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THE HOFFMAN–WIELANDT INEQUALITY FOR QUATERNION MATRICES AND QUATERNION MATRIX POLYNOMIALS

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Abstract. The purpose of this paper is to derive the Hoffman-Wielandt inequality and its generalization for quaternion matrices. Diagonalizability of the block companion matrix of certain quadratic (linear) quaternion matrix polynomials is brought out. As a consequence, we prove that if $Q(\lambda)$ is another quadratic (linear) quaternion matrix polynomial, then under certain conditions on the coefficients, a generalization of the Hoffman-Wielandt inequality for their corresponding block companion matrices holds. We also prove that if $P(\lambda)$ is a quaternion matrix polynomial with unitary coefficients, then any right eigenvalue λ_0 of $P(\lambda)$ lies in the annular region $\frac{1}{2} < |\lambda_0| < 2$.

1. Introduction

Quaternion matrices and quaternion matrix polynomials have been of considerable interest to researchers in the last few years. Noncommutativity of quaternion multiplication makes analysis over the quaternions quite intriguing. A good reference for quaternion matrices is the survey article by Zhang [13] and the references cited therein. On the other hand, literature on quaternion matrix polynomials is quite limited. Recently in [1] and [3], the authors discuss interesting techniques to derive eigenvalue bounds of quaternion matrix polynomials.

Perturbation analysis of matrices over complex numbers and their eigenvalues is an old problem but less studied for quaternion matrices. One of the well known inequalities on this is the Hoffman-Wielandt inequality (see Theorem 1 below). The reader may refer to Theorem 6.3.5 of [8] for a proof. The book by Bhatia [6] gives a detailed account of the spectral variation problem. Possible generalizations of the Hoffman-Wielandt inequality for complex matrices exist in the literature (see for instance Theorems 2, 3, 5, 6, 7 and 8 of [10] and Theorem 2 of [11] and Remark 3.3(2) of [12]). Among these generalizations of Theorem 1, obtained by relaxing normality of one or both the matrices in the above theorem, we focus our attention to the one given in [11, 12]. We state this below as Theorem 2. This was also recently studied in [4]

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in the context of complex matrix polynomials. In [2], the authors give a Bauer-Fike type theorem for the right eigenvalues of quaternion matrices and also discuss perturbations via a block-diagonal decomposition and Jordan canonical form of quaternion matrices. In a recent work, we have also obtained location and perturbation results for coneigenvalues of quaternion matrices [5]. The purpose of this paper is to derive the Hoffman-Wielandt inequality and its generalization for quaternion matrices involving their right eigenvalues. We study these in the context of quaternion matrix polynomials.

The notations $||\cdot||_2$ and $||\cdot||_F$ denote respectively the spectral norm and the Frobenius norm of complex matrices.

THEOREM 1. *Let A and B be two $n\times n$ normal matrices with $\lambda_1,\ldots,\lambda_n$ and $\mu_1,\ldots,\mu_n$ as their eigenvalues respectively given in some order. Then there exists a permutation π of $\{1,\ldots,n\}$ such that*

$$\sum_{i=1}^{n}|\lambda_i-\mu_{\pi(i)}|^2 \leqslant ||A-B||_F^2.$$

THEOREM 2. *Let A be a diagonalizable matrix of order n and B be a normal matrix of order n, with eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ and $\mu_1,\mu_2,\ldots,\mu_n$, respectively. Let X be a nonsingular matrix whose columns are eigenvectors of A. Then, there exists a permutation π of $\{1,\ldots,n\}$ such that*

$$\sum_{i=1}^{n}|\lambda_i-\mu_{\pi(i)}|^2 \leqslant ||X||_2^2||X^{-1}||_2^2||A-B||_F^2.$$

An example to illustrate the importance of normality of one of the matrices in Theorem 2 can be easily provided.

This paper is organized as follows. Section 2 contains notations, preliminaries and necessary results from [13] and [1]. The Hoffman-Wielandt inequality and its generalization for quaternion matrices are proved in Section 3.1. In Section 3.2, we discuss these for block companion matrices of quaternion matrix polynomials. Diagonalizability of the block companion matrix of certain quadratic (linear) quaternion matrix polynomials is brought out. We also prove that if $P(\lambda)$ is a quaternion matrix polynomial with unitary coefficients, then any right eigenvalue λ_0 of $P(\lambda)$ lies in the annular region $\frac{1}{2}<|\lambda_0|<2$.

2. Notations and preliminaries

Throughout this paper, we use the following notation and terminology. The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively. $\mathbb{C}^+$ denotes the closed upper half plane of the complex plane. The set $\mathbb{H} := \{a_0+a_1i+a_2j+a_3k \mid a_i \in \mathbb{R}\}$ with $i^2=j^2=k^2=ijk=-1$, denotes the set of real quaternions. The conjugate and modulus of an element $q\in\mathbb{H}$ are denoted and defined as $\bar{q} := a_0-a_1i-a_2j-a_3k$, $|q| := \sqrt{a_0^2+a_1^2+a_2^2+a_3^2}$ respectively. Define $Re(q)=a_0$ as the real part of q, $Co(q)=a_0+a_1i$, the complex part of q and $Im(q)=a_1i+a_2j+a_3k$, the imaginary part of q. Two quaternions p and q are similar (written as $p \sim q$) if there exists a nonzero quaternion r such that $r^{-1}qr = p$. If $p \sim q$, then $|p| = |q|$. This similarity is an equivalence relation on $\mathbb{H}$.

