality follows from Theorem 13, and the last inequality follows from Lemma 1.

The second inequality is obtained by taking $f(t)=t^r$ for $r\geqslant 1$, and the last inequality follows by replacing r with $-r$ in the second one. $\square$

An application of Theorem 14 is the following generalization of the preceding theorem.

THEOREM 17. *Let $f:[0,\infty)\to[0,\infty)$ be an increasing and convex function, and let $\mathscr{T}_i,\mathscr{S}_i\in\mathscr{B}^{++}(\mathscr{H})$, for all $i=1,\ldots,n$. If $0\leqslant\kappa\leqslant 1$, then*

$$f\left(\sum_{i=1}^n\|\mathscr{T}!_\kappa\mathscr{S}\|^{-1}\right)\leqslant\sum_{i=1}^n\left\|f(\mathscr{T}_i^{-1})\nabla_\kappa f(\mathscr{S}_i^{-1})\right\|.$$

In particular, for $r\geqslant 1$:

$$\left(\sum_{i=1}^n\|\mathscr{T}!_\kappa\mathscr{S}\|^{-1}\right)^r\leqslant\sum_{i=1}^n\left\|\mathscr{T}_i^{-r}\nabla_\kappa\mathscr{S}_i^{-r}\right\|.$$

Proof. Similarly, for any unit vector $x\in\mathscr{H}$,

$$\begin{aligned} f\left(\sum_{i=1}^n\langle(\mathscr{T}_i!_\kappa\mathscr{S}_i)x,x\rangle^{-1}\right) &= f\left(\sum_{i=1}^n\left\langle\left(\kappa\mathscr{T}_i^{-1}+(1-\kappa)\mathscr{S}_i^{-1}\right)^{-1}x,x\right\rangle^{-1}\right)\\ &\leqslant \sum_{i=1}^n\left\langle f\left(\kappa\mathscr{T}_i^{-1}+(1-\kappa)\mathscr{S}_i^{-1}\right)x,x\right\rangle\\ &\leqslant \left\|\sum_{i=1}^n f\left(\kappa\mathscr{T}_i^{-1}+(1-\kappa)\mathscr{S}_i^{-1}\right)\right\|\\ &\leqslant \sum_{i=1}^n\left\|\kappa f(\mathscr{T}_i^{-1})+(1-\kappa)f(\mathscr{S}_i^{-1})\right\|\\ &= \sum_{i=1}^n\left\|f(\mathscr{T}_i^{-1})\nabla_\kappa f(\mathscr{S}_i^{-1})\right\|, \end{aligned}$$

where the first inequality follows from Theorem 14, and the last inequality follows from Lemma 1.

The second inequality is obtained by taking $f(t)=t^r$ for $r\geqslant 1$. $\square$

Concluding remarks

In this paper, we presented a comprehensive study of Jensen and Jensen-Mercer type inequalities within the framework of generalized convexity, particularly focusing on the class of M_Ψ-convex functions. By incorporating nonlinear mean structures via a strictly monotonic function Ψ, we obtained meaningful generalizations and refinements of classical convex inequalities.

Our results include both refinements and reversals of well-known inequalities, which lead to double-sided bounds that enhance the precision of existing estimates. The use of specific transformation functions such as $\Psi(t) = 1/(t-k)$ and $\Psi(t) = \log(t-k)$ enabled us to recover and extend results for k-harmonically and k-geometrically convex functions, further broadening the applicability of our framework.

We also explored operator versions of these inequalities by employing the concept of harmonic convexity. This allowed us to derive Jensen-type inequalities for positive invertible operators, providing spectral norm estimates and operator inequalities that go beyond classical formulations. In particular, we established new inequalities comparing the harmonic and arithmetic means of operators, as well as their refined norm bounds under convex transformations.

The presented results not only unify various strands of convex analysis and operator theory but also provide a flexible and powerful toolbox for future research. Potential applications include areas such as optimization, entropy analysis, and functional inequalities in quantum information theory.

We believe that the introduced methods and results will stimulate further investigation into the structure of generalized convex functions and their applications across mathematical and applied disciplines.

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