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JUNG–TYPE INEQUALITIES AND BLASCHKE–SANTALÓ DIAGRAMS FOR DIFFERENT DIAMETER VARIANTS

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Abstract. There are numerous options for defining the diameter of a convex body that fall apart when we consider non-symmetric gauges. We show that the most natural definitions correspond to different symmetrizations of the gauge, i.e. means of the gauge C and its origin reflection $-C$, which allow geometric inequalities between them. In addition, we study Jung-type and more general geometric inequalities for those diameters with respect to the circumradius and inradius, partly also involving the Minkowski asymmetry of the gauge. The completeness of these systems of inequalities is also tackled by providing (major parts of) the according (r,D,R)-Blaschke-Santaló-diagrams.

1. Introduction

Inequalities between geometric functionals, such as the inradius, circumradius, and diameter, form a central area of convex geometry. They play an important role in many classical works such as [4, 13, 14, 18, 26] and are still of interest today [3, 22, 27, 30]. These geometric inequalities have proven to be useful for many results in convexity and have many applications, too (for example, they provide bounds on the size of so-called core-sets for containment under homothety [11]). One of the first such inequalities has been given by Jung [32]. It provides a bound for the diameter-circumradius ratio in Euclidean spaces. Variants and natural extensions, given e.g. in [2, 12, 16], are typically subsumed under the term *Jung-type*.

A non-empty compact convex set is called a *convex body*. The family of convex bodies in $\mathbb{R}^n$ is denoted by $\mathscr{C}^n$, the convex bodies in $\mathbb{R}^n$ excluding single points by $\bar{\mathscr{C}}^n$, and those which contain 0 in their interior, by $\mathscr{C}^n_0$. For any $X,Y \subset \mathbb{R}^n$ and $\rho \in \mathbb{R}$, let $X+Y := \{x+y : x \in X, y \in Y\}$ denote the *Minkowski sum* of X and Y and $\rho X := \{\rho x : x \in X\}$ the ρ-*dilatation* of X. We abbreviate $\{x\}+Y =: x+Y$ and $(-1)X =: -X$. If $X = -X$, the set X is called *0-symmetric*. If there exists $t \in \mathbb{R}^n$ such that $-(t+X) = t+X$, we say that X is *symmetric*. Segments with endpoints $x,y \in \mathbb{R}^n$ are denoted by $[x,y] := \{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}$.

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This work expands upon the master thesis of Mia Runge, from which several results concerning the properties of diameters and the Blaschke-Santaló diagrams have been adopted.

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Even when the gauge is not symmetric, the in- and circumradius have a unified definition: The *circumradius* of $K \in \mathscr{C}^n$ with respect to $C \in \mathscr{C}^n$ is defined as

$$R(K,C) := \inf\{\rho \geqslant 0 : \exists t \in \mathbb{R}^n \text{ such that } K \subset t + \rho C\}$$

and the *inradius* as

$$r(K,C) := \sup\{\rho \geqslant 0 : \exists t \in \mathbb{R}^n \text{ such that } t + \rho C \subset K\}.$$

One should recognize that the inradius can be expressed as a circumradius: $r(K,C) = R(C,K)^{-1}$ with the usual rules $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$.

However, there are several ways to interpret the diameter in the symmetric case, and we consider different extensions to the non-symmetric case.

First, one may understand the diameter of $K \in \mathscr{C}^n$ with respect to $C \in \mathscr{C}_0^n$ as a kind of "maximal" segment. But how should this "maximality" be defined?

We could measure the distance between two points $x, y \in K$ using the *gauge function* $\|\cdot\|_C : \mathbb{R}^n \to [0,\infty]$, $x \mapsto \|x\|_C := \min\{\lambda \geqslant 0 : x \in \lambda C\}$ and define the diameter as the maximal such distance:

$$D_{\mathrm{MIN}}(K,C) := \max_{x,y \in K} \|x-y\|_C.$$

We call it the *minimum diameter*. This definition has been first studied by Leichtweiss [33].

If C is not symmetric the values of $\|x-y\|_C$ and $\|x-y\|_{-C}$ might differ. Instead of taking the maximum of these two values, one may consider the arithmetic mean of them:

$$D_{\mathrm{HM}}(K,C) := \max_{x,y \in K} \tfrac{1}{2}\left(\|x-y\|_C + \|y-x\|_C\right).$$

We call this the *harmonic diameter*.