Let $M_n(X)$ denote the set of all $n \times n$ matrices whose entries are from X, where X is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Given $A = (a_{ij}) \in M_n(\mathbb{H})$, the conjugate transpose of A denoted by A^* is defined as $A^* = (\bar{a}_{ji})$. $A \in M_n(\mathbb{H})$ is said to be normal if $AA^* = A^*A$, unitary if $AA^* = I = A^*A$, Hermitian if $A^* = A$; and invertible if $AB = BA = I$ for some $B \in M_n(\mathbb{H})$. An $n \times n$ quaternion matrix A is said to be positive (semi)definite if A is Hermitian and $x^*Ax > (\geqslant)0$ for all nonzero vector $x \in \mathbb{H}^n$. Since the set of quaternions is a noncommutative division ring, there exists a notion of right and left eigenvalues for a matrix in $M_n(\mathbb{H})$. For $A \in M_n(\mathbb{H})$, a quaternion $\lambda_0 \in \mathbb{H}$ is called a right (left) eigenvalue if there exists a nonzero vector $x \in \mathbb{H}^n$ such that $Ax = x\lambda_0$ $(Ax = \lambda_0 x)$. Note that if λ_0 is a right eigenvalue of A, then any quaternion similar to λ_0 is also a right eigenvalue of A. Therefore, there can be infinitely many right eigenvalues for $A \in M_n(\mathbb{H})$. However, any matrix $A \in M_n(\mathbb{H})$ has exactly n right eigenvalues, which are complex numbers having nonnegative imaginary parts (Theorem 5.4, [13]). These right eigenvalues are called the standard eigenvalues of A. Any quaternion which is a right eigenvalue of A lies in one of the equivalence classes of these n complex numbers. On the contrary, the number of left eigenvalues up to equivalent classes is still unknown and a quaternion similar to a left eigenvalue need not be a left eigenvalue. Thus in comparison to left eigenvalues, right eigenvalues have received more attention in the literature.

$A \in M_n(\mathbb{H})$ is said to be diagonalizable if there exists an invertible matrix $S \in M_n(\mathbb{H})$ such that $S^{-1}AS = J$, where $J = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and λ_i's are the standard eigenvalues of A. Given a matrix $A \in M_n(\mathbb{H})$, we can write $A = A_1 + A_2 j$, where A_1, $A_2 \in M_n(\mathbb{C})$. We can then associate to A, a $2n \times 2n$ complex block matrix, $\chi_A = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1}. \end{bmatrix}$, called the complex adjoint matrix of A. For $A \in M_n(\mathbb{H})$, the Frobenius norm and the spectral norm are defined as follows:

(1) $||A||_F = \left(\operatorname{trace} A^*A\right)^{1/2}$

(2) $||A||_2 = \sup\limits_{x \neq 0} \left\{ \dfrac{||Ax||_2}{||x||_2} : x \in \mathbb{H}^n \right\}$.

For $A \in M_n(\mathbb{H})$ it is easy to verify that $\sqrt{2}||A||_F = ||\chi_A||_F$ and $||A||_2 = ||\chi_A||_2$. Unless and until specified, all the matrices considered in this paper are quaternion matrices. We now list some fundamental information about matrix A and its complex adjoint matrix χ_A. These results can be found in [13].

PROPOSITION 3. *Let $A, B \in M_n(\mathbb{H})$ and $\alpha \in \mathbb{R}$. Then*

(a) $\chi_{I_n} = I_{2n}$.

(b) $\chi_{AB} = \chi_A \chi_B$.

(c) $\chi_{\alpha A} = \alpha \chi_A$.

(d) $\chi_{A+B} = \chi_A + \chi_B$.

(e) $\chi_{A^*} = (\chi_A)^*$.

(f) $\chi_{A^{-1}} = (\chi_A)^{-1}$, *if* A^{-1} *exists.*

(g) χ_A *is unitary, Hermitian, diagonalizable, invertible or normal if and only if* A *is unitary, Hermitian, diagonalizable, invertible, or normal respectively.*

(h) $\chi_A \chi_B = \chi_B \chi_A$ *if and only if* $AB = BA$.

(i) $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the standard eigenvalues of* A *if and only if* $\lambda_1, \lambda_2, \ldots, \lambda_n$, $\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_n$ *are the eigenvalues of* χ_A.

(j) *A complex matrix* A *is diagonalizable over* $\mathbb{H}$ *if and only if it is diagonalizable over* $\mathbb{C}$.

The following remark is worth pointing out.

REMARK 1.

