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A CLASS OF DISCRETE HILBERT–TYPE INEQUALITIES IN THE WHOLE PLANE WITH A NON–MONOTONIC KERNEL

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Abstract. In this work, we first construct a non-monotonic discrete kernel function in the whole plane, where the parameters in the newly constructed kernel function are restricted to two special subset of the real line. Utilizing some classic techniques from real analysis and converting the weight functions in the whole plane to the first quadrant, we established the estimation formula for the weight functions, and then a class of new Hilbert-type inequality and its equivalent forms are established. In addition, it is proved that the constant factor of the newly obtained inequality is optimal. Moreover, assigning some special values to the parameters in the kernel function, some special inequalities are proved at the end of the paper.

1. Introduction

In this paper, it is assumed that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$, except where specially noted. Let $\boldsymbol{a} = \{a_m\}_{m=1}^{\infty} \in l_2$, $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty} \in l_2$ be two real number sequences, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\|\boldsymbol{a}\|_2\|\boldsymbol{b}\|_2, \tag{1.1}$$

where the constant factor π is the best possible.

Inequality (1.1) is normally referred to as Hilbert inequality [4], which was proposed by the renowned German D. Hilbert in his lecture on integral equations in 1908. In 1925, Hardy [5] established the extended form of (1.1) as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\csc\frac{\pi}{p}\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q, \tag{1.2}$$

where the constant factor $\pi\csc\frac{\pi}{p}$ is also the best possible.

In 1991, Hsu [6] brought foward the weight coefficient method and established the following strengthened form of (1.1), that is,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \|\boldsymbol{a}\|_{\mu,2}\|\boldsymbol{b}\|_{\nu,2}, \tag{1.3}$$

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where $\mu_m = \pi - \frac{\beta_0}{\sqrt{m}}$, $\nu_n = \pi - \frac{\beta_0}{\sqrt{n}}$ $(\beta_0 = 1.1213^+)$.

In the 30 years since the weight coefficient method was proposed by Hsu, researchers have established various extended forms of inequality (1.2) by optimizing Hsu's method, such as the following one which was proposed by M. Krnić and J. Pečarić in 2006 [8]:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\beta}} < B(\beta_1,\beta_2)\,\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{1.4}$$

where $\beta_1+\beta_2=\beta$ $(0<\beta_1,\beta_2\leqslant 2)$, $\mu_m = m^{p(1-\beta_1)-1}$, $\nu_n = n^{q(1-\beta_2)-1}$, and $B(s,t)$ is β-function [17]. In regard to other extensions of (1.2), we refer to [3, 7, 9, 10, 12, 20–22, 26, 27]. Such inequalities as (1.2), (1.3) and (1.4) are frequently referred to as Hilbert-type inequalities. In addition to the above-mentioned inequalities, the following inequalities are also classic [4]:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q, \tag{1.5}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\log\frac{m}{n}}{m-n}a_m b_n < \left(\pi\csc\frac{\pi}{p}\right)^2\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q, \tag{1.6}$$

where the constant factors pq and $\left(\pi\csc\frac{\pi}{p}\right)^2$ are the best possible.

Inequalities as (1.5) and (1.6) are also referred to as Hilbert-type inequalities. In the past 20 years, by applying some techniques of real analysis, especially Euler-Maclaurin summation formula, researchers established a large number of extensions, analogues, reverses and strengthened forms of these classical Hilbert-type inequalities (see [7, 10, 22–25]).

In general, for a discrete Hilbert type inequality, it is easy to establish corresponding integral and half-discrete Hilbert-type inequalities, such as the following two which are the the integral forms of (1.5) and (1.6), respectively:

$$\int_{y\in\mathbb{R}^+}\int_{x\in\mathbb{R}^+}\frac{f(x)g(y)}{\max\{x,y\}}\mathrm{d}x\mathrm{d}y < pq\|f\|_p\|g\|_q, \tag{1.7}$$

$$\int_{y\in\mathbb{R}^+}\int_{x\in\mathbb{R}^+}\frac{\log\frac{x}{y}}{x-y}f(x)g(y)\mathrm{d}x\mathrm{d}y < \left(\pi\csc\frac{\pi}{p}\right)^2\|f\|_p\|g\|_q, \tag{1.8}$$

where $f,g\geqslant 0$, $f\in L^p(\mathbb{R}^+)$, $g\in L^q(\mathbb{R}^+)$, and pq and $\left(\pi\csc\frac{\pi}{p}\right)^2$ are the best possible. Some other integral and half-discrete Hilbert-inequalities can be referred to [1, 2, 7, 10, 11, 13–15, 22, 28–30]. It should be noted that the integral Hilbert-type inequalities are generally established in the first quadrant. By introducing some new construction techniques, researchers established some new results with the kernel functions defined in the whole plane in recent years [16, 18, 31–33]. However, it is not an easy task to establish discrete Hilbert-type inequalities in the whole plane, as we need to consider some influencing factors on kernel functions, such as the non-negativity, monotonicity and integrability. Nevertheless, we can still find some sporadic results appearing in the literature (see [19]).

In this work, we first construct a non-monotonic discrete kernel function in the whole plane by employing a completely different approach from [19]. Then, converting the weight function to the first quadrant for estimation, a discrete Hilbert-type inequality and its equivalent forms are proved . The paper is organized as follows: detailed lemmas are presented in Section 2, and main theorems and some corollaries are presented in Section 3 and Section 4, respectively.

2. Definitions and lemmas

In this section, we first define the following two special sets of real numbers, that is,

$$\Omega_1 := \left\{ u : u = \frac{2m+1}{2n+1}, m,n \in \mathbb{Z}^+ \right\},$$
$$\Omega_2 := \left\{ u : u = \frac{2m}{2n+1}, m,n \in \mathbb{Z}^+ \right\}.$$

LEMMA 2.1. *Let* $\theta \in \{1,-1\}$, $0 < \tau \leqslant 1$, *and* $\lambda \geqslant 0$. *Suppose that* $\theta, \beta, \lambda, \tau$ *satisfy one of the following conditions:*

(a) $\theta = \pm 1$, $\beta, \gamma \in \Omega_1$, *and* $\beta + \gamma + \tau \leqslant 1$;

(b) $\theta = -1$, $\beta, \gamma \in \Omega_2$, *and* $\gamma \geqslant 2\beta$.

Define

$$K(t) := \frac{(\theta t^\beta - 1)\log|t|}{(\theta t^\gamma + 1)\max\{1,|t|^\lambda\}} \tag{2.1}$$

when $t \in \mathbb{R} \setminus \{-1,0,1\}$, *and* $K(t) := \frac{\beta}{\gamma}$ *when* $t \in \{-1,1\}$. *Then* $K(t) > 0$, *and*

$$\rho(t) := [K(t) + K(-t)]t^{\tau-1} \ \left(t \in \mathbb{R}^+\right)$$

decreases monotonically with t.

Proof. We first consider the case that $\theta, \beta, \gamma, \tau$ satisfy condition (a). Then

$$\rho(t) = \left(\frac{\theta t^\beta - 1}{\theta t^\gamma + 1} + \frac{\theta t^\beta + 1}{\theta t^\gamma - 1} \right) \frac{t^{\tau-1}\log t}{\max\{1,t^\lambda\}}$$
$$= \frac{t^{\tau-1} + t^{\beta+\gamma+\tau-1}}{\max\{1,t^\lambda\}} \frac{2\log t}{t^{2\gamma}-1} := \rho_1(t)\rho_2(t).$$

Since $0 < \tau \leqslant 1$, $\beta + \gamma + \tau \leqslant 1$, and $\lambda \geqslant 0$, it can be shown that $\rho_1(t)$ decreases monotonically with t $(t \in \mathbb{R}^+)$. Furthermore, it can be proved that [24] $\frac{\mathrm{d}\rho_2}{\mathrm{d}t} < 0$, and it follows therefore that $\rho_2(t)$ $\left(\rho_2(1) := \frac{1}{\gamma}\right)$ decreases monotonically with t $(t \in \mathbb{R}^+)$ and $\rho_2(t) > 0$. Hence, $\rho(t)$ decreases monotonically with t $(t \in \mathbb{R}^+)$.

