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SHARP IMPROVEMENTS TO THE YOUNG INEQUALITY WITH THE KANTOROVICH CONSTANT AND APPLICATIONS

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Abstract. The aim of the present paper is to show the best possible bound of quadratic fractional type for one-term refinements and reverses of the Young inequality involving the Kantorovich constant. The key tool for obtaining the results is a new L' Hôpital-type higher monotone rule. Some applications to operator inequalities and to the theory of matrices are also discussed.

1. Introduction and statement of main results

The well-known classical inequality between two positive real numbers a and b says that, for all $\nu \in [0,1]$,

$$a\sharp_\nu b := a^{1-\nu}b^\nu \leqslant (1-\nu)a+\nu b =: a\nabla_\nu b$$

with equality sign if and only if $a=b$ or $\nu \in \{0,1\}$, which is attributed to the English mathematician W. H. Young, see [13]. It is also called the weighted geometric-arithmetic mean inequality. The main goal of this paper is to investigate and propose the best possible bounds of quadratic fractional type for one-term refinements and reverses of the Young inequality involving the Kantorovich constant, and then show their applications to operator inequalities and to the theory of matrices.

Let us start this topic with several subtle results, which provide the key motivation for this research. First, using the Kantorovich constant given by $K(h,2) = \frac{(h+1)^2}{4h}$ for $h>0$, Zou et al. [14] proposed a refinement and Liao et al. [6] proved a reverse of the Young inequality in 2011 and 2015 as follows:

$$K(h,2)^{r(\nu)}a\sharp_\nu b \leqslant a\nabla_\nu b \leqslant K(h,2)^{R(\nu)}a\sharp_\nu b, \tag{1.1}$$

here and in the future $a,b>0$, $h=\frac{b}{a}$, $\nu \in [0,1]$, $r(\nu)=\min\{\nu,1-\nu\}$, and $R(\nu) = \max\{\nu,1-\nu\}$. If $a \neq b$ and $\nu \notin \{0,1\}$, both equality signs in (1.1) occur when and only when $\nu = \frac{1}{2}$. This also explains why we usually consider either interval $(0,\frac{1}{2})$ or $(\frac{1}{2},1)$ in one-term refinements and reverses of the Young inequality related to the Kantorovich constant.

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In 2023 and 2024, C. Yang and Z. Wang [10, Theorem 2.2] and X. Yang, C. Yang, H. Li [12, Lemma 2.6 & Theorem 2.7] showed an attractive improvement to the second inequality in (1.1) involving the Kantorovich constant as

$$\frac{K(h,2)^{\nu}a\sharp_{\nu}b-a\nabla_{\nu}b}{K(h,2)^{\tau}a\sharp_{\tau}b-a\nabla_{\tau}b}\leqslant\frac{\nu}{\tau}\leqslant\frac{\nu(1-\nu)}{\tau(1-\tau)},$$

where $\frac{1}{2}<\nu\leqslant\tau<1$. To see that the inequality above is an improvement of the second inequality in (1.1), let us observe that for any $\nu\in(\frac{1}{2},1)$ there exists $\nu_0\in(\frac{1}{2},\nu)$. This, combined with the inequality above, gives us that

$$0\leqslant\frac{\nu}{\nu_0}[K(h,2)^{\nu_0}a\sharp_{\nu_0}b-a\nabla_{\nu_0}b]\leqslant K(h,2)^{\nu}a\sharp_{\nu}b-a\nabla_{\nu}b.$$

It is noticeable that the constant $\frac{\nu(1-\nu)}{\tau(1-\tau)}$ is the best possible bound in the two weight refinement of the Young inequality without the Kantorovich constant (see [9]) but not in the Young inequality with the Kantorovich constant. This raises a natural question: whether the bound $\frac{\nu}{\tau}$ is the best possible? In fact, by employing the L' Hôpital Monotone Rule (or, for short, LMR) given in [1, Theorem 2] and the fact that the equality sign in the second inequality of (1.1) only happens at $\nu=\frac{1}{2}$, the bound $\frac{\nu}{\tau}$ can easily be improved to $\frac{2\nu-1}{2\tau-1}$ to obtain the following result, for all $a,b>0$ and $\frac{1}{2}<\nu\leqslant\tau<1$,

$$\frac{K(h,2)^{\nu}a\sharp_{\nu}b-a\nabla_{\nu}b}{K(h,2)^{\tau}a\sharp_{\tau}b-a\nabla_{\tau}b}\leqslant\frac{2\nu-1}{2\tau-1}\leqslant\frac{\nu}{\tau}.$$

Obviously, it is not too hard to verify that the bound $\frac{2\nu-1}{2\tau-1}$ is the best possible bound of linear fractional type, or equivalently that the above inequality cannot be refined by another bound in the class $\frac{c_1\nu+d_1}{c_2\tau+d_2}$. We will find the best (upper) bound of the form $g(\nu)/g(\tau)$ with g a quadratic polynomial so that for given $\frac{1}{2}<\nu\leqslant\tau<1$ there holds for all $a>b>0$ the inequality

$$\frac{K(h,2)^{\nu}a\sharp_{\nu}b-a\nabla_{\nu}b}{K(h,2)^{\tau}a\sharp_{\tau}b-a\nabla_{\tau}b}\leqslant\frac{g(\nu)}{g(\tau)}.$$

This is also the best lower bound of quadratic fractional type for the left fraction that holds for all $b>a>0$. By considering the function $f(t)=(\frac{c+1}{2})^{2t}-(1-t+ct)$, induced by $K(h,2)^{t}a\sharp_{t}b-a\nabla_{t}b$ for all $t\in(\frac{1}{2},1)$, the desired inequality is reduced to find g such that $\frac{f}{g}$ is increasing (decreasing) on the interval $(\frac{1}{2},1)$ if and only if $c>1$ ($c\in(0,1)$, respectively). In this way and the further idea of the L' Hôpital-type Higher Monotone Rule (see Theorem 2.3 below), we obtain the best possible bound of quadratic fractional type $\frac{\nu(2\nu-1)}{\tau(2\tau-1)}$. Therefore, in this work the best possible bound of quadratic fractional type for one-term refinements and reverses of the Young inequality involving the Kantorovich constant based on the approach via L' Hôpital-type Higher Monotone Rule, are explored and evaluated for the first time. More precisely, we have:

THEOREM 1.1. *Let* $\frac{1}{2} < \nu \leqslant \tau < 1$ *and* $K(h,2) = \frac{(h+1)^2}{4h}$ *be the Kantorovich constant, where* $h = \frac{b}{a} > 0$. *If* $b > a > 0$, *then*

$$\frac{K(h,2)^{\nu} a\sharp_{\nu} b - a\nabla_{\nu} b}{K(h,2)^{\tau} a\sharp_{\tau} b - a\nabla_{\tau} b} \leqslant \frac{\nu(2\nu-1)}{\tau(2\tau-1)} \leqslant \frac{2\nu-1}{2\tau-1}.$$

If $a > b > 0$, *the first inequality is reversed. The fraction* $\frac{\nu(2\nu-1)}{\tau(2\tau-1)}$ *is in both cases attained as the limit of the quotient when* $b \to a$.