1. For a complex matrix A, the eigenvalues over $\mathbb{C}$ and the standard eigenvalues over $\mathbb{H}$ are not the same in general. For example consider $A = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$. Then the eigenvalues of A over $\mathbb{C}$ are $1+i$ and $1-i$, whereas, the standard eigenvalues of A over $\mathbb{H}$ are $\lambda_1 = \lambda_2 = 1+i$.

2. Note that as sets $M_n(\mathbb{C}) \subsetneq M_n(\mathbb{H})$ and every unitary matrix in $M_n(\mathbb{C})$ is a unitary matrix in $M_n(\mathbb{H})$ as well. However, there are more unitary matrices in $M_n(\mathbb{H})$ than in $M_n(\mathbb{C})$. For example the matrix $A = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}$ is unitary in $M_n(\mathbb{H})$ but $A \notin M_n(\mathbb{C})$. The same happens for normal, Hermitian, positive (semi)definite and diagonalizable matrices.

We now define matrix polynomials where the coefficients are from $M_n(\mathbb{H})$. As the multiplication of quaternions is noncommutative, we have the notion of right and left matrix polynomials. A right quaternion matrix polynomial of size n and degree m is a mapping $P : \mathbb{H} \to M_n(\mathbb{H})$ defined by $P(\lambda) = \sum_{i=0}^{m} A_i \lambda^i$, where $A_i \in M_n(\mathbb{H})$ (that is, the indeterminate λ is on the right of the matrix coefficients). A scalar $\lambda_0 \in \mathbb{H}$ is said to be a right eigenvalue of $P(\lambda)$ if $\sum_{i=0}^{m} A_i x \lambda_0^i = 0$ for some nonzero vector $x \in \mathbb{H}^n$. If the leading coefficient A_m is invertible we associate a monic quaternion matrix polynomial $P_U(\lambda) = I\lambda^m + B_{m-1}\lambda^{m-1} + \cdots + B_1\lambda + B_0$, where $B_i = A_m^{-1} A_i$ for $i = 0, 1, \ldots, m-1$. It is easy to verify that the right eigenvalues of $P(\lambda)$ and $P_U(\lambda)$ are the same. The right eigenvalues of P_U are the same as the right eigenvalues of the block companion matrix $C_P = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -B_0 & -B_1 & -B_2 & \cdots & -B_{m-1} \end{bmatrix} \in M_{mn}(\mathbb{H})$ (for details, see [1]). We define the standard eigenvalues of $P(\lambda)$ to be the standard eigenvalues of C_P.

An element $\lambda_0 \in \mathbb{H}$ is called a left eigenvalue of $P(\lambda)$ if $\sum_{i=0}^{m} A_i \lambda_0^i x = 0$ for some nonzero vector $x \in \mathbb{H}^n$. Note that the left eigenvalues of $P(\lambda)$ coincide with the left eigenvalues of C_P (the proof of this statement is very similar to that of Theorem 5.1 of [1]). An $n \times n$ left quaternion matrix polynomial is a map $Q : \mathbb{H} \to M_n(\mathbb{H})$ defined by $Q(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$, where $A_i \in M_n(\mathbb{H})$ (note that in this case, the indeterminate λ is on the left of the matrix coefficients). A quaternion λ_0 is called a left eigenvalue of $Q(\lambda)$ if $\sum_{i=0}^{m} \lambda_0^i A_i x = 0$ for some nonzero vector $x \in \mathbb{H}^n$. However, one cannot define right eigenvalues for left quaternion matrix polynomials in the way we defined them for right quaternion matrix polynomials. Moreover, when the leading coefficient is invertible, it is easy to verify that the left eigenvalues of a left quaternion matrix polynomial are the same as the left eigenvalues of the corresponding block companion matrix (for details, see [1]).

As mentioned earlier, for a quaternion matrix the number of left eigenvalues is still unknown and hence is less studied in the literature. Therefore, as far as this paper goes, we restrict ourselves to the study of right eigenvalues of right quaternion matrix polynomials. We refer to right quaternion matrix polynomials as just quaternion matrix polynomials.

ASSUMPTIONS. Throughout this paper, we assume that the leading coefficient of a quaternion matrix polynomial is invertible.

3. Main results

The key results of this paper are presented in this section, which has been further divided into subsections for reading convenience.

3.1. The Hoffman-Wielandt inequality and its generalization for quaternion matrices

The Hoffman-Wielandt inequality in general does not hold for quaternion matrices if we consider right eigenvalues that are not standard. For example consider normal matrices $A = \begin{bmatrix} 1+i & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$, whose standard eigenvalues are $1+i$, 1 and i, 1 respectively. For the matrix A we consider the right eigenvalues $\mu_1 = 1-i$, $\mu_2 = 1$ and for B we consider $\delta_1 = i$, $\delta_2 = 1$. Then for any permutation π on $\{1,2\}$ the summation $\sum_{i=1}^{2}|\mu_i-\delta_{\pi(i)}|^2>1$, whereas $||A-B||_F^2=1$. We therefore consider the standard right eigenvalues of quaternion matrices and prove the Hoffman-Wielandt inequality for standard eigenvalues below.