Secondly, if $\theta, \beta, \gamma, \tau$ satisfy condition (b), then it follows that $K(t)$ is an even function, and

$$\rho(t) = \frac{2\log t}{t^{\beta}-1}\frac{t^{\tau-1}}{\max\{1,t^{\lambda}\}}\frac{t^{2\beta-1}}{t^{\gamma}-1} := \hat{\rho}_1(t)\hat{\rho}_2(t)\hat{\rho}_3(t).$$

Since $\beta \in \Omega_2$, $0 < \tau \leqslant 1$, and $\lambda \geqslant 0$, it can be obtained that [24] $\hat{\rho}_1(t)$ $\left(\hat{\rho}_1(1) := \frac{2}{\beta}\right)$ and $\hat{\rho}_2(t)$ decreases with t $(t \in \mathbb{R}^+)$. In addition, it can also be proved that

$$\hat{\rho}_3(t) = \frac{t^{2\beta}-1}{t^{\gamma}-1} \quad \left(\hat{\rho}_3(1) := \frac{2\beta}{\gamma}\right)$$

decreases with t $(t \in \mathbb{R}^+)$. In fact,

$$\frac{\mathrm{d}\hat{\rho}_3}{\mathrm{d}u} = \frac{-t^{2\beta-1}}{(t^{\gamma}-1)^2}\left[2\beta + (\gamma-2\beta)t^{\gamma} - \gamma t^{\gamma-2\beta}\right] := \frac{-t^{2\beta-1}}{(t^{\gamma}-1)^2}h(t).$$

In view of that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \gamma(\gamma-2\beta)t^{\gamma-2\beta-1}(t^{2\beta}-1),$$

and $\gamma \geqslant 2\beta > 0$, it follows that $\frac{\mathrm{d}h}{\mathrm{d}t} < 0$ if $t \in (0,1)$, and $\frac{\mathrm{d}h}{\mathrm{d}t} > 0$ if $t \in (1,\infty)$. Thus, $h(t) \geqslant h(1) = 0$, and therefore $\frac{\mathrm{d}\hat{\rho}_3}{\mathrm{d}t} < 0$ $(t \in \mathbb{R}^+ \setminus \{1\})$. Observe that $\hat{\rho}_3(1) = \frac{2\beta}{\gamma}$, then $\hat{\rho}_3(t)$ is continuous on $\mathbb{R}^+$, and therefore $\hat{\rho}_3(t)$ decreases with t $(t \in \mathbb{R}^+)$. From the above discussion, it follows that $\rho(t)$ decreases monotonically with t $(t \in \mathbb{R}^+)$ when $\theta, \beta, \gamma, \tau$ satisfy condition (b). The proof of $K(t) > 0$ is trivial, and we will not provide a detailed proof here. Lemma 2.1 is proved. □

LEMMA 2.2. *Let $\theta \in \{1,-1\}$, $0 < \tau, \kappa \leqslant 1$, and $\lambda \geqslant 0$. Suppose that $\theta, \beta, \gamma, \tau, \kappa$, λ satisfy $\tau + \kappa = \gamma + \lambda - \beta$ and one of the following conditions:*

(a) $\theta = \pm 1$, $\beta, \gamma \in \Omega_1$;

(b) $\theta = -1$, $\beta, \gamma \in \Omega_2$.

Let $K(t)$ be defined by Lemma 2.1, and

$$N(\beta,\gamma,\tau,\kappa) := \begin{cases} 2\sum_{j=0}^{\infty}\left[\frac{1}{(2\gamma j+\tau)^2} + \frac{1}{(2\gamma j+\beta+\gamma+\kappa)^2}\right] \\ +2\sum_{j=0}^{\infty}\left[\frac{1}{(2\gamma j+\kappa)^2} + \frac{1}{(2\gamma j+\beta+\gamma+\tau)^2}\right], & \beta,\gamma \in \Omega_1, \\ \\ 2\sum_{j=0}^{\infty}\left[\frac{1}{(\gamma j+\tau)^2} + \frac{1}{(\gamma j+\beta+\kappa)^2}\right] \\ +2\sum_{j=0}^{\infty}\left[\frac{1}{(\gamma j+\kappa)^2} + \frac{1}{(\gamma j+\beta+\tau)^2}\right], & \beta,\gamma \in \Omega_2. \end{cases} \tag{2.2}$$

Then

$$\int_{t\in\mathbb{R}^+}[K(t)+K(-t)]t^{\tau-1}\mathrm{d}t = N(\beta,\gamma,\tau,\kappa). \tag{2.3}$$

Proof. To begin with, we consider the case where β,γ satisfy condition (a). Observing that $\tau+\kappa=\gamma+\lambda-\beta$, we have

$$\begin{aligned}
&\int_{t\in\mathbb{R}^+}[K(t)+K(-t)]t^{\tau-1}\mathrm{d}t\\
&=\int_0^1\left(\frac{\theta t^{\beta}-1}{\theta t^{\gamma}+1}+\frac{\theta t^{\beta}+1}{\theta t^{\gamma}-1}\right)(\log t)t^{\tau-1}\mathrm{d}t+\int_1^{\infty}\left(\frac{\theta t^{\beta}-1}{\theta t^{\gamma}+1}+\frac{\theta t^{\beta}+1}{\theta t^{\gamma}-1}\right)(\log t)t^{\tau-\lambda-1}\mathrm{d}t\\
&=2\int_0^1\frac{t^{\beta+\gamma+\tau-1}+t^{\tau-1}}{t^{2\gamma}-1}\log t\mathrm{d}t+2\int_1^{\infty}\frac{t^{\beta+\gamma+\tau-\lambda-1}+t^{\tau-\lambda-1}}{t^{2\gamma}-1}\log t\mathrm{d}t\\
&=2\int_0^1\frac{t^{\tau-1}\log t+t^{\beta+\gamma+\kappa-1}\log t+t^{\kappa-1}\log t+t^{\beta+\gamma+\tau-1}\log t}{t^{2\gamma}-1}\mathrm{d}t\\
&:=2(J_1+J_2+J_3+J_4).
\end{aligned} \tag{2.4}$$

Expand $\frac{1}{1-t^{2\gamma}}$ $(0<t<1)$ into Maclaurin series, and employ Lebesgue term-by-term integration theorem, then we have

$$J_1=-\int_0^1\sum_{j=0}^{\infty}\left(t^{2\gamma j+\tau-1}\log t\right)\mathrm{d}t=-\sum_{j=0}^{\infty}\int_0^1 t^{2\gamma j+\tau-1}\log t\mathrm{d}t. \tag{2.5}$$

Set $\log t=\frac{-u}{2\gamma j+\tau}$, then we have

$$\int_0^1 t^{2\gamma j+\tau-1}\log t\mathrm{d}t=-\frac{1}{(2\gamma j+\tau)^2}\int_0^{\infty}ue^{-u}\mathrm{d}u=-\frac{1}{(2\gamma j+\tau)^2}. \tag{2.6}$$

Apply (2.6) to (2.5), we have

$$J_1=\sum_{j=0}^{\infty}\frac{1}{(2\gamma j+\tau)^2}. \tag{2.7}$$

Similarly, it can also be proved that

$$J_2=\sum_{j=0}^{\infty}\frac{1}{(2\gamma j+\beta+\gamma+\kappa)^2}, \tag{2.8}$$

$$J_3=\sum_{j=0}^{\infty}\frac{1}{(2\gamma j+\kappa)^2}, \tag{2.9}$$

$$J_4=\sum_{j=0}^{\infty}\frac{1}{(2\gamma j+\beta+\gamma+\tau)^2}. \tag{2.10}$$

Plugging (2.7), (2.8), (2.9) and (2.10) back into (2.4), and using (2.2), we arrive at (2.3).