Both definitions above differ from the most common diameter which has been studied in [10, 12, 14, 25], where it is defined as twice the maximal circumradius of segments in K:

$$D_{\mathrm{AM}}(K,C) := \max_{x,y \in K} 2R([x,y],C).$$

We call it the *arithmetic diameter* (or standard) diameter. It stems from interpreting it as the best two-point approximation of K in terms of the circumradius (cf. [11]). However, this arithmetic diameter can also be defined as the maximal distance between two parallel supporting hyperplanes of K (i. e. the maximal breadth) relative to the breadth of the gauge in the same direction. Using the *support function* $h_C(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $h_C(a) := \max_{x \in C} a^\top x$ one obtains

$$D_{\mathrm{AM}}(K,C) = \max_{a \in \mathbb{R}^n \setminus \{0\}} \frac{h_K(a) + h_K(-a)}{\frac{1}{2}(h_C(a) + h_C(-a))}.$$

Taking the maximum of $h_C(a)$ and $h_C(-a)$ instead of the arithmetic mean, we obtain the fourth diameter we consider, the *maximum diameter*:

$$D_{\mathrm{MAX}}(K,C) := \max_{a \in \mathbb{R}^n \setminus \{0\}} \frac{h_K(a) + h_K(-a)}{\max(h_C(a), h_C(-a))}.$$

The diameters D_{HM} and D_{MAX} have been first introduced in [7, Appendix]. If the gauge C is 0-symmetric, all these definitions lead to the same (classical) diameter:

$$D(K,C) = \max_{x,y\in K} \|x-y\|_C = \max_{x,y\in K} 2R([x,y],C) = \max_{a\in\mathbb{R}^n\setminus\{0\}} \frac{h_K(a)+h_K(-a)}{h_C(a)}.$$

Thus, if we consider a symmetric gauge, we omit the index in D. It turns out that in all of the four diameter definitions, the second argument can be replaced by a symmetrization of the gauge C: We define the *minimum* symmetrization of C by $C_{\mathrm{MIN}} := C \cap -C$, the *harmonic mean* symmetrization by $C_{\mathrm{HM}} := \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$, where C° denotes the polar set of C, the *arithmetic mean* symmetrization by $C_{\mathrm{AM}} := \frac{C-C}{2}$, and the *maximum* symmetrization by $C_{\mathrm{MAX}} := \mathrm{conv}(C \cup -C)$.

Doing so, for each $\mathrm{M} \in \{\mathrm{MIN},\mathrm{HM},\mathrm{AM},\mathrm{MAX}\}$, one obtains

$$D_{\mathrm{M}}(K,C) = D(K,C_{\mathrm{M}}),$$

see [7, Appendix] and Section 2. These symmetrizations and their relations are studied, for example, in [19, 35] and recently also in [6, 7]. The latter are motivated by the objective of achieving a better understanding of these diameters, in particular by relating them to each other through geometric inequalities. When considering the relations between different convex bodies, we use the following notation. For $K,C \in \mathscr{C}^n$ we say that K is *optimally contained* in C if $K \subset C$ and $R(K,C) = 1$, which is abbreviated by $K \subset^{\mathrm{opt}} C$. The four symmetrizations can be ordered, and the first and third containments are always optimal [6, 19].

PROPOSITION 1.1. *Let $C \in \mathscr{C}^n_0$. Then,*

$$C_{\mathrm{MIN}} \subset^{\mathrm{opt}} C_{\mathrm{HM}} \subset C_{\mathrm{AM}} \subset^{\mathrm{opt}} C_{\mathrm{MAX}}.$$

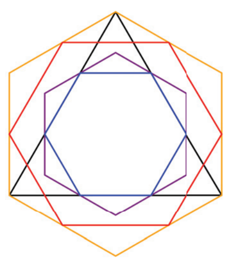

Figure 1: *The equilateral triangle (black) and its symmetrizations: minimum (blue), harmonic mean (purple), arithmetic mean (red), maximum (orange) (cf.* [6]*).*

In contrast to C_{AM}, the symmetrizations C_{MIN}, C_{HM}, and C_{MAX} depend decisively on the position of C and thus the according diameters do, too. However, all diameters and therefore the symmetrizations should coincide if C is symmetric. This is achieved if and only if, in those cases, C is 0-symmetric. Thus, it seems natural to consider in general only "somehow" centered gauges. We use the following measure of asymmetry and centering of the gauge. The *Minkowski asymmetry* of $C \in \mathscr{C}^n$ is defined as $s(C) := R(C,-C)$, and we say that C is *Minkowski-centered* if $C \subset^{\text{opt}} -s(C)C$. For $C \in \bar{\mathscr{C}}^n$, the range of the Minkowski asymmetry is $[1,n]$, where $s(C) = 1$ if and only if C is symmetric and $s(C) = n$ if and only if C is an n-dimensional simplex [24].