THEOREM 4. *Let A and B in $M_n(\mathbb{H})$ be normal matrices. Let $\mu_1,\mu_2\ldots,\mu_n$ and $\delta_1,\delta_2\ldots,\delta_n$ be the standard eigenvalues of A and B respectively. Then there exists a permutation π on the indices $1,2,\ldots n$ such that*

$$\sum_{i=1}^{n}|\mu_i-\delta_{\pi(i)}|^2\leqslant||A-B||_F^2. \tag{3.1}$$

Proof. As A and B are normal, we see from Proposition 3 (g) that the matrices χ_A and χ_B are normal matrices in $M_{2n}(\mathbb{C})$. Therefore, the eigenvalues of χ_A and χ_B are $\mu_1,\mu_2,\ldots,\mu_n,\bar{\mu}_1,\bar{\mu}_2,\ldots,\bar{\mu}_n$ and $\delta_1,\delta_2,\ldots,\delta_n,\bar{\delta}_1,\bar{\delta}_2,\ldots,\bar{\delta}_n$ respectively. Let us rename these eigenvalues as $\mu_1,\mu_2,\ldots,\mu_{2n}$ and $\delta_1,\delta_2,\ldots,\delta_{2n}$ respectively in the same order. Then, from Theorem 1 we infer that there exists a permutation σ on the indices $1,2,\ldots,2n$ such that

$$\sum_{i=1}^{2n}|\mu_i-\delta_{\sigma(i)}|^2\leqslant||\chi_A-\chi_B||_F^2.$$

Consider,

$$\begin{aligned}||\chi_A-\chi_B||_F^2&\geqslant\sum_{i=1}^{2n}|\mu_i-\delta_{\sigma(i)}|^2\\&=\sum_{i=1}^{n}|\mu_i-\delta_{\sigma(i)}|^2+\sum_{i=n+1}^{2n}|\bar{\mu}_{i-n}-\delta_{\sigma(i)}|^2.\end{aligned}$$

Define

$$\gamma_i:=\begin{cases}\delta_{\sigma(i)}, & \text{if } 1\leqslant\sigma(i)\leqslant n\\ \bar{\delta}_{\sigma(i)}, & \text{if } n+1\leqslant\sigma(i)\leqslant 2n.\end{cases}$$

Recalling that for any two elements $a,b\in\mathbb{C}^+$, $|\bar{a}-b|\geqslant|a-b|$, and $|\bar{a}-\bar{b}|=|a-b|$, we see that

$$\sum_{i=1}^{n}|\mu_i-\delta_{\sigma(i)}|^2+\sum_{i=n+1}^{2n}|\bar{\mu}_{i-n}-\delta_{\sigma(i)}|^2\geqslant\sum_{i=1}^{n}|\mu_i-\gamma_i|^2+\sum_{i=n+1}^{2n}|\mu_{i-n}-\gamma_i|^2.$$

Since $2||A-B||_F^2=||\chi_A-\chi_B||_F^2$, we have,

$$\sum_{i=1}^{n}|\mu_i-\gamma_i|^2+\sum_{i=n+1}^{2n}|\mu_{i-n}-\gamma_i|^2\leqslant 2||A-B||_F^2. \tag{3.2}$$

Note that $\{\gamma_1, \ldots, \gamma_{2n}\} = \{\delta_1, \ldots, \delta_n\}$, where each δ_i repeats exactly twice in inequality (3.2).

Let $S_1 = \sum_{i=1}^{n} |\mu_i - \gamma_i|^2$ and $S_2 = \sum_{i=n+1}^{2n} |\mu_{i-n} - \gamma_i|^2$.

Claim. It is possible to rearrange the summations S_1 and S_2 by interchanging the summands in S_1 (if necessary) with the summands in S_2 to get two new summations in which both μ_i's and γ_i's are distinct.

Proof of the Claim. Let $k \leqslant n$ be the smallest integer such that $\gamma_1, \gamma_2, \ldots, \gamma_k$ are distinct in S_1. If $k = n$, we are done. Assume that $k < n$. Consider the $(k+1)^{th}$ summand, $|\mu_{k+1} - \gamma_{k+1}|^2$ of S_1. Then $\gamma_{k+1} = \gamma_q$ for some $1 \leqslant q \leqslant k$. Swap $|\mu_{k+1} - \gamma_{k+1}|^2$ with the $(k+1)^{th}$ summand $|\mu_{k+1} - \gamma_{n+k+1}|^2$ of S_2. Then, the first $(k+1)$ summands of S_1 is

$$\sum_{i=1}^{k} |\mu_i - \gamma_i|^2 + |\mu_{k+1} - \gamma_{n+k+1}|^2.$$

If $\gamma_{n+k+1} \neq \gamma_r$ for any $1 \leqslant r \leqslant k$, we stop here. If not, then $\gamma_{n+k+1} = \gamma_r$ for some $1 \leqslant r \leqslant k$. Note that $r \neq q$, because each γ_i repeats exactly twice and we already have $\gamma_{k+1} = \gamma_q$. Now swap the summand $|\mu_r - \gamma_r|^2$ of S_1 with the r^{th} summand, $|\mu_r - \gamma_{n+r}|^2$ of S_2 and proceed this procedure. After $(k+1)$ repetitions, the first $(k+1)$ terms of S_1 is

