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JUNG–TYPE INEQUALITIES AND BLASCHKE–SANTALÓ DIAGRAMS FOR DIFFERENT DIAMETER VARIANTS

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Abstract. There are numerous options for defining the diameter of a convex body that fall apart when we consider non-symmetric gauges. We show that the most natural definitions correspond to different symmetrizations of the gauge, i.e. means of the gauge C and its origin reflection $-C$, which allow geometric inequalities between them. In addition, we study Jung-type and more general geometric inequalities for those diameters with respect to the circumradius and inradius, partly also involving the Minkowski asymmetry of the gauge. The completeness of these systems of inequalities is also tackled by providing (major parts of) the according (r,D,R)-Blaschke-Santaló-diagrams.

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