A first Jung-type inequality is that of Bohnenblust [2], comparing the circumradius and the diameter in general Minkowski spaces. A generalization allowing also non-symmetric gauges is shown in [12]. For $K \in \mathscr{C}^n$ and $C \in \mathscr{C}_0^n$, we have

$$D_{\text{AM}}(K,C) \geqslant \frac{2(s(K)+1)}{s(K)(s(C)+1)} R(K,C) \geqslant \frac{2}{n} R(K,C)$$

with equality from left to right if and only if C is an n-simplex and $K = -C$ [12].

The following theorems collect several Jung-type inequalities for the different diameters.

THEOREM 1.2. *Let $K \in \mathscr{C}^n$ and $C \in \mathscr{C}_0^n$. Then,*

$$D_{\text{MIN}}(K,C) \geqslant \frac{s(K)+1}{s(K)} R(K,C).$$

Using $s(K) \leqslant n$, the Jung-type inequality by Leichtweiss [33],

$$D_{\text{MIN}}(K,C) \geqslant \frac{n+1}{n} R(K,C),$$

can be obtained from Theorem 1.2. For D_{HM} and D_{MAX}, we obtain non-tight bounds using containment factors between the symmetrizations in the respective sections. For D_{MAX}, we provide a detailed analysis for the planar case.

THEOREM 1.3. *Let $K,C \in \mathscr{C}^2$.*

i) If C is Minkowski-centered and $\dim(C) = 2$,

$$D_{\text{MAX}}(K,C) \geqslant R(K,C).$$

ii) If $0 \in C$ *and* $\dim(C) = 2$,

$$D_{\text{MAX}}(K,C) \geqslant \frac{2}{3} R(K,C).$$

As part of our investigation, we also consider the following question: given a Minkowski-centered gauge $C \in \mathscr{C}_0^2$ and values (r,D,R), is there a convex body $K \in \mathscr{C}^2$ such that its inradius is r, its diameter is D, and its circumradius is R (all w. r. t. C)? This kind of question can be answered by giving a system of inequalities such that for every triple fulfilling these inequalities there exists such a convex body K. Such systems have been considered first by Blaschke for the volume, surface area, and mean width in 3-space [1] and by Santaló for some triples of functionals out of area, perimeter, circumradius, inradius, diameter, and width for the Euclidean planar case [38]. Later, systems for many triples from the list of Santaló could be completed [5, 15, 28, 29] and even four of these functionals [8, 39] have been taken into account. Other functionals, such as the first Dirichlet eigenvalue [21] or the Cheeger constant [20], have been included as well.

To describe the values the inradius, circumradius, and diameter may have when we consider a fixed, Minkowski-centered gauge, we study the following mappings.

DEFINITION 1.4. For $\mathrm{M} \in \{\mathrm{MIN}, \mathrm{HM}, \mathrm{AM}, \mathrm{MAX}\}$, define

$$f_{\mathrm{M}} : \overline{\mathscr{C}^n} \times \mathscr{C}_0^n \to \mathbb{R}^2, \quad f_{\mathrm{M}}(K,C) = \left(\frac{r(K,C)}{R(K,C)}, \frac{D_{\mathrm{M}}(K,C)}{2R(K,C)} \right).$$

As with the diameter we simply write f if C is 0-symmetric. The set $f_{\mathrm{M}}(\overline{\mathscr{C}^n},C)$ is called the (r,D_{M},R)-diagram – the *Blaschke-Santaló diagram* for the inradius, circumradius, and diameter (depending on the respective definitions) with regard to the gauge C.

In [9] a complete system of inequalities for the (r,D_{AM},R)-diagram with triangular gauge S is derived. It turns out that $f_{\mathrm{AM}}(\overline{\mathscr{C}^n},S) = f_{\mathrm{AM}}(\overline{\mathscr{C}^n},\mathscr{C}_0^n)$, which means that $f_{\mathrm{AM}}(\overline{\mathscr{C}^n},S)$ provides inequalities being valid for all possible gauges. Likewise, we describe the diagrams $f_{\mathrm{MIN}}(\overline{\mathscr{C}^n},S)$, $f_{\mathrm{HM}}(\overline{\mathscr{C}^n},S)$, and $f_{\mathrm{MAX}}(\overline{\mathscr{C}^n},S)$.