$$\sum_{i=1}^{k+1} |\mu_i - \gamma_{j_i}|^2,$$

where $\gamma_{j_s} \neq \gamma_{j_t}$ if $s \neq t$. Thus, by mathematical induction, we can rearrange the summands of S_1 and S_2 such that

$$S_1 + S_2 = \sum_{i=1}^{n} |\mu_i - \gamma_{j_i}|^2 + \sum_{i=1}^{n} |\mu_i - \gamma_{k_i}|^2,$$

where $\{\gamma_{j_1}, \gamma_{j_2}, \ldots, \gamma_{j_n}\} = \{\gamma_{k_1}, \gamma_{k_2}, \ldots, \gamma_{k_n}\} = \{\delta_1, \delta_2, \ldots, \delta_n\}$. This proves the claim.

Define permutations σ_1 and σ_2 on $\{1, 2, \ldots, n\}$ by

$$\sigma_1(i) = j_i \text{ and } \sigma_2(i) = k_i.$$

Then,

$$\sum_{i=1}^{n} |\mu_i - \delta_{\sigma_1(i)}|^2 + \sum_{i=1}^{n} |\mu_i - \delta_{\sigma_2(i)}|^2 = \sum_{i=1}^{n} |\mu_i - \gamma_{j_i}|^2 + \sum_{i=1}^{n} |\mu_i - \gamma_{k_i}|^2 \leqslant 2||A - B||_F^2.$$

Assuming without loss of generality that

$$\sum_{i=1}^{n} |\mu_i - \delta_{\sigma_1(i)}|^2 \leqslant \sum_{i=1}^{n} |\mu_i - \delta_{\sigma_2(i)}|^2,$$

we thus conclude that

$$2\sum_{i=1}^{n} |\mu_i - \delta_{\sigma_1(i)}|^2 \leqslant 2||A - B||_F^2.$$

This implies

$$\sum_{i=1}^{n} |\mu_i - \delta_{\pi(i)}|^2 \leqslant ||A-B||_F^2,$$

where $\pi = \sigma_1$. $\square$

We now prove a generalization of the Hoffman-Wielandt inequality along the one stated in Theorem 2.

THEOREM 5. *Let A be a diagonalizable matrix in $M_n(\mathbb{H})$ and B be a normal matrix in $M_n(\mathbb{H})$. Let $\mu_1,\mu_2,\dots,\mu_n$ and $\delta_1,\delta_2,\dots,\delta_n$ be the standard eigenvalues of A and B respectively. Let X be a nonsingular matrix such that $X^{-1}AX = D = diag(\mu_1,\mu_2,\dots,\mu_n)$. Then there exists a permutation π on $\{1,2,\dots,n\}$ such that*

$$\sum_{i=1}^{n} |\mu_i - \delta_{\pi(i)}|^2 \leqslant ||X||_2^2||X^{-1}||_2^2||A-B||_F^2.$$

Proof. Note that the eigenvalues of χ_A and χ_B are $\mu_1,\mu_2,\dots,\mu_n,\bar{\mu}_1,\bar{\mu}_2,\dots,\bar{\mu}_n$ and $\delta_1,\delta_2,\dots,\delta_n,\bar{\delta}_1,\bar{\delta}_2,\dots,\bar{\delta}_n$ respectively. Let us rename these as $\mu_1,\mu_2,\dots,\mu_{2n}$ and $\delta_1,\delta_2,\dots,\delta_{2n}$ respectively, in the same order. Notice that χ_B is normal as B is normal. Consider the matrix, $\chi_{X^{-1}AX} = \chi_D = \begin{bmatrix} D & 0 \\ 0 & \bar{D} \end{bmatrix}$. Since

$$\chi_{X^{-1}AX} = \chi_{X^{-1}}\chi_A\chi_X = (\chi_X)^{-1}\chi_A\chi_X,$$

we have

$$(\chi_X)^{-1}\chi_A\chi_X = \mathrm{diag}(\mu_1,\mu_2,\dots,\mu_n,\bar{\mu}_1,\bar{\mu}_2,\dots,\bar{\mu}_n).$$