In a recent enjoyable work by C. Yang and G. Zhang [11, Theorems 2.1 & 2.3], the authors showed that, if $\frac{1}{6} < \nu \leqslant \tau < \frac{1}{2}$ and $b > a > 0$, then

$$\frac{\nu}{\tau} \leqslant \frac{a\nabla_{\nu} b - K(h,2)^{-\nu} a\sharp_{\nu} b}{a\nabla_{\tau} b - K(h,2)^{-\tau} a\sharp_{\tau} b} \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}.$$

As mentioned above, whereas almost all one-term improvements to the Young inequality with the Kantorovich constant have been established on either $(0,\frac{1}{2})$ or $(\frac{1}{2},1)$, the interval $(\frac{1}{6},\frac{1}{2})$ in the above inequality is not natural. This motivates us to investigate the double inequality on some larger subinterval of $(0,1)$. In fact, by employing the L' Hôpital-type Higher Monotone Rule, we can extend it to the largest interval $(0,1)$ on which $a\nabla_{\nu} b$ and $a\sharp_{\nu} b$ are still weighted arithmetic and geometric means of a and b, respectively. Furthermore, a much more interesting issue is that in the same way as before the upper bound $\frac{\nu(1-\nu)}{\tau(1-\tau)}$ can be improved to the best possible upper bound of quadratic fractional type $\frac{\nu(3-2\nu)}{\tau(3-2\tau)}$; of course, it also becomes the best possible lower bound of quadratic fractional type when $a > b > 0$. More precisely, we obtain the following.

THEOREM 1.2. *Let* $0 < \nu \leqslant \tau < 1$. *If* $b > a > 0$ *and* $h = \frac{b}{a}$, *then*

$$\frac{\nu}{\tau} \leqslant \frac{a\nabla_{\nu} b - K(h,2)^{-\nu} a\sharp_{\nu} b}{a\nabla_{\tau} b - K(h,2)^{-\tau} a\sharp_{\tau} b} \leqslant \frac{\nu(3-2\nu)}{\tau(3-2\tau)} \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}.$$

Moreover, the second inequality is reversed when $a > b > 0$.

To extend Theorems 1.1 and 1.2 to more general forms of convex functions, we utilize the original idea and techniques given in the following.

THEOREM 1.3. ([8]) *Let* a_1, a_2, b_1, b_2 *with* $a_1 \neq a_2$, $b_1 \neq b_2$ *be real numbers selected from an interval* J *on which* φ *is a strictly increasing convex function. Then, the following statements hold.*

(i) *If* $a_2, b_1 \in [a_1, b_2]$, *then*

$$\frac{a_2 - a_1}{b_2 - b_1} \geqslant \frac{\varphi(a_2) - \varphi(a_1)}{\varphi(b_2) - \varphi(b_1)}. \tag{1.2}$$

(ii) *If* $a_1, b_2 \in [b_1, a_2]$, *the inequality* (1.2) *is reversed.*

By taking $a_1 = a\nabla_\nu b,\ a_2 = K(h,2)^\nu a\sharp_\nu b,\ b_1 = a\nabla_\tau b,$ and $b_2 = K(h,2)^\tau a\sharp_\tau b$ in Theorem 1.3 and using Theorem 1.1, we obtain the following theorem, which is a generalization and refinement of [12, Theorem 2.7(a)]. More precisely, we have:

THEOREM 1.4. *Let* $\frac{1}{2} < \nu \leqslant \tau < 1$ *and* $K(h,2) = \frac{(h+1)^2}{4h}$ *be the Kantorovich constant, where* $h = \frac{b}{a} > 0$. *Let* φ *be a strictly increasing convex function defined on* $(0,\infty)$. *If* $b > a > 0$, *then*

$$\frac{\varphi(K(h,2)^\nu a\sharp_\nu b) - \varphi(a\nabla_\nu b)}{\varphi(K(h,2)^\tau a\sharp_\tau b) - \varphi(a\nabla_\tau b)} \leqslant \frac{\nu(2\nu-1)}{\tau(2\tau-1)}.$$

Conversely, if $a > b > 0$, *the inequality is reversed.*

By taking $a_2 = a\nabla_\nu b,\ a_1 = K(h,2)^{-\nu} a\sharp_\nu b,\ b_2 = a\nabla_\tau b,$ and $b_1 = K(h,2)^{-\tau} a\sharp_\tau b$ in Theorem 1.3 and employing Theorem 1.2, we get the following theorem, which is a generalization and refinement of [11, Theorems 2.2 & 2.3]. Namely, the following holds.

THEOREM 1.5. *Let* $0 < \nu \leqslant \tau < 1$, *the Kantorovich constant* $K(h,2)$ *and the function* φ *be as in Theorem 1.4. If* $b > a > 0$, *then*

$$\frac{\varphi(a\nabla_\nu b) - \varphi(K(h,2)^{-\nu} a\sharp_\nu b)}{\varphi(a\nabla_\tau b) - \varphi(K(h,2)^{-\tau} a\sharp_\tau b)} \leqslant \frac{\nu(3-2\nu)}{\tau(3-2\tau)}.$$

Conversely, the inequality is reversed when $a > b > 0$.

The key tool for proving Theorems 1.1 and 1.2 is the LMR given in [1, Theorem 2] and a new L' Hôpital-type Higher Monotone Rule in Theorem 2.3. This helps us significantly reduce complicated calculations. Furthermore, thanks to this tool and some subtle techniques, we can also find the best possible bound of quadratic fractional type as seen above. The best possible constant $\frac{3}{2}$ in the quadratic fractional type bound $\frac{\nu(3-2\nu)}{\tau(3-2\tau)} = \frac{\nu(\frac{3}{2}-\nu)}{\tau(\frac{3}{2}-\tau)}$ is the limit of the function $\phi(x) = \frac{1}{x}\ln\left(\frac{e^x-1}{x(2-e^x)}\right)$ at $x = 0$, which will be found in Lemma 2.7 below.

We conclude this section by outlining the structure of the present paper consisting of four sections as follows. In Section 2, we extend the LMR to the L' Hôpital-type Higher Monotone Rule and use them to prove the theorems given in Section 1. In Section 3, we present operator versions of the obtained inequalities as mentioned above. Some applications of the main results in the first section to the theory of matrices are present in the final section.

2. L' Hôpital-type higher monotone rule and proofs of the main results

The main goal of this section is to prove the theorems in the previous section. As mentioned before, the key tool for proving Theorems 1.1 and 1.2 is the L' Hôpital Monotone Rule. However, we will extend the LMR to a more general case of the L' Hôpital-Type Higher Monotone Rule (or, for short, LHMR).

2.1. L' Hôpital-type higher monotone rule

For the reader's convenience, we cite LMR here.

LEMMA 2.1. ([1, Theorem 2]) *Let $-\infty < a < b < \infty$, and $f, g : [a,b] \to \mathbb{R}$ be two continuous functions differentiable in (a,b). Assume g' never vanishes in (a,b) and that one of the conditions $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$ holds. Then if f'/g' is increasing or decreasing on (a,b), then so is f/g.*

In the following the function

$$H = H_{f,g} = \frac{g^2}{|g'|}\left(\frac{f}{g}\right)'$$

will be used for various functions f,g, the hypotheses of which will be stated in each case and guarantee that H is well defined. The following is said in [1, Theorem 5] and stated here in a more explicit form.

THEOREM 2.2. ([1, Theorem 5]) *Let f,g be differentiable functions on the interval (a,b). Then:*

(i) *If $gg' > 0$ on (a,b), then the function $H_{f,g}$ is increasing on (a,b) if f'/g' is increasing on (a,b).*

(i') *If $gg' > 0$ on (a,b), then the function $H_{f,g}$ is decreasing on (a,b) if f'/g' is decreasing on (a,b).*

(ii) *If $gg' < 0$ on (a,b), then the function $H_{f,g}$ is decreasing on (a,b) if f'/g' is increasing on (a,b).*

(ii') *If $gg' < 0$ on (a,b), then the function $H_{f,g}$ is increasing on (a,b) if f'/g' is decreasing on (a,b).*

So, in a nutshell, the main content of Theorem 2.2 is that there holds

$$(gg' > 0 \text{ and } f'/g' \text{ increasing on } (a,b)) \Rightarrow (H_{f,g} \text{ increasing})$$

and whenever we change at the positions of "$>$" or "increasing" in the hypothesis the symbol or word there into its opposite, so the conclusion will change into its opposite.