Secondly, consider the case where β,γ satisfy condition (b). Then, $K(t)=K(-t)$, and

$$\begin{aligned}
&\int_{t\in\mathbb{R}^+}[K(t)+K(-t)]t^{\tau-1}\mathrm{d}t\\
&=2\int_0^1\frac{t^{\tau-1}+u^{\beta+\tau-1}}{t^{\gamma}-1}\log t\mathrm{d}t+2\int_1^{\infty}\frac{t^{\tau-\lambda-1}+t^{\tau+\beta-\lambda-1}}{t^{\gamma}-1}\log t\mathrm{d}t\\
&=2\int_0^1\frac{t^{\tau-1}\log t+t^{\beta+\kappa-1}\log t}{t^{\gamma}-1}\mathrm{d}t+2\int_0^1\frac{t^{\kappa-1}\log t+t^{\beta+\tau-1}\log t}{t^{\gamma}-1}\mathrm{d}t\\
&:=2(I_1+I_2).
\end{aligned}\tag{2.11}$$

Similar to the above discussion, we conclude that

$$I_1=2\sum_{j=0}^{\infty}\left[\frac{1}{(\gamma j+\tau)^2}+\frac{1}{(\gamma j+\beta+\kappa)^2}\right],\tag{2.12}$$

$$I_2=2\sum_{j=0}^{\infty}\left[\frac{1}{(\gamma j+\kappa)^2}+\frac{1}{(\gamma j+\beta+\tau)^2}\right].\tag{2.13}$$

Inserting (2.12) and (2.13) back into (2.11), we can also arrive at (2.3). Lemma 2.2 is proved. □

LEMMA 2.3. [28] *Let* $s,t,z>0$, $s+t=z$, *and* $\psi(x)=\csc^2 x$. *Then*

$$\sum_{j=0}^{\infty}\left[\frac{1}{(zj+s)^2}+\frac{1}{(zj+t)^2}\right]=\frac{\pi^2}{z^2}\psi\left(\frac{s\pi}{z}\right).\tag{2.14}$$

LEMMA 2.4. *Let* $\beta,\gamma\in\Omega_1$, $\tau,\kappa>0$ *and* $\tau+\kappa=3\gamma-\beta$. *Let*

$$\begin{aligned}
\bar{N}(\beta,\gamma,\tau,\kappa):=2\sum_{j=0}^{\infty}&\left[\frac{1}{(2\gamma j+\tau)^2}+\frac{1}{(2\gamma j+\beta+\gamma+\kappa)^2}\right]\\
&+2\sum_{j=0}^{\infty}\left[\frac{1}{(2\gamma j+\kappa)^2}+\frac{1}{(2\gamma j+\beta+\gamma+\tau)^2}\right].
\end{aligned}\tag{2.15}$$

Then

$$\bar{N}(\beta,\gamma,\tau,\kappa)=\begin{cases}\frac{\pi^2}{2\gamma^2}\psi\left(\frac{\kappa\pi}{2\gamma}\right)+\frac{\pi^2}{6\gamma^2}-Q_1,\ \tau=2\gamma,\\[2ex] \frac{\pi^2}{2\gamma^2}\left[\psi\left(\frac{\tau\pi}{2\gamma}\right)+\psi\left(\frac{\kappa\pi}{2\gamma}\right)\right]-Q_0,\ \tau\neq 2\gamma\ \text{ and }\ \kappa\neq 2\gamma,\\[2ex] \frac{\pi^2}{2\gamma^2}\psi\left(\frac{\tau\pi}{2\gamma}\right)+\frac{\pi^2}{6\gamma^2}-Q_2,\ \kappa=2\gamma,\end{cases}$$

where $Q_1=\frac{2}{(2\gamma-\kappa)^2}$, $Q_2=\frac{2}{(2\gamma-\tau)^2}$, $Q_0=Q_1+Q_2$.

Proof. If $0<\tau<2\gamma$, $0<\kappa<2\gamma$, then we have $\beta+\kappa>\gamma$ and $\beta+\tau>\gamma$. By Lemma 2.3, we have

$$\begin{aligned}
&\sum_{j=0}^{\infty}\left[\frac{1}{(2\gamma j+\tau)^2}+\frac{1}{(2\gamma j+\beta+\gamma+\kappa)^2}\right]\\
=&\sum_{j=0}^{\infty}\left[\frac{1}{(2\gamma j+\tau)^2}+\frac{1}{(2\gamma j+\beta+\kappa-\gamma)^2}\right]-\frac{1}{(\beta+\kappa-\gamma)^2}\\
=&\sum_{j=0}^{\infty}\left[\frac{1}{(2\gamma j+\tau)^2}+\frac{1}{(2\gamma j+2\gamma-\tau)^2}\right]-\frac{1}{(2\gamma-\tau)^2}\\
=&\frac{\pi^2}{4\gamma^2}\psi\left(\frac{\tau\pi}{2\gamma}\right)-\frac{1}{(2\gamma-\tau)^2}.
\end{aligned}\tag{2.16}$$

Similarly, we have

$$\sum_{j=0}^{\infty}\left[\frac{1}{(2\gamma j+\kappa)^2}+\frac{1}{(2\gamma j+\beta+\gamma+\tau)^2}\right]=\frac{\pi^2}{4\gamma^2}\psi\left(\frac{\kappa\pi}{2\gamma}\right)-\frac{1}{(2\gamma-\kappa)^2}.\tag{2.17}$$

Combine (2.16) and (2.17), then we have

$$\bar{N}(\beta,\gamma,\tau,\kappa)=\frac{\pi^2}{2\gamma^2}\left[\psi\left(\frac{\tau\pi}{2\gamma}\right)+\psi\left(\frac{\kappa\pi}{2\gamma}\right)\right]-Q_0.\tag{2.18}$$

If $\tau=2\gamma$, $0<\kappa<2\gamma$, then we have $\beta+\kappa=\gamma$ and $\beta+\tau>\gamma$. Thus

$$\sum_{j=0}^{\infty}\left[\frac{1}{(2\gamma j+\tau)^2}+\frac{1}{(2\gamma j+\gamma+\beta+\kappa)^2}\right]=\frac{1}{4\gamma^2}\sum_{j=0}^{\infty}\frac{2}{(j+1)^2}=\frac{\pi^2}{12\gamma^2}.\tag{2.19}$$

Combine (2.17) and (2.19), then we have

$$\bar{N}(\beta,\gamma,\tau,\kappa)=\frac{\pi^2}{2\gamma^2}\psi\left(\frac{\kappa\pi}{2\gamma}\right)+\frac{\pi^2}{6\gamma^2}-Q_1.$$