This implies that χ_A is diagonalizable over $\mathbb{C}$ through χ_X. Therefore by Theorem 2 there exists a permutation σ on the indices $1,2,\dots,2n$ such that $\sum_{i=1}^{2n} |\mu_i - \delta_{\sigma(i)}|^2 \leqslant ||\chi_X||_2^2||\chi_X^{-1}||_2^2||\chi_A - \chi_B||_F^2$. Since $||\chi_X||_2 = ||X||_2$, $||\chi_X^{-1}||_2 = ||X^{-1}||_2$ and $2||A-B||_F^2 = ||\chi_A - \chi_B||_F^2$, this implies $\sum_{i=1}^{2n} |\mu_i - \delta_{\sigma(i)}|^2 \leqslant 2||X||_2^2||X^{-1}||_2^2||A-B||_F^2$. As in the proof of the Theorem 4, there exists a permutation π on $\{1,2,\dots,n\}$ such that $2\sum_{i=1}^{n} |\mu_i - \delta_{\pi(i)}|^2 \leqslant \sum_{i=1}^{2n} |\mu_i - \delta_{\sigma(i)}|^2$. Therefore $\sum_{i=1}^{n} |\mu_i - \delta_{\pi(i)}|^2 \leqslant ||X||_2^2||X^{-1}||_2^2||A-B||_F^2$. $\square$

3.2. The Hoffman-Wielandt inequality for quaternion matrix polynomials

In this section, we investigate the Hoffman-Wielandt and its generalization for block companion matrices of quaternion matrix polynomials. We begin with the following lemma, whose proof is a routine computation.

LEMMA 1. *Let* $P(\lambda) = I\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ *be a monic matrix polynomial with block companion matrix* C_P.

(i) If $m = 1$, *then* C_P *is normal if and only if* A_0 *is normal.*

(ii) If $m = 2$, *then* C_P *is normal if and only if* A_0 *is unitary,* A_1 *is normal and* $A_1^* A_0 = -A_1$.

(iii) If $m \geqslant 3$, *then* C_P *is normal if and only if* A_0 *is unitary and* $A_1 = \cdots = A_{m-1} = 0$.

The inequality (3.1) does not generally hold for block companion matrices of linear quaternion matrix polynomials whose coefficients are normal matrices. We illustrate this with an example taken from [4].

EXAMPLE 1. Let $P(\lambda) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\lambda + \begin{bmatrix} 2 & 2 \\ 2 & -14 \end{bmatrix}$, $Q(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{5}{4} \end{bmatrix}\lambda + \begin{bmatrix} 2 & 5 \\ 5 & -\frac{30}{4} \end{bmatrix}$ whose respective block companion matrices are $C_P = \begin{bmatrix} -1 & -1 \\ 1 & -7 \end{bmatrix}$ and $C_Q = \begin{bmatrix} -2 & -5 \\ 4 & -6 \end{bmatrix}$. The standard eigenvalues of C_P and C_Q are respectively $\lambda_1 = -4 - 2\sqrt{2}$, $\lambda_2 = -4 + 2\sqrt{2}$ and $\mu_1 = -4 + 4i$, $\mu_2 = -4 + 4i$. It is easy to verify that $||C_P - C_Q||_F^2 = 27$, whereas for any permutation π on $\{1,2\}$, $\sum_{i=1}^{2} |\lambda_i - \mu_{\pi(i)}|^2 = 48$.

However, for linear quaternion matrix polynomials with unitary coefficients, the inequality (3.1) for the corresponding block companion matrices follows from Theorem 4, because the respective block companion matrices are unitary. For quadratic quaternion matrix polynomials whose coefficients are either normal or unitary matrices, the inequality (3.1) in general fails to hold. The following example from [4] illustrates this.

EXAMPLE 2. Let $P(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\lambda^2 + \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\lambda + \begin{bmatrix} \frac{4}{\sqrt{41}} & \frac{5}{\sqrt{41}} \\ \frac{5}{\sqrt{41}} & -\frac{4}{\sqrt{41}} \end{bmatrix}$ and $Q(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\lambda^2 + \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\lambda + \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$. Consider the respective block companion matrices C_P and C_Q. The standard eigenvalues of C_P and C_Q are $\lambda_1 = 1.6163$, $\lambda_2 = -0.4969 + 0.8643i$, $\lambda_3 = -0.4969 + 0.8643i$, $\lambda_4 = -0.6225$ and $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = -1$, $\mu_4 = -1$ respectively. Note that $||C - D||_F^2 = 4$. However, $\sum_{i=1}^{4} |\mu_{\pi(i)} - \lambda_i|^2 \geqslant 4.5102 > 4$ for any permutation π on $\{1,2,3,4\}$.

We now derive, as a consequence of Lemma 1 and Theorem 4, the Hoffman-Wielandt inequality for block companion matrices of certain types of quaternion matrix polynomials.

THEOREM 6. *Let* $P(\lambda)$ *and* $Q(\lambda)$ *be monic matrix polynomials of degree* m *and size* n *which satisfy the conditions of Lemma 1. If* C_P *and* C_Q *are the block companion* *matrices of $P(\lambda)$ and $Q(\lambda)$, then there exists a permutation π on $\{1,2,\dots,mn\}$ such that*

$$\sum_{i=1}^{mn} |\lambda_i - \mu_{\pi(i)}|^2 \leqslant ||C_P - C_Q||_F^2, \tag{3.3}$$

where $\{\lambda_i\}$ and $\{\mu_i\}$ are the standard eigenvalues of C_P and C_Q respectively.