The LMR requires that the first-order derivative of the denominator function does not vanish on the open interval of the domain. However, in the following LHMR the denominator function is allowed to degenerate at some point.

In the following statement we use the notation $H_{f,g}(c^-)$ as a shorthand for $\lim\limits_{t\to c^-} H_{f,g}(t)$. We use analogous understanding of limits for other similar situations where necessary.

THEOREM 2.3. *Let $f,g \in C^2[a,b]$ satisfy $gg'' \neq 0$ on (a,b) and assume there exists a unique point $c \in (a,b)$ in which g' changes sign. Suppose furthermore for all $a < t_1 < c < t_2 < b$ that $g'(t_1)g''(t_1)H_{f',g'}(c) \leqslant 0 \leqslant g'(t_2)g''(t_2)H_{f',g'}(c)$ and that f''/g'' is increasing on (a,b). Then the following hold:*

(i) *If* $gg'' > 0$ *on* (a,b) *and* $H_{f,g}(c^{\pm}) \geqslant 0$, *then* f/g *is increasing.*

(ii) *If* $gg'' < 0$ *on* (a,b) *and* $H_{f,g}(c^{\pm}) \leqslant 0$, *then* f/g *is decreasing.*

Proof. (i) In this case we have $\operatorname{sgn} g = \operatorname{sgn} g''$ is a constant function equal -1 or $+1$ on (a,b). It follows that gg' and $g'g''$ change sign exactly in c. One of two possibilities then is $g'(t_1)g''(t_1) > 0 > g'(t_2)g''(t_2)$; the other is having $<$ in place of the $>$. For the selected case we find from the hypothesis that $H_{f',g'}(c) \leqslant 0$. Do in Theorem 2.2(i) the replacements $g \to g',\ g' \to g'',\ f \to f',\ f' \to f''$ and the interval replacement $(a,b) \to (a,c)$. As by our current hypothesis we have that f''/g'' is increasing in (a,c), we get that $H_{f',g'}$ is increasing on (a,c). Next note that on the interval (c,b), we have $gg' < 0$ yet still f''/g'' is increasing. So, using Theorem 2.2(ii), we get that $H_{f',g'}$ is decreasing on (c,b). Thus: $H_{f',g'}(t_1) \leqslant 0 \geqslant H_{f',g'}(t_2)$.

By the definition of $H_{f',g'}$, the negativity of $H_{f',g'}$, on any $t_1 < c < t_2$ implies that $\left(\frac{f'}{g'}\right)' < 0$ on such t_1, t_2 which means that f'/g' is decreasing on (a,b).

We now use the sign changing property of gg' in c and have from the possibility chosen above for $g'g''$, that there holds $g(t_1)g'(t_1) > 0 > g(t_2)g'(t_2)$. We apply again Theorem 2.2 this time for $H_{f,g}$ but again for intervals (a,c) and (c,b). As noted, f'/g' is decreasing on (a,c). Hence by Theorem 2.2(i'), $H_{f,g}$ is decreasing on (a,c) and as f'/g' is also decreasing on (c,b), by Theorem 2.2(ii'), $H_{f,g}$ is increasing on (c,b). Hence $H_{f,g}(t_1) \geqslant H_{f,g}(c^-) \geqslant 0 \leqslant H_{f,g}(c^+) \leqslant H_{f,g}(t_2)$. So by the definition of $H_{f,g}$, the nonnegativity of this function on (a,b) yields $(f/g)'$ is positive and so f/g is increasing on (a,b). The possibility $g'(t_1)g''(t_1) < 0 < g'(t_2)g''(t_2)$ is treated similarly.

The Case (ii) can be treated by adaptions of the proof for Part (i). □

REMARK 2.4. The conclusions of the theorem remain valid if the hypotheses are changed to $g'(t_1)g''(t_1)H_{f',g'}(c) \geqslant 0 \geqslant g'(t_2)g''(t_2)H_{f',g'}(c)$, and f''/g'' is decreasing', with everything else equal.

Note that if the equation $g'(t) = 0$ has no solution in the interval (a,b), then, using a similar approach as in the proof above for each of the intervals (a,c) and (c,b), we obtain the following result.

COROLLARY 2.5. *Let* $f, g \in C^2([a,b])$ *satisfy that* $gg'g'' \neq 0$ *on* (a,b). *Suppose* $g'g''H_{f',g'}(a^+) \geqslant (\leqslant)\ 0$ *on* (a,b) *or* $g'g''H_{f',g'}(b^-) \leqslant (\geqslant)\ 0$ *on* (a,b), *and* $\frac{f''}{g''}$ *is increasing (decreasing, respectively) on* (a,b). *Then, the following statements hold.*

(i) *If* $gg'' > 0$ *on* (a,b), *and* $H_{f,g}(a^+) \geqslant (\leqslant)\ 0$ *or* $H_{f,g}(b^-) \geqslant (\leqslant)\ 0$, *then* $\frac{f}{g}$ *is increasing (decreasing, respectively) on* (a,b).

(ii) *If* $gg'' < 0$ *on* (a,b), *and* $H_{f,g}(a^+) \leqslant (\geqslant)\ 0$ *or* $H_{f,g}(b^-) \leqslant (\geqslant)\ 0$, *then* $\frac{f}{g}$ *is decreasing (increasing, respectively) on* (a,b).

2.2. Proof of Theorems 1.1 and 1.4

In order to prove Theorem 1.1, we need the following result.

LEMMA 2.6. *Let* $\psi(x) = x\ln^2 x - 2x\ln x + 2x - 2$ *on* $(0,\infty)$. *Then, we have* $\psi(x_2) > 0 > \psi(x_1)$ *for all* $\infty > x_2 > 1 > x_1 > 0$.

Proof. First, by denoting $t = \ln x$ for each $x \in (0,\infty)$, the function ψ becomes $g(t) = t^2e^t - 2te^t + 2e^t - 2$ on $(-\infty,\infty)$ with the first-order derivative $g'(t) = t^2e^t > 0$ on $(-\infty,\infty)$. Hence, the function g is strictly increasing on $(-\infty,\infty)$ with $g(0) = 0$. Therefore, for all $-\infty < t_1 < 0 < t_2 < \infty$ we have $g(t_1) < 0 < g(t_2)$, which are equivalent to the desired inequalities. $\square$

Proof of Theorem 1.1. Clearly, for $c := \frac{b}{a} \in (0,\infty)$ the inequalities in Theorem 1.1 are equivalent to

$$\frac{(\frac{1+c}{2})^{2\nu} - (1-\nu+\nu c)}{(\frac{1+c}{2})^{2\tau} - (1-\tau+\tau c)} \leqslant (\geqslant) \frac{\nu(2\nu-1)}{\tau(2\tau-1)}, \tag{2.1}$$

when $c > 1$ ($c \in (0,1)$, respectively). By letting $f(t) = (\frac{1+c}{2})^{2t} - (1-t+ct)$ and $g(t) = t(2t-1)$ for all $t \in [\frac{1}{2},1]$, the inequality (2.1) reduces to proving that the function $\frac{f}{g}$ is increasing (decreasing) on $(\frac{1}{2},1)$ for each fixed $c > 1$ ($0 < c < 1$, respectively). Indeed, by computing directly we obtain the following derivatives $f'(t) = 2(\frac{1+c}{2})^{2t}\ln(\frac{1+c}{2}) + 1 - c$, $f''(t) = 4(\frac{1+c}{2})^{2t}\ln^2(\frac{1+c}{2})$, $g'(t) = 4t-1$ and $g''(t) = 4$ for each fixed $c \in (0,\infty)$. Obviously, the function $\frac{f''}{g''} = (\frac{1+c}{2})^{2t}\ln^2(\frac{1+c}{2})$ is increasing (decreasing) on $(\frac{1}{2},1)$ if and only if $c > 1$ ($c \in (0,1)$, respectively). On the other hand, we also have $H_{f,g}\left(\frac{1}{2}\right) = \lim_{t\to\frac{1}{2}^+} H_{f,g}(t) = 0$, and