THEOREM 1.5. *For every Minkowski-centered triangle $S \in \mathscr{C}^2$, the diagram $f_{\mathrm{MAX}}(\overline{\mathscr{C}^2},S)$ is fully described by the inequalities*

$$\begin{aligned} D_{\mathrm{MAX}}(K,S) &\leqslant 2R(K,S) \\ 2r(K,S) &\leqslant D_{\mathrm{MAX}}(K,S) \\ 0 &\leqslant r(K,S) \\ R(K,S) &\leqslant D_{\mathrm{MAX}}(K,S) \\ \left(\frac{D_{\mathrm{MAX}}(K,S)}{R(K,S)} - \frac{1}{2}\right)\left(\frac{3}{2} - \frac{D_{\mathrm{MAX}}(K,S)}{R(K,S)}\right) &\leqslant \frac{r(K,S)}{R(K,S)}. \end{aligned}$$

As it turns out, the descriptions of the diagrams for D_{HM} and D_{MIN} follow from the ones of D_{MAX} and D_{AM}, respectively, by using containment factors between the symmetrizations of triangles.

COROLLARY 1.6. *For every Minkowski-centered triangle $S \in \mathscr{C}^2$, the diagram $f_{\mathrm{HM}}(\overline{\mathscr{C}^2},S)$ is fully described by the inequalities*

$$\begin{aligned} D_{\mathrm{HM}}(K,S) &\leqslant 3R(K,S), \\ 3r(K,S) &\leqslant D_{\mathrm{HM}}(K,S), \\ 0 &\leqslant r(K,S), \end{aligned}$$

$$R(K,S) \leqslant \frac{2}{3} D_{\mathrm{HM}}(K,S),$$
$$\frac{4}{9}\left(\frac{D_{\mathrm{HM}}(K,S)}{R(K,S)} - \frac{3}{4}\right)\left(\frac{9}{4} - \frac{D_{\mathrm{HM}}(K,S)}{R(K,S)}\right) \leqslant \frac{r(K,S)}{R(K,S)}.$$

COROLLARY 1.7. *For every Minkowski-centered triangle $S \in \mathscr{C}^2$, the diagram $f_{\mathrm{MIN}}(\overline{\mathscr{C}}^2, S)$ is fully described by the inequalities*

$$D_{\mathrm{MIN}}(K,S) \leqslant 3R(K,C),$$
$$2r(K,S) + R(K,S) \leqslant D_{\mathrm{MIN}}(K,S),$$
$$\frac{D_{\mathrm{MIN}}(K,C)}{3R(K,C)}\left(1 - \frac{D_{\mathrm{MIN}}(K,C)}{3R(K,C)}\right) \leqslant \frac{r(K,C)}{R(K,C)}.$$

The paper is organized as follows. Section 2 provides more details on the different diameter variants and Section 3 presents general properties of Blaschke-Santaló diagrams. Sections 4, 5, and 6 each deal with one of the three diameter variants. In Section 4, D_{MAX} is discussed and Theorems 1.3 and 1.5 are proven. In Section 5, we prove the results concerning D_{MIN}, including Theorem 1.2 and Corollary 1.7. Finally, Section 6 provides the results for D_{HM}, including Corollary 1.6.

2. Properties of the diameters

We give more detailed definitions which help to analyze the different diameters.

DEFINITION 2.1. Let $K \in \mathscr{C}^n$, $C \in \mathscr{C}^n_0$, and $a \in \mathbb{R}^n \setminus \{0\}$.

i) The a-length $l_{a,\mathrm{AM}}$ is defined as

$$l_{a,\mathrm{AM}}(K,C) := 2 \max_{x-y \in (K-K)\cap \mathrm{lin}(a)} R([x,y],C).$$

ii) The a-breadth $b_{a,\mathrm{AM}}$ is defined as

$$b_{a,\mathrm{AM}}(K,C) := 2 \cdot \frac{h_K(a) + h_K(-a)}{h_C(a) + h_C(-a)}.$$

In [12] the following properties of D_{AM}, which are well known for symmetric gauges (cf. [23]), are proven.