Proof. By assumptions on $P(\lambda)$ and $Q(\lambda)$, the matrices C_P and C_Q are normal. The desired conclusion follows from Theorem 4. □

We now investigate a generalization of the Hoffman-Wielandt inequality for the block companion matrices of quaternion matrix polynomials, along similar lines as stated in Theorem 2.

Given an $n \times n$ quaternion matrix polynomial $P(\lambda) = \sum_{i=0}^{m} A_i \lambda^i$ we associate a $2n \times 2n$ complex matrix polynomial $P_\chi : \mathbb{C} \to M_{2n}(\mathbb{C})$ defined by $P_\chi(\lambda) = \sum_{i=0}^{m} \chi_{A_i} \lambda^i$. We call $P_\chi(\lambda)$ as the complex adjoint matrix polynomial of $P(\lambda)$. We have the following relation between the standard eigenvalues of $P(\lambda)$ and the eigenvalues of $P_\chi(\lambda)$.

THEOREM 7. *Let $P(\lambda) = I\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ be a quaternion matrix polynomial. Then, $\lambda_1, \lambda_2, \dots, \lambda_{mn}$ are standard eigenvalues of $P(\lambda)$ if and only if $\lambda_1, \lambda_2, \dots, \lambda_{mn}, \bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_{mn}$ are eigenvalues of $P_\chi(\lambda)$.*

Proof. Consider $C_P = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{m-1} \end{bmatrix}$, the block companion matrix corresponding to $P(\lambda)$. Let $A_i = A_{i1} + A_{i2}j$, where $A_{i1}, A_{i2} \in M_n(\mathbb{C})$ for $i = 0, 1, \dots, m-1$. Then

$$C_P = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_{01} & -A_{11} & -A_{21} & \cdots & -A_{(m-1)1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -A_{02} & -A_{12} & -A_{22} & \cdots & -A_{(m-1)2} \end{bmatrix} j$$
$$= C_{P_1} + C_{P_2} j.$$

Therefore the complex adjoint matrix of C_P is $\chi_{C_P} = \begin{bmatrix} C_{P_1} & C_{P_2} \\ -\bar{C}_{P_2} & \bar{C}_{P_1} \end{bmatrix}$. The corresponding block companion matrix of $P_\chi(\lambda)$ is

$$C_{P_\chi} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -\chi_{A_0} & -\chi_{A_1} & -\chi_{A_2} & \cdots & -\chi_{A_{m-1}} \end{bmatrix}.$$

It is easy to verify that $\chi_{C_P} = PC_{P_\chi}P^{-1}$, where P is the permutation matrix given by $P = E_{11} + E_{23} + E_{35} + \cdots + E_{m(2m-1)} + E_{(m+1)2} + E_{(m+2)4} + \cdots + E_{(2m)(2m)}$, where E_{ij} is a block matrix of size $2mn \times 2mn$ with each block of size $n \times n$ such that the ij^{th} block is I_n (the identity matrix of size n) and the remaining blocks being zeros for $1 \leqslant i, j \leqslant 2m$. The proof now follows from Proposition 3 (i). □

We have the following result as a consequence of the above theorem.

THEOREM 8. *Let $P(\lambda)$ be a quaternion matrix polynomial all of whose coefficients are unitary matrices and let λ_0 be one of its right eigenvalues. Then $\frac{1}{2} < |\lambda_0| < 2$.*

Proof. Since the coefficients of $P(\lambda)$ are quaternion unitary matrices, the coefficients of its complex adjoint matrix polynomial, $P_\chi(\lambda)$ are complex unitary matrices. Therefore by Theorem 3.2 of [7], if μ_0 is an eigenvalue of $P_\chi(\lambda)$, then $\frac{1}{2} < |\mu_0| < 2$. Now the result follows from the previous theorem and Corollary 6.1 from [13]. □

We now examine diagonalizability of the block companion matrices of certain classes of quaternion matrix polynomials. We begin with linear quaternion matrix polynomials.

THEOREM 9. *Let $P(\lambda)$ be a linear quaternion matrix polynomial whose coefficients are either (a) unitary matrices or (b) diagonal matrices or (c) positive definite matrices. Then the corresponding block companion matrix C_P of $P(\lambda)$ is diagonalizable.*

Proof. Let $P(\lambda) = A_1\lambda + A_0$. If the coefficients of $P(\lambda)$ are either unitary or diagonal matrices then the result follows trivially. Suppose A_1 and A_0 are positive definite matrices. Consider the corresponding complex adjoint matrix polynomial $P_\chi(\lambda) = \chi_{A_1}\lambda + \chi_{A_0}$. Observe that χ_{A_1} and χ_{A_0} are positive definite complex matrices. Therefore by part (3) of Theorem 2.2 of [4], the block companion matrix $C_{P_\chi} = -\chi_{A_1}^{-1}\chi_{A_0}$ is diagonalizable. Since χ_{C_P} is similar to C_{P_χ}, it follows that χ_{C_P} is diagonalizable. Hence it follows from Proposition 3 (g) that $C_P = -A_1^{-1}A_0$ is diagonalizable. □

REMARK 2. When the coefficients of a linear quaternion matrix polynomial $P(\lambda)$ are either normal matrices or upper(lower) triangular matrices, the corresponding block companion matrix C_P is not diagonalizable in general. The examples given in Remark $2.3(1)$ of [4] along with Proposition 3 (j) serve the purpose.