$$\begin{aligned} H_{f',g'}\left(\frac{1}{2}\right) &= \lim_{t\to\frac{1}{2}^+} H_{f',g'}(t) = \frac{c+1}{2}\ln^2\left(\frac{c+1}{2}\right) - 2\frac{c+1}{2}\ln\left(\frac{c+1}{2}\right) + 2\frac{c+1}{2} - 2 \\ &= s\ln^2 s - 2s\ln s + 2s - 2 = \psi(s), \end{aligned}$$

where $s = \frac{c+1}{2}$ and ψ is given in Lemma 2.6. Hence, it follows from Lemma 2.6 that $H_{f',g'}(\frac{1}{2}) \geqslant (\leqslant)\ 0$ if and only if $c > 1$ ($c \in (0,1)$, respectively). This, together with $gg' > 0$, $g'g'' > 0$ on $(\frac{1}{2},1)$, and Corollary 2.5, implies that $\frac{f}{g}$ is increasing (decreasing) on $(\frac{1}{2},1)$ if and only if $c > 1$ ($c \in (0,1)$, respectively).

It remains to prove that the leftmost expression in (2.1) converges to the right quotient when $c \searrow 1$ (or $c \nearrow 1$). Indeed, by the general binomial series (for real exponents) around $c = 1$ we have

$$\begin{aligned} &\lim_{c\to 1} \frac{(\frac{1+c}{2})^{2\nu} - (1-\nu+\nu c)}{(\frac{1+c}{2})^{2\tau} - (1-\tau+\tau c)} \\ &= \lim_{c\to 1} \frac{1+\nu(c-1)+\nu(2\nu-1)2^{-1}(c-1)^2 + o((c-1)^3) - (1-\nu+\nu c)}{1+\tau(c-1)+\tau(2\tau-1)2^{-1}(c-1)^2 + o((c-1)^3) - (1-\tau+\tau c)}. \end{aligned}$$

Applying L'Hospital's rule in this limit, we derive

$$\lim_{c\to 1}\frac{(\frac{1+c}{2})^{2\nu}-(1-\nu+\nu c)}{(\frac{1+c}{2})^{2\tau}-(1-\tau+\tau c)}=\frac{\nu(2\nu-1)}{\tau(2\tau-1)}.$$

Hence, the proof is now complete. $\square$

Proof of Theorem 1.4. For any $a,b>0$ and $\frac{1}{2}<\nu\leqslant\tau<1$, we take appropriate mean values $a_1=a\nabla_\nu b$, $a_2=K(h,2)^\nu a\sharp_\nu b$, $b_1=a\nabla_\tau b$, and $b_2=K(h,2)^\tau a\sharp_\tau b$. By the inequality (1.1), we have $a_1<a_2$ and $b_1<b_2$. Moreover, it follows from $b>a>0$ and $\frac{1}{2}<\nu\leqslant\tau<1$ that

$$b_1-a_1=(1-\tau)a+\tau b-[(1-\nu)a+\nu b]=(\tau-\nu)(b-a)\geqslant 0,$$

and

$$b_2-a_2=K(h,2)^\tau a\sharp_\tau b-K(h,2)^\nu a\sharp_\nu b=K(h,2)^\nu a\sharp_\nu b\left[\left(K(h,2)\frac{b}{a}\right)^{\tau-\nu}-1\right]\geqslant 0,$$

which lead to $b_1\geqslant a_1$ and $b_2\geqslant a_2$. Clearly, these two inequalities are reversed when $a>b>0$. Therefore, we obtain $a_2,b_1\in[a_1,b_2]$ for the case $b>a>0$, and $a_1,b_2\in[b_1,a_2]$ for the case $a>b>0$. By Theorems 1.1 and 1.3 we deduce that, for any $\frac{1}{2}<\nu\leqslant\tau<1$ and $b>a>0$ ($a>b>0$, respectively),

$$\frac{\varphi(a_2)-\varphi(a_1)}{\varphi(b_2)-\varphi(b_1)}\leqslant(\geqslant)\ \frac{a_2-a_1}{b_2-b_1}\leqslant(\geqslant)\ \frac{\nu(2\nu-1)}{\tau(2\tau-1)},$$

which finishes the proof of Theorem 1.4. $\square$

2.3. Proofs of Theorems 1.2 and 1.5

The following two lemmas give us some auxiliary results for proving Theorem 1.2.

LEMMA 2.7. *The function* $\phi(x)=\frac{1}{x}\ln\left(\frac{e^x-1}{x(2-e^x)}\right)$ *is strictly increasing on each interval* $(-\infty,0)$ *and* $(0,\ln 2)$; *moreover*, $\lim\limits_{x\to 0}\phi(x)=\frac{3}{2}$.

Proof. First, we have $\lim\limits_{x\to 0}\frac{e^x-1}{x(2-e^x)}=\lim\limits_{x\to 0}\frac{1}{2-e^x}\lim\limits_{x\to 0}\frac{e^x-1}{x}=1$. Therefore, by Lemma 2.1, the monotonic property of ϕ on each interval $(-\infty,0)$ and $(0,\ln 2)$ is the same as that of the derivative of $x\phi(x)$ which is $\Phi(x)$ given by

$$\Phi(x)=\frac{xe^x-e^x+1}{x(e^x-1)}+\frac{e^x}{2-e^x}=:\Phi_1(x)+\Phi_2(x).$$

Clearly, the function Φ_2 is strictly increasing on $(-\infty,\ln 2)$. Hence, it suffices to prove that the function Φ_1 is strictly increasing on each interval $(-\infty,0)$ and $(0,\ln 2)$.

Now, if setting $f(x)=xe^x-e^x+1$ and $g(x)=x(e^x-1)$ on $(-\infty,\ln 2)$, we then have $f'(x)=xe^x$, $f''(x)=e^x(1+x)$, $g'(x)=xe^x+e^x-1$, and $g''(x)=e^x(2+x)$; moreover, $f(0)=g(0)=0$ and $f'(0)=g'(0)=0$. This, together with Lemma 2.1, yields that the functions $\Phi_1=\frac{f}{g}$, $\frac{f'}{g'}$, and $\frac{f''}{g''}$ are monotonic in the same direction on each interval $(-2,0)$ and $(0,\ln 2)$ and as the function $\frac{f''}{g''}=\frac{1+x}{2+x}$ is strictly increasing on each interval $(-2,0)$ and $(0,\ln 2)$, it follows that Φ_1 is strictly increasing on each interval $(-2,0)$ and $(0,\ln 2)$.

Next, in order to prove that Φ_1 is strictly increasing on $(-\infty,0)$, we will show that Φ_1 is strictly increasing on an interval $(-\infty,-1]$. Indeed, by computing directly the first-order derivative of $\frac{f'}{g'}$, we obtain

$$\left(\frac{f'}{g'}\right)'=\frac{f''g'-f'g''}{[g']^2}=\frac{e^x(e^x-1-x)}{(xe^x+e^x-1)^2}>0$$

on $(-\infty,-1]$, namely, the function $\frac{f'}{g'}$ is strictly increasing on $(-\infty,-1]$. This, combined with the fact that $\frac{f'}{g'}$ is strictly increasing on $(-2,0)$ as seen above implies that $\frac{f'}{g'}$ is strictly increasing on $(-\infty,0)$. Therefore, combining with Lemma 2.1, we deduce that the function Φ_1 is strictly increasing on $(-\infty,0)$. Hence, $\phi(x)$ is strictly increasing on each interval $(-\infty,0)$ and $(0,\ln 2)$.