If $2\gamma<\tau<3\gamma$, $0<\kappa<2\gamma$, then we have $0<\beta+\kappa<\gamma$ and $\beta+\tau>\gamma$. By Lemma 2.3, we have

$$\begin{aligned}
&\sum_{j=0}^{\infty}\left[\frac{1}{(2\gamma j+\tau)^2}+\frac{1}{(2\gamma j+\beta+\gamma+\kappa)^2}\right]\\
=&\sum_{j=0}^{\infty}\left[\frac{1}{(2\gamma j+\tau-2\gamma)^2}+\frac{1}{(2\gamma j+\beta+\gamma+\kappa)^2}\right]-\frac{1}{(2\gamma-\tau)^2}\\
=&\sum_{j=0}^{\infty}\left[\frac{1}{(2\gamma j+\tau-2\gamma)^2}+\frac{1}{(2\gamma j+4\gamma-\tau)^2}\right]-\frac{1}{(2\gamma-\tau)^2}\\
=&\frac{\pi^2}{4\gamma^2}\psi\left(\frac{\tau\pi}{2\gamma}\right)-\frac{1}{(2\gamma-\tau)^2}.
\end{aligned}\tag{2.20}$$

Combine (2.17) and (2.20), it can also be proved that (2.18) holds.

If $0<\tau<2\gamma$, $\kappa=2\gamma$, then we have $\beta+\kappa>\gamma$ and $\beta+\tau=2\gamma$, and it can be proved that

$$\bar{N}(\beta,\gamma,\tau,\kappa)=\frac{\pi^2}{2\gamma^2}\psi\left(\frac{\tau\pi}{2\gamma}\right)+\frac{\pi^2}{6\gamma^2}-\frac{1}{(2\gamma-\tau)^2}-Q_2.$$

If $0<\tau<2\gamma$, $2\gamma<\kappa<3\gamma$, then inequality (2.18) follows obviously. Lemma 2.4 is proved. $\square$

LEMMA 2.5. *Let $\beta,\gamma\in\Omega_2$, $\tau,\kappa>0$ and $\tau+\kappa=2\gamma-\beta$. Let*

$$\begin{aligned}\hat{N}(\beta,\gamma,\tau,\kappa):=&\,2\sum_{j=0}^{\infty}\left[\frac{1}{(\gamma j+\tau)^2}+\frac{1}{(\gamma j+\beta+\kappa)^2}\right]\\&+2\sum_{j=0}^{\infty}\left[\frac{1}{(\gamma j+\kappa)^2}+\frac{1}{(\gamma j+\beta+\tau)^2}\right].\end{aligned}\tag{2.21}$$

Then

$$\hat{N}(\beta,\gamma,\tau,\kappa)=\begin{cases}\frac{2\pi^2}{\gamma^2}\psi\left(\frac{\kappa\pi}{\gamma}\right)+\frac{2\pi^2}{3\gamma^2}-T_1, & \tau=\gamma,\\[2ex] \frac{2\pi^2}{\gamma^2}\left[\psi\left(\frac{\tau\pi}{\gamma}\right)+\psi\left(\frac{\kappa\pi}{\gamma}\right)\right]-T_0, & \tau\neq\gamma \quad\text{and}\quad \kappa\neq\gamma,\\[2ex] \frac{2\pi^2}{\gamma^2}\psi\left(\frac{\tau\pi}{\gamma}\right)+\frac{2\pi^2}{3\gamma^2}-T_2, & \kappa=\gamma,\end{cases}$$

where $T_1=\frac{2}{(\gamma-\kappa)^2}$, $T_2=\frac{2}{(\gamma-\tau)^2}$, $T_0=T_1+T_2$.

Proof. If $0<\tau<\gamma$, $0<\kappa<\gamma$, then we have $\beta+\kappa>\gamma$ and $\beta+\tau>\gamma$. By Lemma 2.3, we have

$$\begin{aligned}&\sum_{j=0}^{\infty}\left[\frac{1}{(\gamma j+\tau)^2}+\frac{1}{(\gamma j+\beta+\kappa)^2}\right]\\=&\sum_{j=0}^{\infty}\left[\frac{1}{(\gamma j+\tau)^2}+\frac{1}{(\gamma j+\beta+\kappa-\gamma)^2}\right]-\frac{1}{(\beta+\kappa-\gamma)^2}\\=&\sum_{j=0}^{\infty}\left[\frac{1}{(\gamma j+\tau)^2}+\frac{1}{(\gamma j+\gamma-\tau)^2}\right]-\frac{1}{(\gamma-\tau)^2}\\=&\frac{\pi^2}{\gamma^2}\psi\left(\frac{\tau\pi}{\gamma}\right)-\frac{1}{(\gamma-\tau)^2}.\end{aligned}\tag{2.22}$$

Similarly, we have

$$\sum_{j=0}^{\infty}\left[\frac{1}{(\gamma j+\kappa)^2}+\frac{1}{(\gamma j+\beta+\tau)^2}\right]=\frac{\pi^2}{\gamma^2}\psi\left(\frac{\kappa\pi}{\gamma}\right)-\frac{1}{(\gamma-\kappa)^2}.\tag{2.23}$$

Combine (2.22) and (2.23), then

$$\hat{N}(\beta,\gamma,\tau,\kappa) = \frac{2\pi^2}{\gamma^2}\left[\psi\left(\frac{\tau\pi}{\gamma}\right)+\psi\left(\frac{\kappa\pi}{\gamma}\right)\right]-T_0. \tag{2.24}$$

If $\tau=\gamma,\ 0<\kappa<\gamma$, then

$$\sum_{j=0}^{\infty}\left[\frac{1}{(\gamma j+\tau)^2}+\frac{1}{(\gamma j+\beta+\kappa)^2}\right]=\frac{1}{\gamma^2}\sum_{j=0}^{\infty}\frac{2}{(j+1)^2}=\frac{\pi^2}{3\gamma^2}. \tag{2.25}$$

It follows from (2.23) and (2.25) that

$$\hat{N}(\beta,\gamma,\tau,\kappa) = \frac{2\pi^2}{\gamma^2}\psi\left(\frac{\kappa\pi}{\gamma}\right)+\frac{2\pi^2}{3\gamma^2}-T_1.$$

If $\gamma<\tau<2\gamma,\ 0<\kappa<\gamma$, then it can be proved that (2.24) holds true.

If $0<\tau<\gamma,\ \kappa=\gamma$, then

$$\hat{N}(\beta,\gamma,\tau,\kappa) = \frac{2\pi^2}{\gamma^2}\psi\left(\frac{\tau\pi}{\gamma}\right)+\frac{2\pi^2}{3\gamma^2}-T_2.$$

If $0<\tau<\gamma,\ \gamma<\kappa<2\gamma$, then it can also be proved that (2.24) holds true. From the above discussions, it can be concluded that Lemma 2.5 holds. □

3. Main results

THEOREM 3.1. *Let $\theta\in\{1,-1\}$ and $0<\tau,\kappa\leqslant 1$, and $\lambda\geqslant 0$. Suppose that $\theta,\beta,\gamma,\tau,\kappa,\lambda$ satisfy $\tau+\kappa=\gamma+\lambda-\beta$ and one of the following conditions:*

(a) $\theta=\pm 1,\ \beta,\gamma\in\Omega_1$, *and* $\max\{\tau,\kappa\}\leqslant 1-\beta-\gamma$*;*

(b) $\theta=-1,\ \beta,\gamma\in\Omega_2$, *and* $\gamma\geqslant 2\beta$.