PROPOSITION 2.2. *Let $K \in \mathscr{C}^n$, $C \in \mathscr{C}^n_0$, and $a \in \mathbb{R}^n \setminus \{0\}$.*

i)
$$D_{\mathrm{AM}}(K,C) = \max_{a \in \mathbb{R}^n \setminus \{0\}} l_{a,\mathrm{AM}}(K,C).$$

ii)
$$D_{\mathrm{AM}}(K,C) = \max_{a \in \mathbb{R}^n \setminus \{0\}} b_{a,\mathrm{AM}}(K,C).$$

iii)

$$l_{a,\mathrm{AM}}(K,C) = l_{a,\mathrm{AM}}\left(K_{\mathrm{AM}},C\right) = l_{a,\mathrm{AM}}\left(K,C_{\mathrm{AM}}\right)$$
$$= l_{a,\mathrm{AM}}\left(K_{\mathrm{AM}},C_{\mathrm{AM}}\right).$$

iv)

$$b_{a,\mathrm{AM}}(K,C) = b_{a,\mathrm{AM}}\left(K_{\mathrm{AM}},C\right) = b_{a,\mathrm{AM}}\left(K,C_{\mathrm{AM}}\right)$$
$$= b_{a,\mathrm{AM}}\left(K_{\mathrm{AM}},C_{\mathrm{AM}}\right).$$

v)

$$D_{\mathrm{AM}}(K,C) = D_{\mathrm{AM}}\left(K_{\mathrm{AM}},C\right) = D\left(K,C_{\mathrm{AM}}\right)$$
$$= D\left(K_{\mathrm{AM}},C_{\mathrm{AM}}\right) = 2R\left(K_{\mathrm{AM}},C_{\mathrm{AM}}\right).$$

vi)

$$\min_{a\in\mathbb{R}^n\setminus\{0\}} l_{a,\mathrm{AM}}(K,C) = \min_{a\in\mathbb{R}^n\setminus\{0\}} b_{a,\mathrm{AM}}(K,C) = 2r\left(K_{\mathrm{AM}},C_{\mathrm{AM}}\right).$$

Next, we introduce a-lengths and a-breadths for the other diameters.

DEFINITION 2.3. Let $K \in \mathscr{C}^n$, $C \in \mathscr{C}^n_0$, and $a \in \mathbb{R}^n \setminus \{0\}$.

i) The asymmetric *a-length $l'_{a,\mathrm{MIN}}$* is defined as

$$l'_{a,\mathrm{MIN}}(K,C) := \max_{x-y\in(K-K)\cap\mathrm{pos}(a)} \|x-y\|_C.$$

The symmetric *a-length $l_{a,\mathrm{MIN}}$* is defined as

$$l_{a,\mathrm{MIN}}(K,C) := \max_{x-y\in(K-K)\cap\mathrm{lin}(a)} \|x-y\|_C.$$

ii) The *a-length $l_{a,\mathrm{HM}}$* is defined as

$$l_{a,\mathrm{HM}}(K,C) := \max_{x-y\in(K-K)\cap\mathrm{lin}(a)} \frac{1}{2}(\|x-y\|_C + \|x-y\|_{-C}).$$

iii) The *a-breadth $b_{a,\mathrm{MAX}}$* is defined as

$$b_{a,\mathrm{MAX}}(K,C) := \frac{h_K(a)+h_K(-a)}{\max(h_C(a),h_C(-a))}.$$

The following lemma states analogous results to Proposition 2.2 for the three other diameter definitions.

LEMMA 2.4. *Let $K \in \mathscr{C}^n$, $C \in \mathscr{C}^n_0$, and $a \in \mathbb{R}^n \setminus \{0\}$.*

i)

$$D_{\mathrm{MIN}}(K,C) = \max_{a \in \mathbb{R}^n \setminus \{0\}} l'_{a,\mathrm{MIN}}(K,C) = \max_{a \in \mathbb{R}^n \setminus \{0\}} l_{a,\mathrm{MIN}}(K,C),$$
$$D_{\mathrm{HM}}(K,C) = \max_{a \in \mathbb{R}^n \setminus \{0\}} l_{a,\mathrm{HM}}(K,C),$$
$$D_{\mathrm{MAX}}(K,C) = \max_{a \in \mathbb{R}^n \setminus \{0\}} b_{a,\mathrm{MAX}}(K,C).$$