Finally, by using the Maclaurin series $\ln(x+1)=x-\frac{x^2}{2}+\cdots$ and the series expansion $\frac{e^x-1}{x(2-e^x)}=1+\frac{3}{2}x+\frac{13}{6}x^2+\cdots$ we derive the limit $\lim\limits_{x\to 0}\phi(x)=\frac{3}{2}$. $\square$

LEMMA 2.8. *For all $x\in(0,\infty)$ we let $f(x)=x-1-2\left(\frac{2x}{1+x}\right)^{\frac{3}{2}}\ln\left(\frac{2x}{1+x}\right)$. Then, we have $f(x_1)<0<f(x_2)$ for all $0<x_1<1<x_2<\infty$.*

Proof. First, it follows from Lemma 2.7 and the strictly increasing property of the function $\ln(\frac{2x}{1+x})$ on $(0,\infty)$ and of the function ϕ on each interval $(-\infty,0)$ and $(0,\ln 2)$ that $\Phi(x)=\phi\circ\ln(\frac{2x}{1+x})=\ln(\frac{x-1}{2\ln(\frac{2x}{1+x})})\left[\ln(\frac{2x}{1+x})\right]^{-1}$ is strictly increasing on each interval $(0,1)$ and $(1,\infty)$. Hence, by letting $t=\ln(\frac{2x}{1+x})$ we deduce that, for all $\infty>x_2>1>x_1>0$,

$$\Phi(x_2)>\Phi(1)>\Phi(x_1), \tag{2.2}$$

where $\Phi(1):=\lim\limits_{x\to 1}\Phi(x)=\lim\limits_{t\to 0}\phi(t)=\frac{3}{2}$. The double inequality (2.2) is equivalent to the inequality $\frac{x-1}{2\ln(\frac{2x}{1+x})}>\left(\frac{2x}{1+x}\right)^{3/2}$ for all $x\in(0,1)\cup(1,\infty)$, which implies $f(x_1)<0<f(x_2)$ for all $0<x_1<1<x_2<\infty$. $\square$

Proof of Theorem 1.2. The third inequality follows from the fact that the function $\frac{3-2t}{1-t}$ is increasing on $(0,1)$. The other inequalities are equivalent to

$$\frac{\nu}{\tau}\leqslant\frac{1-\nu+c\nu-(\frac{2c}{1+c})^{2\nu}}{1-\tau+c\tau-(\frac{2c}{1+c})^{2\tau}}\leqslant\frac{\nu(3-2\nu)}{\tau(3-2\tau)}, \tag{2.3}$$

where $c = \frac{b}{a}$. By letting $f(t) = 1 - t + ct - (\frac{2c}{1+c})^{2t}$ and $g(t) = t(3-2t)$ for all $t \in [0,1]$ for each fixed $c > 0$, the first inequality in (2.3) reduces to proving that the function $\frac{f(t)}{t}$ is decreasing on $(0,1)$ whereas the second inequality in (2.3) is based on showing that the function $\frac{f}{g}$ is increasing on $(0,1)$ if $c > 1$ otherwise decreasing. By $f(0) = 0$ and Lemma 2.1, the functions $\frac{f(t)}{t}$ and $f'(t) = c - 1 - 2(\frac{2c}{1+c})^{2t}\ln(\frac{2c}{1+c})$ are monotonic in the same sense on $(0,1)$. Clearly, the function f' is decreasing on $(0,1)$ for any fixed $c > 0$. Thus, the function $\frac{f(t)}{t}$ is decreasing on $(0,1)$ as desired.

Next, we have $f''(t) = -4(\frac{2c}{1+c})^{2t}\ln^2(\frac{2c}{1+c})$, $g'(t) = 3 - 4t$, and $g''(t) = -4$ for each fixed $c \in (0,\infty)$ and all $t \in (0,1)$. Clearly, $\frac{f''(t)}{g''(t)} = (\frac{2c}{1+c})^{2t}\ln^2(\frac{2c}{1+c})$ is increasing (decreasing) on $(0,1)$ if and only if $c > 1$ ($c \in (0,1)$, respectively). Moreover, by computing directly and using Lemma 2.8 we obtain

$$H_{f',g'}\Big(\frac{3}{4}\Big) = c - 1 - 2\Big(\frac{2c}{1+c}\Big)^{\frac{3}{2}}\ln\Big(\frac{2c}{1+c}\Big) \geqslant (\leqslant)\ 0$$

if and only if $c > 1$ ($c \in (0,1)$, respectively). Similarly, it follows from Lemma 2.8, $H_{f,g}(t) = \frac{f'(t)}{g'(t)}g(t) - f(t)$ for all $t \in (0,\frac{3}{4})$, and $H_{f,g}(t) = -\frac{f'(t)}{g'(t)}g(t) + f(t)$ for all $t \in (\frac{3}{4},1)$ that

$$H_{f,g}\Big(\Big(\frac{3}{4}\Big)^{\pm}\Big) = \begin{cases} \infty & \text{if } c > 1, \\ -\infty & \text{if } 0 < c < 1. \end{cases}$$

On the other hand, we have $g > 0 > g''$ on $(0,1)$, $g' > 0$ on $(0,\frac{3}{4})$ and $g' < 0$ on $(\frac{3}{4},1)$. Thus, we have $g'(t_1)g''(t_1) < 0 < g'(t_2)g''(t_2)$ for all $0 < t_1 < \frac{3}{4} < t_2 < 1$. This implies $g'(t_1)g''(t_1)H_{f',g'}(\frac{3}{4}) \leqslant (\geqslant)\ 0 \leqslant (\geqslant)\ g'(t_2)g''(t_2)H_{f',g'}(\frac{3}{4})$ for all $0 < t_1 < \frac{3}{4} < t_2 < 1$ if and only if $c > 1$ ($c \in (0,1)$, respectively). Finally, based on the property $gg'' < 0$ on $(0,1)$ and Theorem 2.3(ii), we deduce that $\frac{f}{g}$ is increasing (decreasing) on $(0,1)$ if and only if $c > 1$ ($c \in (0,1)$, respectively). This completes the proof. $\square$

Proof of Theorem 1.5. For any $a,b > 0$ and $0 < \nu \leqslant \tau < 1$, we let $a_2 = a\nabla_\nu b$, $a_1 = K(h,2)^{-\nu}a\sharp_\nu b$, $b_2 = a\nabla_\tau b$, and $b_1 = K(h,2)^{-\tau}a\sharp_\tau b$. Clearly, $a_2 > a_1$ and $b_2 > b_1$ by the Young inequality and $K(h,2) \geqslant 1$ for all $h > 0$. Similar to the proof of Theorem 1.4, by $0 < \nu \leqslant \tau < 1$ we have $a_2 \leqslant (\geqslant)\ b_2$ if and only if $b > a > 0$ ($a > b > 0$, respectively). Let us consider the difference

$$b_1 - a_1 = K(h,2)^{-\tau}a\sharp_\tau b - K(h,2)^{-\nu}a\sharp_\nu b = a\Big(\frac{1}{(\frac{1+c}{2})^{2\tau}} - \frac{1}{(\frac{1+c}{2})^{2\nu}}\Big),$$

where $c = 1/h$. One has $b_1 - a_1 \leqslant (\geqslant)\ 0$ if and only if $c = \frac{a}{b} > 1$ ($c = \frac{a}{b} < 1$, respectively), or equivalently, $b_1 \leqslant (\geqslant)\ a_1$ if and only if $a > b > 0$ ($b > a > 0$, respectively).