Define

$$G(m,n):=\frac{\left(\theta m^\beta-n^\beta\right)\log|\frac{m}{n}|}{(\theta m^\gamma+n^\gamma)\max\{|m|^\lambda,|n|^\lambda\}} \tag{3.1}$$

when $m,n\neq 0$ and $m\neq\pm n$, and $G(m,n):=\frac{\beta}{\gamma}|n|^{\beta-\gamma-\lambda}$ when $m=\pm n$. Suppose that $a_m,b_n>0,\ \boldsymbol{a}=\{a_m\}_{m\in\mathbb{Z}^0}\in l_{p,\mu}$ and $\boldsymbol{b}=\{b_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$, where $\mu_m=|m|^{p(1-\kappa)-1}$, $\nu_n=|n|^{q(1-\tau)-1}$. Let $N(\beta,\gamma,\tau,\kappa)$ be defined by (2.2). Then

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0}G(m,n)a_mb_n<N(\beta,\gamma,\tau,\kappa)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{3.2}$$

where the constant factor $N(\beta,\gamma,\tau,\kappa)$ in (3.2) is the best possible.

Proof. By Hölder's inequality, we have

$$\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} G(m,n)a_m b_n$$
$$=\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}\left\{[G(m,n)]^{\frac{1}{p}}|n|^{\frac{\tau-1}{p}}|m|^{\frac{1-\kappa}{q}}a_m\right\}\left\{[G(m,m)]^{\frac{1}{q}}|m|^{\frac{\kappa-1}{q}}|n|^{\frac{1-\tau}{p}}b_n\right\}$$
$$\leqslant\left\{\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}G(m,n)|n|^{\tau-1}|m|^{\frac{p(1-\kappa)}{q}}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0}G(m,n)|m|^{\kappa-1}|n|^{\frac{q(1-\tau)}{p}}b_n^q\right\}^{\frac{1}{q}}$$
$$=\left\{\sum_{m\in\mathbb{Z}^0}A_m|m|^{\frac{p(1-\kappa)}{q}}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n\in\mathbb{Z}^0}B_n|n|^{\frac{q(1-\tau)}{p}}b_n^q\right\}^{\frac{1}{q}}, \tag{3.3}$$

where

$$A_m=\sum_{n\in\mathbb{Z}^0}G(m,n)|n|^{\tau-1},\ B_n=\sum_{m\in\mathbb{Z}^0}G(m,n)|m|^{\kappa-1}.$$

Observing that $\gamma-\beta\in\Omega_2$, whether $\beta,\gamma\in\Omega_1$ or $\beta,\gamma\in\Omega_2$, we have

$$A_m=\sum_{n\in\mathbb{Z}^0}G(m,n)|n|^{\tau-1}=\sum_{n\in\mathbb{Z}^+}[G(m,n)+G(m,-n)]n^{\tau-1}$$
$$=\sum_{n\in\mathbb{Z}^+}|m|^{\beta-\gamma-\lambda}\left[K\left(\frac{n}{m}\right)+K\left(-\frac{n}{m}\right)\right]n^{\tau-1}$$
$$=\sum_{n\in\mathbb{Z}^+}|m|^{\beta+\tau-\gamma-\lambda-1}\left[K\left(\frac{n}{|m|}\right)+K\left(-\frac{n}{|m|}\right)\right]\left(\frac{n}{|m|}\right)^{\tau-1}.$$

By Lemma 2.1, it can be shown that

$$\left[K\left(\frac{n}{|m|}\right)+K\left(-\frac{n}{|m|}\right)\right]\left(\frac{n}{|m|}\right)^{\tau-1}$$

decreases monotonically with $n(n\in\mathbb{Z}^+)$ for a fixed m. Therefore, it follows from Lemma 2.2 that

$$A_m<|m|^{\beta+\tau-\gamma-\lambda-1}\int_{t\in\mathbb{R}^+}\left[K\left(\frac{t}{|m|}\right)+K\left(-\frac{t}{|m|}\right)\right]\left(\frac{t}{|m|}\right)^{\tau-1}\mathrm{d}t$$
$$=|m|^{\beta+\tau-\gamma-\lambda}\int_{t\in\mathbb{R}^+}[K(t)+K(-t)]t^{\tau-1}\mathrm{d}t$$
$$=N(\beta,\gamma,\tau,\kappa)|m|^{\beta+\tau-\gamma-\lambda}. \tag{3.4}$$

Similar discussions leads to

$$B_m<|n|^{\beta+\kappa-\gamma-\lambda}\int_{t\in\mathbb{R}^+}[K(t)+K(-t)]t^{\kappa-1}\mathrm{d}t.$$

By Lemma 2.2 and observing that $\tau+\kappa=\gamma+\lambda-\beta$, we have

$$\int_{t\in\mathbb{R}^+}[K(t)+K(-t)]t^{\kappa-1}\mathrm{d}t=N(\beta,\gamma,\tau,\kappa),$$

and it follows that

$$B_m < N(\beta,\gamma,\tau,\kappa)|n|^{\beta+\kappa-\gamma-\lambda}. \tag{3.5}$$

Inserting (3.4) and (3.5) into (3.3), and observing that $\tau+\kappa=\gamma+\lambda-\beta$, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} G(m,n)a_m b_n &< N(\beta,\gamma,\tau,\kappa)\left\{\sum_{m\in\mathbb{Z}^0}|m|^{\frac{p(1-\kappa)-q\kappa}{q}}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n\in\mathbb{Z}^0}|n|^{\frac{q(1-\tau)-p\tau}{p}}b_n^q\right\}^{\frac{1}{q}}\\ &= N(\beta,\gamma,\tau,\kappa)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}.\end{aligned}$$

Hence, we arrive at (3.2). In what follows, it will be proved that the constant factor $N(\beta,\gamma,\tau,\kappa)$ in (3.2) is the best possible. Suppose that there exists a positive real number L such that

$$L \leqslant N(\beta,\gamma,\tau,\kappa), \tag{3.6}$$

and

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0} G(m,n)a_m b_n < L\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{3.7}$$

then it can be proved that

$$L = N(\beta,\gamma,\tau,\kappa). \tag{3.8}$$

In fact, let

$$\begin{aligned}\boldsymbol{a}' &:= \{a'_m\}_{m\in\mathbb{Z}^0} := \left\{|m|^{\kappa-1-\frac{2}{pz}}\right\}_{m\in\mathbb{Z}^0},\\ \boldsymbol{b}' &:= \{b'_n\}_{n\in\mathbb{Z}^0} := \left\{|n|^{\tau-1-\frac{2}{qz}}\right\}_{n\in\mathbb{Z}^0},\end{aligned}$$

where the natural number z is positive and large enough. Then

$$\begin{aligned}\sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} G(m,n)a'_m b'_n &= \sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+} G(m,n)a'_m b'_n + \sum_{n\in\mathbb{Z}^-}\sum_{m\in\mathbb{Z}^+} G(m,n)a'_m b'_n\\ &\quad + \sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^-} G(m,n)a'_m b'_n + \sum_{n\in\mathbb{Z}^-}\sum_{m\in\mathbb{Z}^-} G(m,n)a'_m b'_n\\ &= \sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+}[G(m,n)+G(-m,n)]m^{\kappa-1-\frac{2}{pz}}n^{\tau-1-\frac{2}{qz}}\\ &\quad + \sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+}[G(m,-n)+G(-m,-n)]m^{\kappa-1-\frac{2}{pz}}n^{\tau-1-\frac{2}{qz}}\\ &:= \sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+}\Phi(m,n) + \sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+}\Psi(m,n).\end{aligned} \tag{3.9}$$