We now consider quadratic quaternion matrix polynomials.

THEOREM 10. *Let $P(\lambda) = I\lambda^2 + U_1\lambda + U_0$ be an $n \times n$ quaternion matrix polynomial where the coefficients U_0 and U_1 are commuting unitary matrices. Then the corresponding block companion matrix C_P of $P(\lambda)$ is diagonalizable.*

Proof. Let χ_{U_1} and χ_{U_0} be the complex adjoint matrices of U_1 and U_0 respectively. Consider the corresponding complex adjoint matrix polynomial of $P(\lambda)$, given by $P_\chi(\lambda) = I\lambda^2 + \chi_{U_1}\lambda + \chi_{U_0}$ with the block companion matrix C_{P_χ}. Since U_1 and U_0 are commuting unitary matrices, the matrices χ_{U_1} and χ_{U_0} are commuting complex unitary matrices. Then by Theorem 2.4 of [4], C_{P_χ} is diagonalizable. Since χ_{C_P} and C_{P_χ} are similar, χ_{C_P} is diagonalizable. Proposition 3 (g) then implies that C_P is diagonalizable. $\square$

REMARK 3.

1. Note that if either the commutativity condition is removed or if $\deg P(\lambda) > 2$, then Theorem 10 is not true in general. The examples given in Remark 2.5 of [4] illustrate this.

2. The term $\kappa(X) = ||X||_2||X^{-1}||_2$, that appears in the general form of the Hoffman-Wielandt inequality, is the spectral condition number of a square matrix X. In Theorem 10, though we know that χ_{C_P} is diagonalizable through a matrix X for which $\kappa(X) < 2$ (see Section 2.4.1 of [4] for details), we do not know the matrix which diagonalizes C_P. Similarly, in part (c) of Theorem 9, the matrix χ_{C_P} is diagonalizable through a positive definite matrix, but we do not know the matrix which diagonalizes the block companion matrix C_P. Hence it is difficult to estimate the condition number for these matrices. The block companion matrices, in parts (a) and (b) of Theorem 9 are diagonalizable through a unitary matrix and identity matrix respectively. The condition number of these matrices is 1.

We end this paper with the proof of the general form of the Hoffman-Wielandt inequality for block companion matrices of quaternion matrix polynomials.

THEOREM 11. *Let $P(\lambda)$ and $Q(\lambda)$ be quadratic quaternion matrix polynomials of the same size. Let C_P and C_Q be the corresponding block companion matrices. If the coefficients of $P(\lambda)$ are commuting unitary matrices and $Q(\lambda)$ satisfies the condition of Lemma 1 (ii), then there exists a permutation π of the indices $1,2,\ldots,2n$ such that*

$$\sum_{i=1}^{2n} |\lambda_i - \mu_{\pi(i)}|^2 \leqslant ||X||_2^2||X^{-1}||_2^2||C_P - C_Q||_F^2$$

where, $\{\lambda_i\}$ and $\{\mu_i\}$ are the standard eigenvalues of C_P and C_Q respectively, and X is a nonsingular matrix.

Proof. By Theorem 10 the block companion matrix C_P is diagonalizable and by Lemma 1 (ii), the matrix C_Q is normal. The result then follows from Theorem 5. $\square$

REMARK 4. In a similar line, one can prove the general form of the Hoffman-Wielandt inequality for the block companion matrices of linear quaternion matrix polynomials. If $P(\lambda)$ and $Q(\lambda)$ are linear matrix polynomials of same size, where $P(\lambda)$ satisfies either of the conditions in Theorem 9 and $Q(\lambda)$ satisfies the condition (i) of Lemma 1, then the general form of Hoffman-Wielandt inequality follows.

4. Concluding remarks

We have derived the Hoffman-Wielandt inequality and its generalization for quaternion matrices. Under certain assumptions on the coefficients, diagonalizability of block companion matrices of quaternion matrix polynomials is proved. Similarly, a characterization to determine when the block companion matrix of a quaternion matrix polynomial is normal is given. As a consequence, a generalization of the Hoffman-Wielandt inequality for the corresponding block companion matrices of such quaternion matrix polynomials is derived. In addition, bound for the right eigenvalues of quaternion matrix polynomials whose coefficient matrices are unitary is given. The results presented in this paper lead to some interesting questions/problems, one of which has already been pointed out in Remark 3(2). Another interesting question concerns the spectral variation involving left eigenvalues of quaternion matrices and quaternion matrix polynomials; the paper by Huang and So [9] contains some interesting results on finding/computing left eigenvalues of quaternion matrices.

Declarations

- The authors declare that there is no conflict of interest in this work.
- There is no data used in this work.

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