For $b > a > 0$, we have $a_2, b_1 \in [a_1, b_2]$. This, combined with Theorem 1.3(i) and Theorem 1.2, gives us that, for all $0 < \nu \leqslant \tau < 1$ and $b > a > 0$,

$$\frac{\varphi(a_2) - \varphi(a_1)}{\varphi(b_2) - \varphi(b_1)} \leqslant \frac{a_2 - a_1}{b_2 - b_1} \leqslant \frac{\nu(3-2\nu)}{\tau(3-2\tau)}.$$

Conversely, for $a > b > 0$ we get $a_1, b_2 \in [b_1, a_2]$, which, together with Theorem 1.3(ii) and Theorem 1.2, implies that, for all $0 < \nu \leqslant \tau < 1$ and $a > b > 0$,

$$\frac{\varphi(a_2) - \varphi(a_1)}{\varphi(b_2) - \varphi(b_1)} \geqslant \frac{a_2 - a_1}{b_2 - b_1} \geqslant \frac{\nu(3 - 2\nu)}{\tau(3 - 2\tau)}.$$

This finishes the proof of the theorem. $\square$

3. Operator inequalities

The main goal of this section is to give operator versions of the inequalities obtained in Section 1. To this end, we recall some necessary preliminaries to the theory of operators.

Let $\mathscr{B}(H)$ be the C^*-algebra of all bounded linear operators acting on a complex separable Hilbert spaces $(H, \langle\cdot,\cdot\rangle)$ and I be the identity operator on H. A selfadjoint operator $A \in \mathscr{B}(H)$ is called:

- positive semi-definite, written $A \geqslant 0$, if $\langle Ax, x\rangle \geqslant 0$ for all $x \in H$,
- positive definite, written $A > 0$, if $\langle Ax, x\rangle > 0$ for all $x \in H, x \neq 0$.

For selfadjoint operators A and B in $\mathscr{B}(H)$, we write $A \geqslant B$ if $A - B \geqslant 0$ and $A > B$ if $A - B > 0$. If $A \in \mathscr{B}(H)$ is a selfadjoint operator, $\mathrm{Sp}(A) \subset [m, M]$ implies $mI \leqslant A \leqslant MI$, where the set $\mathrm{Sp}(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible in } \mathscr{B}(H)\}$ is the spectrum of A. In addition, if $f, g \in C(\mathrm{Sp}(A))$ and $f(t) \geqslant g(t)$ for all $t \in \mathrm{Sp}(A)$, then $f(A) \geqslant g(A)$, see [2, Chapter 1] for more details.

Let $A, B \in \mathscr{B}(H)$ be positive definite with the spectrum $\mathrm{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})$ in the interval J. Using the functional calculus for continuous functions, one defines the generalized geometric mean of A and B as

$$A\sharp_f B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}},$$

where f is a continuous function on J, see [4] for more details. For each $\nu \in [0,1]$, by taking the two functions $f_{\sharp_\nu}(t) = t^\nu$ and $f_{\nabla_\nu}(t) = 1 - \nu + \nu t$ on $(0, \infty)$, respectively for f we obtain ν-weighted geometric and arithmetic means respectively as

$$A\sharp_\nu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}} \quad \text{and} \quad A\nabla_\nu B = (1-\nu)A + \nu B.$$

By employing the functional calculus for continuous functions or the well-known theorem of Kubo and Ando [5] for presenting functions $f_{\sharp_\nu}$ and f_{∇_ν}, we easily deduce the operator version of the Young inequality for two positive definite operators $A, B \in \mathscr{B}(H)$ as $A\sharp_\nu B \leqslant A\nabla_\nu B$ for all $\nu \in [0,1]$. Its recent improvement via the Kantorovich constant can be found in [11, Theorem 3.1].

The three theorems below are generalizations and refinements of the operator Young inequality in the literature.

THEOREM 3.1. *Let $\frac{1}{2} < \nu \leqslant \tau < 1$, $0 < m \leqslant \alpha < \beta \leqslant M < \infty$, A and B be positive invertible operators satisfying one of the following conditions:*

(i) $0 < mI \leqslant A \leqslant \alpha I < \beta I \leqslant B \leqslant MI$,

(ii) $0 < mI \leqslant B \leqslant \alpha I < \beta I \leqslant A \leqslant MI$,

where I is the identity operator. Let $\Phi_s^+(t) = \varphi(K(m/M,2)^s t)$ and $\Phi_s^-(t) = \varphi(K(\alpha/\beta,2)^s t)$ for all $s \in (0,1)$ and all $t \in (0,\infty)$, where φ and $K(\cdot,2)$ are as in Theorem 1.4. If the condition (i) holds, the following inequality holds:

$$A\sharp_{\Phi_\nu^-}(A\sharp_\nu B) - A\sharp_\varphi(A\nabla_\nu B) \leqslant \frac{\nu(2\nu-1)}{\tau(2\tau-1)}\Big[A\sharp_{\Phi_\tau^+}(A\sharp_\tau B) - A\sharp_\tau(A\nabla_\tau B)\Big].$$

Conversely, if the condition (ii) holds, then

$$A\sharp_{\Phi_\nu^+}(A\sharp_\nu B) - A\sharp_\varphi(A\nabla_\nu B) \geqslant \frac{\nu(2\nu-1)}{\tau(2\tau-1)}\Big[A\sharp_{\Phi_\tau^-}(A\sharp_\tau B) - A\sharp_\tau(A\nabla_\tau B)\Big].$$

Proof. First, it follows from Theorem 1.4 that, for all $b > 1$ ($b \in (0,1)$, respectively) and $\frac{1}{2} < \nu \leqslant \tau < 1$,

$$\varphi(K(b,2)^\nu b^\nu) - \varphi(1-\nu+\nu b) \leqslant (\geqslant) \frac{\nu(2\nu-1)}{\tau(2\tau-1)}[\varphi(K(b,2)^\tau b^\tau) - \varphi(1-\tau+\tau b)].$$

Since $K(\cdot,2)$ is decreasing on $(0,1)$ and $K(h,2) = K(\frac{1}{h},2)$ for all $h \in (0,\infty)$, we have

$$\max_{b\in[\frac{m}{M},\frac{\alpha}{\beta}]\cup[\frac{\beta}{\alpha},\frac{M}{m}]} K(b,2) = K\Big(\frac{m}{M},2\Big) \quad \text{and} \quad \min_{b\in[\frac{m}{M},\frac{\alpha}{\beta}]\cup[\frac{\beta}{\alpha},\frac{M}{m}]} K(b,2) = K\Big(\frac{\alpha}{\beta},2\Big).$$

Hence, due to the strictly increasing property of φ on $(0,\infty)$ we deduce that, for all $b \in [\frac{\beta}{\alpha},\frac{M}{m}]$,

$$\Phi_\nu^-(b^\nu) - \varphi(1-\nu+\nu b) \leqslant \frac{\nu(2\nu-1)}{\tau(2\tau-1)}[\Phi_\tau^+(b^\tau) - \varphi(1-\tau+\tau b)], \tag{3.1}$$

and, for all $b \in [\frac{m}{M},\frac{\alpha}{\beta}]$,

$$\Phi_\nu^+(b^\nu) - \varphi(1-\nu+\nu b) \geqslant \frac{\nu(2\nu-1)}{\tau(2\tau-1)}[\Phi_\tau^-(b^\tau) - \varphi(1-\tau+\tau b)]. \tag{3.2}$$

If the condition (i) holds, then $\mathrm{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subset [\frac{\beta}{\alpha},\frac{M}{m}]$. Similarly, if the condition (ii) holds, then $\mathrm{Sp}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subset [\frac{m}{M},\frac{\alpha}{\beta}]$. Therefore, it suffices to replace b with $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ corresponding to inequalities in (3.1) and (3.2), and then multiplying both-sides of the obtained inequalities by $A^{\frac{1}{2}}$, we get the desired inequalities. $\square$

By using Theorem 1.5 and similar arguments as in the proof of Theorem 3.1, we obtain the following results.