Since $\gamma-\beta\in\Omega_2$ and $\tau+\kappa=\gamma+\lambda-\beta$, we have

$$\Phi(m,n) = m^{-2-\frac{2}{z}}\left[K\left(\frac{n}{m}\right)+K\left(-\frac{n}{m}\right)\right]\left(\frac{n}{m}\right)^{\tau-1-\frac{2}{qz}},$$

and it follows from Lemma 2.1 that $\Phi(m,n)$ decreases monotonically with $n(n\in\mathbb{Z}^{+})$ for a fixed $m(m\in\mathbb{Z}^{+})$. Similarly, it can be proved that $\Phi(m,n)$ decreases monotonically with $m(m\in\mathbb{Z}^{+})$ for a fixed $n(n\in\mathbb{Z}^{+})$. Thus, we have

$$\sum_{n\in\mathbb{Z}^{+}}\sum_{m\in\mathbb{Z}^{+}}\Phi(m,n)>\int_{1}^{\infty}\int_{1}^{\infty}u^{-2-\frac{2}{z}}\left[K\left(\frac{v}{u}\right)+K\left(-\frac{v}{u}\right)\right]\left(\frac{v}{u}\right)^{\tau-1-\frac{2}{qz}}\mathrm{d}v\mathrm{d}u.$$

Set $v=ut$, and use Fubini's theorem, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}^{+}}\sum_{m\in\mathbb{Z}^{+}}\Phi(m,n)&>\int_{1}^{\infty}u^{-1-\frac{2}{z}}\int_{\frac{1}{u}}^{\infty}[K(t)+K(-t)]t^{\tau-1-\frac{2}{qz}}\mathrm{d}t\mathrm{d}u\\&=\int_{1}^{\infty}u^{-1-\frac{2}{z}}\int_{1}^{\infty}[K(t)+K(-t)]t^{\tau-1-\frac{2}{qz}}\mathrm{d}t\mathrm{d}u\\&\quad+\int_{1}^{\infty}u^{-1-\frac{2}{z}}\int_{\frac{1}{u}}^{1}[K(t)+K(-t)]t^{\tau-1-\frac{2}{qz}}\mathrm{d}t\mathrm{d}u\\&=\frac{z}{2}\int_{1}^{\infty}[K(t)+K(-t)]t^{\tau-1-\frac{2}{qz}}\mathrm{d}t\\&\quad+\int_{0}^{1}[K(t)+K(-t)]t^{\tau-1-\frac{2}{qz}}\int_{\frac{1}{t}}^{\infty}u^{-1-\frac{2}{z}}\mathrm{d}u\mathrm{d}t\\&=\frac{z}{2}\left[\int_{0}^{1}[K(t)+K(-t)]t^{\tau-1+\frac{2}{pz}}\mathrm{d}t+\int_{1}^{\infty}[K(t)+K(-t)]t^{\tau-1-\frac{2}{qz}}\mathrm{d}t\right].\end{aligned}\tag{3.10}$$

Similarly, it can be proved that

$$\Psi(m,n)>\frac{z}{2}\left[\int_{0}^{1}[K(t)+K(-t)]t^{\tau-1+\frac{2}{pz}}\mathrm{d}t+\int_{1}^{\infty}[K(t)+K(-t)]t^{\tau-1-\frac{2}{qz}}\mathrm{d}t\right].\tag{3.11}$$

Inserting (3.10) and (3.11) into (3.9), we have

$$\begin{aligned}&\sum_{n\in\mathbb{Z}^{0}}\sum_{m\in\mathbb{Z}^{0}}G(m,n)a_{m}'b_{n}'\\&>z\left[\int_{0}^{1}[K(t)+K(-t)]t^{\tau-1+\frac{2}{pz}}\mathrm{d}t+\int_{1}^{\infty}[K(t)+K(-t)]t^{\tau-1-\frac{2}{qz}}\mathrm{d}t\right].\end{aligned}\tag{3.12}$$

On the other hand, it follows from (3.7) that

$$\begin{aligned}\sum_{n\in\mathbb{Z}^{0}}\sum_{m\in\mathbb{Z}^{0}}G(m,n)a_{m}'b_{n}'&<L\|\boldsymbol{a}'\|_{p,\mu}\|\boldsymbol{b}'\|_{q,\nu}\\&=L\left(\sum_{m\in\mathbb{Z}^{0}}|m|^{-1-\frac{2}{z}}\right)^{\frac{1}{p}}\left(\sum_{n\in\mathbb{Z}^{0}}|n|^{-1-\frac{2}{z}}\right)^{\frac{1}{q}}\\&=L\sum_{m\in\mathbb{Z}^{0}}|m|^{-1-\frac{2}{z}}=2L\left(1+\sum_{m=2}^{\infty}m^{-1-\frac{2}{z}}\right)\\&<2L\left(1+\int_{1}^{\infty}t^{-1-\frac{2}{z}}\mathrm{d}t\right)=2L+Lz.\end{aligned}\tag{3.13}$$

Combine (3.12) and (3.13), we have

$$\int_0^1 [K(t)+K(-t)]t^{\tau-1+\frac{2}{pz}}\mathrm{d}t+\int_1^\infty [K(t)+K(-t)]t^{\tau-1-\frac{2}{qz}}\mathrm{d}t<\frac{2L}{z}+L.$$

By Fatou's lemma, we have

$$\begin{aligned}\int_{t\in\mathbb{R}^+}[K(t)+K(-t)]t^{\tau-1}\mathrm{d}t &= \int_0^1 \varliminf_{z\to\infty}[K(t)+K(-t)]t^{\tau-1+\frac{2}{pz}}\mathrm{d}t\\ &\quad+\int_1^\infty \varliminf_{z\to\infty}[K(t)+K(-t)]t^{\tau-1-\frac{2}{qz}}\mathrm{d}t\\ &\leqslant \varliminf_{z\to\infty}\left[\int_0^1 [K(t)+K(-t)]t^{\tau-1+\frac{2}{pz}}\mathrm{d}t\right.\\ &\quad\left.+\int_1^\infty [K(t)+K(-t)]t^{\tau-1-\frac{2}{qz}}\mathrm{d}t\right]\\ &\leqslant \varliminf_{z\to\infty}\left(\frac{2L}{z}+L\right)=L. \end{aligned} \tag{3.14}$$

Insert (2.3) into (3.14), then we have

$$L\geqslant N(\beta,\gamma,\tau,\kappa). \tag{3.15}$$

Combining (3.6) and (3.15), we have (3.8), and therefore the constant factor $N(\beta,\gamma,\tau,\kappa)$ in (3.2) is the best possible. Theorem 3.1 is proved. □

Theorem 3.1 implies the following two inequalities of Hardy-type.

THEOREM 3.2. *Under the conditions of Theorem 3.1, we have*

$$\sum_{n\in\mathbb{Z}^0}|n|^{p\tau-1}\left(\sum_{m\in\mathbb{Z}^0}G(m,n)a_m\right)^p<[N(\beta,\gamma,\tau,\kappa)]^p\,\|\boldsymbol{a}\|_{p,\mu}^p, \tag{3.16}$$

$$\sum_{m\in\mathbb{Z}^0}|m|^{q\kappa-1}\left(\sum_{n\in\mathbb{Z}^0}G(m,n)b_n\right)^q<[N(\beta,\gamma,\tau,\kappa)]^q\,\|\boldsymbol{b}\|_{q,\nu}^p, \tag{3.17}$$

where the constant factors $[N(\beta,\gamma,\tau,\kappa)]^p$ and $[N(\beta,\gamma,\tau,\kappa)]^q$ are the best possible.