ii) For $\mathrm{M} \in \{\mathrm{MIN}, \mathrm{HM}\}$,

$$l_{a,\mathrm{M}}(K,C) = \max_{x-y \in (K-K) \cap \mathrm{lin}(a)} \|x-y\|_{C_{\mathrm{M}}} = l_{a,\mathrm{M}}(K, C_{\mathrm{M}})$$
$$= l_{a,\mathrm{M}}(K_{\mathrm{AM}}, C_{\mathrm{M}}) = l_{a,\mathrm{M}}(K_{\mathrm{AM}}, C) = l_{a,\mathrm{AM}}(K, C_{\mathrm{M}}).$$

iii)

$$b_{a,\mathrm{MAX}}(K,C) = \frac{h_K(a) + h_K(-a)}{h_{\mathrm{conv}(C \cup (-C))}(a)} = b_{a,\mathrm{MAX}}(K, C_{\mathrm{MAX}})$$
$$= b_{a,\mathrm{AM}}(K, C_{\mathrm{MAX}}) = b_{a,\mathrm{MAX}}(K_{\mathrm{AM}}, C_{\mathrm{MAX}})) = b_{a,\mathrm{MAX}}(K_{\mathrm{AM}}, C).$$

iv) For $\mathrm{M} \in \{\mathrm{MIN}, \mathrm{HM}, \mathrm{MAX}\}$,

$$D_{\mathrm{M}}(K,C) = D(K, C_{\mathrm{M}}) = D(K_{\mathrm{AM}}, C_{\mathrm{M}}) = D_{\mathrm{M}}(K_{\mathrm{AM}}, C).$$

Proof.

i) Since we maximize over the a-lengths to obtain the diameter, both, $l'_{a,\mathrm{MIN}}$ and $l_{a,\mathrm{MIN}}$, lead to the same diameter:

$$D_{\mathrm{MIN}}(K,C) = \max_{a \in \mathbb{R}^n \setminus \{0\}} l_{a,\mathrm{MIN}}(K,C) = \max_{a \in \mathbb{R}^n \setminus \{0\}} l'_{a,\mathrm{MIN}}(K,C).$$

The remaining identities follow directly from the definitions of the diameters and a-lengths.

ii) Since $\|v\|_{C \cap -C} = \max(\|v\|_C, \|-v\|_C)$ for every $v \in \mathbb{R}^n$, we obtain

$$\max_{x-y \in (K-K) \cap \mathrm{pos}(a)} \max(\|x-y\|_C, \|y-x\|_C) = \max_{x-y \in (K-K) \cap \mathrm{lin}(a)} \|x-y\|_C$$
$$= \max_{x-y \in (K-K) \cap \mathrm{lin}(a)} \|x-y\|_{C \cap -C} = \max_{x-y \in (K-K) \cap \mathrm{lin}(a)} 2R([x,y], C \cap -C).$$

This shows the claim for $\mathrm{M} = \mathrm{MIN}$.

For the first equation in the case $\mathrm{M} = \mathrm{HM}$, we use the fact that $\frac{1}{2}(\|a\|_C + \|-a\|_C) = \|a\|_{\left(\frac{C^\circ - C^\circ}{2}\right)^\circ}$ for any $a \in \mathbb{R}^n$ and $C \in \mathscr{C}^n_0$. The remaining identities follow directly from the definitions of the diameters and a-lengths.

iii) The first equation follows from the fact that $\max(h_C(a), h_C(-a)) = h_{C_{\mathrm{MAX}}}(a)$. The remaining identities follow directly from the definitions of the diameters and a-breadths.

iv) This follows from applying parts ii) and iii) for the different diameter definitions. $\square$

For all diameter definitions, we say that $x, y \in K$ is a *diametric pair* if $D_{\mathrm{M}}(K,C) = D_{\mathrm{M}}([x,y],C)$.

REMARK 2.5. Width-definitions can be done analogously to those of the diameters, using the different definitions of the a-lengths and -breadths. For $g \in \{l, b\}$ such that $g_{a,\mathrm{M}}$ is defined, the corresponding *width* of $K \in \mathscr{C}^n$ with respect to $C \in \mathscr{C}_0^n$ is defined as

$$w_{\mathrm{M}}(K,C) := \min_{a \in \mathbb{R}^n \setminus \{0\}} g_{a,\mathrm{M}}(K,C).$$

A segment $L \subset K$ that attains $w_{\mathrm{M}}(L,C) = w(K,C)$ is called a *width chord* of K. Please note, in the standard case $\mathrm{M} = \mathrm{AM}$ it does not make a difference whether we minimize over the a-length or -breadth. Moreover, using Lemma 2.4 one could symmetrize the arguments of the width as well.