THEOREM 3.2. *Let* $0 < \nu \leqslant \tau < 1$, $0 < m \leqslant \alpha < \beta \leqslant M < \infty$, $\Psi_s^-(t) = \varphi(K(m/M,2)^{-s}t)$ *and* $\Psi_s^+(t) = \varphi(K(\alpha/\beta,2)^{-s}t)$ *for all* $s \in (0,1)$ *and all* $t \in (0,\infty)$, *where* φ *and* $K(\cdot,2)$ *are as in Theorem 1.4. If the operators* A *and* B *satisfy the condition (i) of Theorem 3.1, the following inequality holds:*

$$A\sharp_{\Psi_\nu^-}(A\sharp_\nu B) - A\sharp_\varphi(A\nabla_\nu B) \leqslant \frac{\nu(3-2\nu)}{\tau(3-2\tau)}\Big[A\sharp_{\Psi_\tau^+}(A\sharp_\tau B) - A\sharp_\varphi(A\nabla_\tau B)\Big].$$

If the operators A *and* B *satisfy the condition (ii) of Theorem 3.1, the following inequality holds:*

$$A\sharp_{\Psi_\nu^+}(A\sharp_\nu B) - A\sharp_\varphi(A\nabla_\nu B) \geqslant \frac{\nu(3-2\nu)}{\tau(3-2\tau)}\Big[A\sharp_{\Psi_\tau^-}(A\sharp_\tau B) - A\sharp_\varphi(A\nabla_\tau B)\Big].$$

4. Some applications to matrices

The aim of this section is to establish some inequalities for the determinant, trace and unitarily invariant norms of matrices. Before stating and proving the main results, we recall necessary results about the theory of matrices; see, for example, [3, 8].

Let $\mathbb{M}_n$ be a set of all $n \times n$ complex matrices. We write $\mathbb{M}_n^+ = \{A \in \mathbb{M}_n : A \geqslant 0\}$ and $\mathbb{M}_n^{++} = \{A \in \mathbb{M}_n : A > 0\}$, where $A \geqslant 0$ and $A > 0$ are understood in the sense of operators as in the previous section. A norm $|||\cdot|||$ on $\mathbb{M}_n$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all matrices $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$. In particular, if $V = U^*$, this norm is said to be unitarily weak invariant. Clearly, the trace and determinant norms are unitarily weak invariant on $\mathbb{M}_n$, namely, $\operatorname{tr}(UAU^*) = \operatorname{tr}(A)$ and $|UAU^*| = |A|$, here and in what follows, $|X|$ stands for the determinant of the matrix $X \in \mathbb{M}_n$.

4.1. Determinant inequalities

The determinant version of Young's inequality says that $|A\nabla_\nu B| \geqslant |A\sharp_\nu B|$ for all $\nu \in [0,1]$ and any positive definite matrices A and B in $\mathbb{M}_n$. Its improvements via the Kantorovich constant are given in the two theorems below.

THEOREM 4.1. *Let* $\frac{1}{2} < \nu \leqslant \tau < 1$, $0 < m \leqslant \alpha < \beta \leqslant M < \infty$, φ *be as in Theorem 1.4,* $I \in \mathbb{M}_n$ *be the unit matrix,* $\mathscr{K}(t) = K\big(\frac{|A|^{1/n}}{|B|^{1/n}}, 2\big)^{nt}$, $\mathscr{H}(t) = \big(\frac{\beta}{M}\frac{m\nabla_t M}{\alpha\nabla_t\beta}\big)^n$, *and* $\mathscr{L}(t) = \big(\frac{m}{\alpha}\frac{\beta\nabla_\tau\alpha}{M\nabla_\tau m}\big)^n$, *where* $K(\cdot,2)$ *is the Kantorovich constant. If* $A, B \in \mathbb{M}_n$ *are real matrices satisfying* $MI \geqslant B \geqslant \beta I > \alpha I \geqslant A \geqslant mI > 0$, *the following inequalities hold:*

$$\begin{aligned}\varphi(\mathscr{K}(\nu)|A\sharp_\nu B|) - \varphi(|A\nabla_\nu B|) &\leqslant \frac{\nu(2\nu-1)}{\tau(2\tau-1)}\big[\varphi(\mathscr{K}(\tau)|A\sharp_\tau B|) - \varphi([|A|^{\frac{1}{n}}\nabla_\tau |B|^{\frac{1}{n}}]^n)\big]\\ &\leqslant \frac{\nu(2\nu-1)}{\tau(2\tau-1)}\big[\varphi(\mathscr{K}(\tau)|A\sharp_\tau B|) - \varphi(\mathscr{H}(\tau)|A\nabla_\tau B|)\big],\end{aligned}$$

If $A,B \in \mathbb{M}_n$ are real matrices satisfying $MI \geqslant A \geqslant \beta I > \alpha I \geqslant B \geqslant mI > 0$, the following inequalities hold:

$$\begin{aligned}\varphi(\mathscr{K}(\tau)|A\sharp_\tau B|) - \varphi(|A\nabla_\tau B|) &\leqslant \frac{\nu(2\nu-1)}{\tau(2\tau-1)}\left[\varphi(\mathscr{K}(\nu)|A\sharp_\nu B|) - \varphi([|A|^{\frac{1}{n}}\nabla_\nu |B|^{\frac{1}{n}}]^n)\right]\\ &\leqslant \frac{\nu(2\nu-1)}{\tau(2\tau-1)}\left[\varphi(\mathscr{K}(\nu)|A\sharp_\nu B|) - \varphi(\mathscr{L}(\nu)|A\nabla_\nu B|)\right].\end{aligned}$$

Proof. First, since $MI \geqslant B \geqslant \beta I > \alpha I \geqslant A \geqslant mI > 0$, we have $\frac{\alpha}{\beta} I \geqslant B^{-1}A \geqslant \frac{m}{M} I$ (see, for example, the proof of [8, Theorem 3.3]). This double inequality implies that $t^{-1}(1-t)\frac{\alpha}{\beta} \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant t^{-1}(1-t)\frac{m}{M} > 0$, where λ_i's are the eigenvalues of the matrix $t^{-1}(1-t)B^{-1}A$ for each fixed $t \in (0,1)$. Also, since $B > A > 0$, we have $|B| > |A| > 0$. On the other hand, from Minkowski's inequality and its reverse (see [7]), it follows that, for all $t \in (0,1)$,

$$|A\nabla_t B| \geqslant [|A|^{\frac{1}{n}}\nabla_t |B|^{\frac{1}{n}}]^n \geqslant \left(\frac{1+\lambda_n}{1+\lambda_1}\right)^n |A\nabla_t B| \geqslant \mathscr{H}(t)|A\nabla_t B|,$$

where $\mathscr{H}(t) = \left(\frac{\beta}{M}\frac{m\nabla_t M}{\alpha\nabla_t \beta}\right)^n$. Besides, it is easy to show that $|A\sharp_t B| = |A|\sharp_t |B|$ for all $t \in (0,1)$. Now, letting $\Phi(t) = \varphi(t^n)$ for all $t \in (0,\infty)$ and employing Theorem 1.4 with $b = |B|^{\frac{1}{n}} > a = |A|^{\frac{1}{n}}$, we deduce that, for all $\frac{1}{2} < \nu \leqslant \tau < 1$,