Proof. Let $\boldsymbol{x}:=\{x_n\}_{n\in\mathbb{Z}^0}$, where

$$x_n:=|n|^{p\tau-1}\left(\sum_{m\in\mathbb{Z}^0}G(m,n)\,a_m\right)^{p-1}.$$

By Theorem 3.1, we have

$$\begin{aligned} J := \sum_{n\in\mathbb{Z}^0} |n|^{p\tau-1}\left(\sum_{m\in\mathbb{Z}^0} G(m,n)a_m\right)^p \\ &= \sum_{n\in\mathbb{Z}^0}\sum_{m\in\mathbb{Z}^0} G(m,n)a_m x_n \\ &< N(\beta,\gamma,\tau,\kappa)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{x}\|_{q,\nu} \\ &= N(\beta,\gamma,\tau,\kappa)\|\boldsymbol{a}\|_{p,\mu} J^{\frac{1}{q}}. \end{aligned}$$

It implies that inequality (3.16) holds true naturally. Similarly, it can be proved (3.17) holds. Theorem 3.2 is proved. □

4. Some corollaries

Suppose that $\theta = 1$, $\lambda = 0$ and $\beta,\gamma \in \Omega_1$ in Theorem 3.1, and use Lemma 2.3, then we have the following corollary.

COROLLARY 4.1. *Suppose that $\beta,\gamma\in\Omega_1$, $0<\tau,\kappa\leqslant 1$, $\tau+\kappa=\gamma-\beta$, $\max\{\tau,\kappa\} \leqslant 1-\beta-\gamma$, and $\psi(x) = \csc^2 x$. Let $a_m, b_n > 0$, $\boldsymbol{a} = \{a_m\}_{m\in\mathbb{Z}^0} \in l_{p,\mu}$ and $\boldsymbol{b} = \{b_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$, where $\mu_m = |m|^{p(1-\kappa)-1}$, $\nu_n = |n|^{q(1-\tau)-1}$. Then*

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0} \frac{\left(m^\beta - n^\beta\right)\log|\frac{m}{n}|}{m^\gamma + n^\gamma} a_m b_n < \frac{\pi^2}{2\gamma^2}\left[\psi\left(\frac{\tau\pi}{2\gamma}\right) + \psi\left(\frac{\kappa\pi}{2\gamma}\right)\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}. \tag{4.1}$$

Since $\beta,\gamma\in\Omega_1$, we have the following special case by setting $\gamma = 3\beta$ in inequality (4.1):

$$\begin{aligned} \sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0} \frac{m^\beta - n^\beta}{m^\beta + n^\beta}\cdot\frac{\log|\frac{m}{n}|}{m^{2\beta} - m^\beta n^\beta + m^{2\beta}} a_m b_n &< \frac{\pi^2}{18\beta^2}\left[\psi\left(\frac{\tau\pi}{6\beta}\right) + \psi\left(\frac{\kappa\pi}{6\beta}\right)\right] \\ &\quad \times \|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \end{aligned} \tag{4.2}$$

where $0<\tau,\kappa\leqslant 1$, $\tau+\kappa = 2\beta$, and $\max\{\tau,\kappa\}\leqslant 1-4\beta$ $(\beta\in\Omega_1)$.

Let $\tau=\kappa=\beta$ in (4.2), then (4.2) reduces to

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0} \frac{m^\beta - n^\beta}{m^\beta + n^\beta}\cdot\frac{\log|\frac{m}{n}|}{m^{2\beta} - m^\beta n^\beta + m^{2\beta}} a_m b_n < \frac{4\pi^2}{9\beta^2}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{4.3}$$

where $0<\beta\leqslant\frac{1}{5}$ $(\beta\in\Omega_1)$, $\mu_m = |m|^{p(1-\beta)-1}$, and $\nu_n = |n|^{q(1-\beta)-1}$.

Suppose that $\theta = 1$, $\lambda = 2\gamma$ and $\beta,\gamma\in\Omega_1$ in Theorem 3.1, and use Lemma 2.4, then we have Corollary 4.2.

COROLLARY 4.2. *Suppose that $\beta,\gamma\in\Omega_1$, $0<\tau,\kappa\leqslant 1$, $\tau+\kappa = 3\gamma-\beta$, and $\max\{\tau,\kappa\}$*

$\leqslant 1-\beta-\gamma$. *Let* $a_m, b_n > 0$, $\boldsymbol{a} = \{a_m\}_{m\in\mathbb{Z}^0} \in l_{p,\mu}$ *and* $\boldsymbol{b} = \{b_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$, *where* $\mu_m = |m|^{p(1-\kappa)-1}$, $\nu_n = |n|^{q(1-\tau)-1}$. *Let* $\bar{N}(\beta,\gamma,\tau,\kappa)$ *be defined by Lemma 2.4. Then*

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0} \frac{\left(m^\beta - n^\beta\right)\log\left|\frac{m}{n}\right|}{(m^\gamma + n^\gamma)\max\{|m|^{2\gamma},|n|^{2\gamma}\}} a_m b_n < \bar{N}(\beta,\gamma,\tau,\kappa)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}. \tag{4.4}$$

Set $\gamma = 3\beta$, $\tau = 6\beta$, $\kappa = 2\beta$, then we have $0 < \beta \leqslant \frac{1}{10}$ $(\beta\in\Omega_1)$, and (4.4) is transformed into

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0} \frac{\left(m^\beta - n^\beta\right)\log\left|\frac{m}{n}\right|}{\left(m^{3\beta} + n^{3\beta}\right)\max\{|m|^{6\beta},|n|^{6\beta}\}} a_m b_n < \frac{20\pi^2-27}{216\beta^2}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{4.5}$$

where $\mu_m = |m|^{p(1-2\beta)-1}$, $\nu_n = |n|^{q(1-6\beta)-1}$.

Set $\gamma = 3\beta$, $\tau = \kappa = 4\beta$, then we have $0 < \beta \leqslant \frac{1}{8}$ $(\beta\in\Omega_1)$, and (4.4) is transformed into

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0} \frac{\left(m^\beta - n^\beta\right)\log\left|\frac{m}{n}\right|}{\left(m^{3\beta} + n^{3\beta}\right)\max\{|m|^{6\beta},|n|^{6\beta}\}} a_m b_n < \frac{8\pi^2-27}{27\beta^2}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{4.6}$$

where $\mu_m = |m|^{p(1-4\beta)-1}$, $\nu_n = |n|^{q(1-4\beta)-1}$.

Suppose that $\theta = -1$, $\lambda = 0$ and $\beta,\gamma\in\Omega_1$ in Theorem 3.1, and use Lemma 2.3, then the following corollary holds true.