$$\begin{aligned}\varphi(|A\nabla_\nu B|) &\geqslant \varphi([|A|^{\frac{1}{n}}\nabla_\nu |B|^{\frac{1}{n}}]^n) = \Phi(|A|^{\frac{1}{n}}\nabla_\nu |B|^{\frac{1}{n}})\\ &\geqslant \Phi(K(h,2)^\nu |A|^{\frac{1}{n}}\sharp_\nu |B|^{\frac{1}{n}}) - \frac{\nu(2\nu-1)}{\tau(2\tau-1)}\left[\Phi(K(h,2)^\tau |A|^{\frac{1}{n}}\sharp_\tau |B|^{\frac{1}{n}}) - \Phi(|A|^{\frac{1}{n}}\nabla_\tau |B|^{\frac{1}{n}})\right]\\ &= \varphi(K(h,2)^{n\nu}|A\sharp_\nu B|) - \frac{\nu(2\nu-1)}{\tau(2\tau-1)}\left[\varphi(K(h,2)^{n\tau}|A\sharp_\tau B|) - \varphi([|A|^{\frac{1}{n}}\nabla_\tau |B|^{\frac{1}{n}}]^n)\right]\\ &\geqslant \varphi(K(h,2)^{n\nu}|A\sharp_\nu B|) - \frac{\nu(2\nu-1)}{\tau(2\tau-1)}\left[\varphi(K(h,2)^{n\tau}|A\sharp_\tau B|) - \varphi(\mathscr{H}(\tau)|A\nabla_\tau B|)\right],\end{aligned}$$

which are the desired inequalities. The other inequalities are proved similarly and so we omit the details. □

Thanks to similar arguments as in the proof of Theorem 4.1, we obtain the following theorem.

THEOREM 4.2. *Let $0 < \nu \leqslant \tau < 1$, $0 < m \leqslant \alpha < \beta \leqslant M < \infty$ and $I \in \mathbb{M}_n$ be the unit matrix. Let $\mathscr{H}, \mathscr{K}, \mathscr{L}$ and φ be as in Theorem 4.1. If $A,B \in \mathbb{M}_n$ satisfy $MI \geqslant B \geqslant \beta I > \alpha I \geqslant A \geqslant mI > 0$, then*

$$\begin{aligned}\varphi(|A\nabla_\tau B|) - \varphi(\mathscr{K}(-\tau)|A\sharp_\nu B|) &\geqslant \frac{\tau(3-2\tau)}{\nu(3-2\nu)}\left[\varphi([|A|^{\frac{1}{n}}\nabla_\nu |B|^{\frac{1}{n}}]^n) - \varphi(\mathscr{K}(-\nu)|A\sharp_\nu B|)\right]\\ &\geqslant \frac{\tau(3-2\tau)}{\nu(3-2\nu)}\left[\varphi(\mathscr{H}(\nu)|A\nabla_\nu B|) - \varphi(\mathscr{K}(-\nu)|A\sharp_\nu B|)\right].\end{aligned}$$

If $A, B \in \mathbb{M}_n$ *satisfy* $MI \geqslant A \geqslant \beta I > \alpha I \geqslant B \geqslant mI > 0$, *then*

$$\begin{aligned}\varphi(|A\nabla_\nu B|) - \varphi(\mathscr{K}(-\nu)|A\sharp_\nu B|) &\geqslant \frac{\nu(3-2\nu)}{\tau(3-2\tau)}\left[\varphi([|A|^{\frac{1}{n}}\nabla_\tau|B|^{\frac{1}{n}}]^n) - \varphi(\mathscr{K}(-\tau)|A\sharp_\tau B|)\right]\\ &\geqslant \frac{\nu(3-2\nu)}{\tau(3-2\tau)}\left[\varphi(\mathscr{L}(\tau)|A\nabla_\tau B|) - \varphi(\mathscr{K}(-\tau)|A\sharp_\tau B|)\right].\end{aligned}$$

4.2. Trace and unitarily invariant norm inequalities

Based on Theorems 1.4 and 1.5, as well as the arguments as in the proofs of [8, Theorems 3.7 & 3.8], we obtain the following results, which generalize and refine [12, Theorems 3.3 & 3.4] and [11, Theorems 3.2 & 5.1].

THEOREM 4.3. *Let* $A, B \in \mathbb{M}_n$ *be two positive definite matrices, the parameter* $\mathfrak{T}(t) = K\left(\frac{\mathrm{tr}(A)}{\mathrm{tr}(B)}, 2\right)^t$ *for any* $t \in (-1, 1)$, *and* φ *be as in Theorem 1.4. Then, the following statements hold:*

(i) *If* $B > A$ *and* $\frac{1}{2} < \nu \leqslant \tau < 1$, *then*

$$\begin{aligned}&\varphi(\mathfrak{T}(\nu)\,\mathrm{tr}(A)\sharp_\nu\,\mathrm{tr}(B)) - \varphi\circ\mathrm{tr}(A\nabla_\nu B)\\ &\qquad\leqslant \frac{\nu(2\nu-1)}{\tau(2\tau-1)}\left[\varphi(\mathfrak{T}(\tau)\,\mathrm{tr}(A)\sharp_\tau\,\mathrm{tr}(B)) - \varphi\circ\mathrm{tr}(A\nabla_\tau B)\right].\end{aligned}$$

(ii) *If* $B > A$ $(A > B$, *respectively*$)$ *and* $0 < \nu \leqslant \tau < 1$, *then*

$$\begin{aligned}&\varphi\circ\mathrm{tr}(A\nabla_\nu B) - \varphi(\mathfrak{T}(-\nu)\,\mathrm{tr}(A)\sharp_\nu\,\mathrm{tr}(B))\\ &\qquad\leqslant(\geqslant)\ \frac{\nu(3-2\nu)}{\tau(3-2\tau)}\left[\varphi\circ\mathrm{tr}(A\nabla_\tau B) - \varphi(\mathfrak{T}(-\tau)\,\mathrm{tr}(A)\sharp_\tau\,\mathrm{tr}(B))\right].\end{aligned}$$

THEOREM 4.4. *Let* $A, B, X \in \mathbb{M}_n$ *satisfy the conditions* $AX \neq 0$ *and* $XB \neq 0$. *Let* $\mathfrak{N}(t) = K\left(\frac{|||AX|||}{|||XB|||}, 2\right)^t$ *for any* $t \in (-1, 1)$, *and* φ *be as in Theorem 1.4. Then, the following statements hold:*

(i) *If* $|||XB||| > |||AX|||$ *and* $\frac{1}{2} < \nu \leqslant \tau < 1$, *then*

$$\begin{aligned}&\varphi(\mathfrak{N}(\nu)\,|||AX|||\,\sharp_\nu\,|||XB|||) - \varphi(|||AX|||\,\nabla_\nu\,|||XB|||)\\ &\qquad\leqslant \frac{\nu(2\nu-1)}{\tau(2\tau-1)}\left[\varphi(\mathfrak{N}(\tau)\,|||AX|||\,\sharp_\tau\,|||XB|||) - \varphi(|||AX|||\,\nabla_\tau\,|||XB|||)\right].\end{aligned}$$

(ii) *If* $|||XB||| > |||AX|||$ $(|||XB||| < |||AX|||$, *respectively*$)$ *and* $0 < \nu \leqslant \tau < 1$, *then*

$$\begin{aligned}&\varphi(|||AX|||\,\nabla_\nu\,|||XB|||) - \varphi(\mathfrak{N}(-\nu)\,|||AX|||\,\sharp_\nu\,|||XB|||)\\ &\qquad\leqslant(\geqslant)\ \frac{\nu(3-2\nu)}{\tau(3-2\tau)}\left[\varphi(|||AX|||\,\nabla_\tau\,|||XB|||) - \varphi(\mathfrak{N}(-\tau)\,|||AX|||\,\sharp_\tau\,|||XB|||)\right].\end{aligned}$$

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