COROLLARY 4.3. *Suppose that* $\beta,\gamma\in\Omega_1$, $0<\tau,\kappa\leqslant 1$, $\tau+\kappa=\gamma-\beta$, $\max\{\tau,\kappa\} \leqslant 1-\beta-\gamma$, *and* $\psi(x) = \csc^2 x$. *Let* $a_m, b_n > 0$, $\boldsymbol{a} = \{a_m\}_{m\in\mathbb{Z}^0} \in l_{p,\mu}$ *and* $\boldsymbol{b} = \{b_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$, *where* $\mu_m = |m|^{p(1-\kappa)-1}$, $\nu_n = |n|^{q(1-\tau)-1}$. *Then*

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0} \frac{\left(m^\beta + n^\beta\right)\log\left|\frac{m}{n}\right|}{m^\gamma - n^\gamma} a_m b_n < \frac{\pi^2}{2\gamma^2}\left[\psi\left(\frac{\tau\pi}{2\gamma}\right)+\psi\left(\frac{\kappa\pi}{2\gamma}\right)\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}. \tag{4.7}$$

Set $\gamma = 3\beta$, $\tau = \kappa = \beta$, then $0 < \beta \leqslant \frac{1}{5}$ $(\beta\in\Omega_1)$, and (4.7) reduces to

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0} \frac{m^\beta + n^\beta}{m^\beta - n^\beta}\cdot\frac{\log\left|\frac{m}{n}\right|}{m^{2\beta}+m^\beta n^\beta + m^{2\beta}} a_m b_n < \frac{4\pi^2}{9\beta^2}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{4.8}$$

where $\mu_m = |m|^{p(1-\beta)-1}$, and $\nu_n = |n|^{q(1-\beta)-1}$.

Suppose that $\theta = -1$, $\lambda = 2\gamma$, and $\beta,\gamma\in\Omega_1$ in Theorem 3.1, and use Lemma 2.4, then we have the following corollary similar to Corollary 4.2.

COROLLARY 4.4. *Suppose that* $\beta,\gamma\in\Omega_1$, $0<\tau,\kappa\leqslant 1$, $\tau+\kappa=3\gamma-\beta$, *and* $\max\{\tau,\kappa\}\leqslant 1-\beta-\gamma$. *Let* $a_m, b_n > 0$, $\boldsymbol{a} = \{a_m\}_{m\in\mathbb{Z}^0} \in l_{p,\mu}$ *and* $\boldsymbol{b} = \{b_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$, *where* $\mu_m = |m|^{p(1-\kappa)-1}$, $\nu_n = |n|^{q(1-\tau)-1}$. *Let* $\bar{N}(\beta,\gamma,\tau,\kappa)$ *be defined by Lemma 2.4. Then*

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0} \frac{\left(m^\beta + n^\beta\right)\log\left|\frac{m}{n}\right|}{(m^\gamma - n^\gamma)\max\{|m|^{2\gamma},|n|^{2\gamma}\}} a_m b_n < \bar{N}(\beta,\gamma,\tau,\kappa)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}. \tag{4.9}$$

Suppose that $\theta=-1$, $\lambda=0$, and $\beta,\gamma\in\Omega_2$ in Theorem 3.1, and use Lemma 2.3, then Theorem 3.1 reduces to the following corollary.

COROLLARY 4.5. *Suppose that $\beta,\gamma\in\Omega_2$, $0<\tau,\kappa\leqslant 1$, $\tau+\kappa=\gamma-\beta$, $\gamma\geqslant 2\beta$, and $\psi(x)=\csc^2 x$. Let $a_m,b_n>0$, $\boldsymbol{a}=\{a_m\}_{m\in\mathbb{Z}^0}\in l_{p,\mu}$ and $\boldsymbol{b}=\{b_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$, where $\mu_m=|m|^{p(1-\kappa)-1}$, $\nu_n=|n|^{q(1-\tau)-1}$. Then*

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0}\frac{\left(m^\beta+n^\beta\right)\log|\frac{m}{n}|}{m^\gamma-n^\gamma}a_m b_n<\frac{2\pi^2}{\gamma^2}\left[\psi\left(\frac{\tau\pi}{\gamma}\right)+\psi\left(\frac{\kappa\pi}{\gamma}\right)\right]\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}. \tag{4.10}$$

Set $\gamma=2\beta$, then $\tau+\kappa=\beta$ $(\beta\in\Omega_1)$, and

$$\psi\left(\frac{\tau\pi}{\gamma}\right)+\psi\left(\frac{\kappa\pi}{\gamma}\right)=\frac{1}{\sin^2\frac{\tau\pi}{\gamma}}+\frac{1}{\cos^2\frac{\tau\pi}{\gamma}}=4\psi\left(\frac{\tau\pi}{\beta}\right).$$

Thus, (4.10) is transformed into

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0}\frac{\log|\frac{m}{n}|}{m^\beta-n^\beta}a_m b_n<\frac{2\pi^2}{\beta^2}\psi\left(\frac{\tau\pi}{\beta}\right)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{4.11}$$

where $0<\tau,\kappa\leqslant 1$, and $\tau+\kappa=\beta$ $(\beta\in\Omega_1)$.

Set $\gamma=4\beta$, $\tau=2\beta$, $\kappa=\beta$, then $0<\beta\leqslant\frac{1}{2}$ $(\beta\in\Omega_1)$, and (4.10) reduces to

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0}\frac{\log|\frac{m}{n}|}{\left(m^\beta-n^\beta\right)\left(m^{2\beta}+n^{2\beta}\right)}a_m b_n<\frac{3\pi^2}{8\beta^2}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{4.12}$$

where $\mu_m=|m|^{p(1-\beta)-1}$, $\nu_n=|n|^{q(1-2\beta)-1}$.

Suppose that $\theta=-1$, $\lambda=\gamma$ and $\beta,\gamma\in\Omega_2$ in Theorem 3.1, and use Lemma 2.5, then Theorem 3.1 reduces to the following corollary.

COROLLARY 4.6. *Suppose that, $\beta,\gamma\in\Omega_2$, $0<\tau,\kappa\leqslant 1$, $\tau+\kappa=2\gamma-\beta$, and $\gamma\geqslant 2\beta$. Let $a_m,b_n>0$, $\boldsymbol{a}=\{a_m\}_{m\in\mathbb{Z}^0}\in l_{p,\mu}$ and $\boldsymbol{b}=\{b_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$, where $\mu_m=|m|^{p(1-\kappa)-1}$, $\nu_n=|n|^{q(1-\tau)-1}$. Let $\hat{N}(\beta,\gamma,\tau,\kappa)$ be defined by Lemma 2.5. Then*

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0}\frac{\left(m^\beta+n^\beta\right)\log|\frac{m}{n}|}{(m^\gamma-n^\gamma)\max\{|m|^\gamma,|n|^\gamma\}}a_m b_n<\hat{N}(\beta,\gamma,\tau,\kappa)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}. \tag{4.13}$$

Set $\gamma=2\beta$, $\tau=\beta$, $\kappa=2\beta$, then we have $0<\beta\leqslant\frac{1}{2}$ $(\beta\in\Omega_2)$, and (4.13) reduces to

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0}\frac{\log|\frac{m}{n}|}{\left(m^\beta-n^\beta\right)\max\{|m|^{2\beta},|n|^{2\beta}\}}a_m b_n<\frac{2\pi^2-6}{3\beta^2}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{4.14}$$

where $\mu_m=|m|^{p(1-\beta)-1}$, $\nu_n=|n|^{q(1-2\beta)-1}$.

Set $\gamma = 2\beta$, $\tau = \kappa = \frac{3\beta}{2}$, then we have $0 < \beta \leqslant \frac{2}{3}$ $(\beta \in \Omega_2)$, and (4.13) reduces to

$$\sum_{m\in\mathbb{Z}^0}\sum_{n\in\mathbb{Z}^0}\frac{\log|\frac{m}{n}|}{\left(m^\beta-n^\beta\right)\max\{|m|^{2\beta},|n|^{2\beta}\}}a_mb_n < \frac{2\pi^2-16}{\beta^2}\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{4.15}$$

where $\mu_m = |m|^{p\left(1-\frac{3\beta}{2}\right)-1}$, $\nu_n = |n|^{q\left(1-\frac{3\beta}{2}\right)-1}$.

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