The following proposition is taken from [11]. It characterizes optimal containment. We use the following notation: We write $\mathrm{bd}(C)$ for the *boundary* of a convex body C. A *half-space* $H^{\leqslant}_{(a,\beta)} := \left\{x \in \mathbb{R}^n : a^\top x \leqslant \beta\right\}$ *supports* a convex body C in $p \in \mathrm{bd}(C)$ if $C \subset H^{\leqslant}_{(a,\beta)}$ and $a^\top p = \beta$. A point $p \in \mathrm{bd}(C)$ is *extreme* if $p \notin \mathrm{conv}(C \setminus \{p\})$. The set of extreme points in denoted by $\mathrm{ext}(C)$.

PROPOSITION 2.6. *Let $K, C \in \mathscr{C}^n$ and C be full-dimensional. Then $K \subset^{\mathrm{opt}} C$ if and only if*

i) $K \subset C$ and

ii) for some $k \in \{2, \ldots, n+1\}$, there exist touching points $p^1, \ldots, p^k \in \mathrm{ext}(K) \cap \mathrm{bd}(C)$ and half-spaces $H^{\leqslant}_{(a^i, h_C(a^i))}$ supporting C in p^i with affinely independent outer normals $a^i \in \mathbb{R}^n \setminus \{0\}$, $i \in \{1, \ldots, k\}$, such that $0 \in \mathrm{conv}(\{a^1, \ldots, a^k\})$.

In the following, we study properties of the diameters and how concepts such as completeness translate when using the different definitions.

REMARK 2.7. The inradius, circumradius and diameter are increasing and homogeneous of degree 1 in the first argument and decreasing and homogeneous of degree -1 in the second argument.

For convex combinations of the first argument we obtain the next lemma.

LEMMA 2.8. *Let* $K_1, K_2 \in \mathscr{C}^n$, $C \in \mathscr{C}^n_0$, *and* $\lambda \in [0,1]$.

i) If $c^1 + r_1 C \subset^{\text{opt}} K_1$ *and* $c^2 + r_2 C \subset^{\text{opt}} K_2$ *for some* $c^1, c^2 \in \mathbb{R}^n$ *and* $r_1, r_2 \geqslant 0$, *then*

$$r(\lambda K_1 + (1-\lambda)K_2, C) \geqslant \lambda r_1 + (1-\lambda) r_2.$$

Moreover, equality is attained if the optimal containment of the homothets of C *within* K_1 *and* K_2 *is characterizable by the touching points* $c^1 + r_1 p^i$ *and* $c^2 + r_2 p^i$, *with* $p^i \in \text{bd}(C)$, *and corresponding outer normals* a^i *for* $i \in \{1, \dots, k\}$, *for some* $k \in \{1, \dots, n+1\}$, *using Proposition 2.6.*

ii) If $K_1 \subset^{\text{opt}} c^1 + R_1 C$ *and* $K_2 \subset^{\text{opt}} c^2 + R_2 C$ *for some* $c^1, c^2 \in \mathbb{R}^n$ *and* $r_1, r_2 \geqslant 0$, *then*

$$R(\lambda K_1 + (1-\lambda)K_2, C) \leqslant \lambda R_1 + (1-\lambda) R_2.$$

Moreover, equality is attained if the optimal containment of the homothets of K_1 *and* K_2 *within* $R_1 C$ *and* $R_2 C$, *respectively, is characterizable by the touching points* $c^1 + R_1 p^i$ *and* $c^2 + R_2 p^i$, *with* $p^i \in \text{bd}(C)$, *with* $i \in \{1, \dots, k\}$, *for some* $k \in \{1, \dots, n+1\}$, *using Proposition 2.6.*

iii) For any $\mathrm{M} \in \{\mathrm{MIN}, \mathrm{HM}, \mathrm{AM}, \mathrm{MAX}\}$, *if* $D_1 = D_{\mathrm{M}}(K_1, C)$ *and* $D_2 = D_{\mathrm{M}}(K_2, C)$, *then*

$$D_{\mathrm{M}}(\lambda K_1 + (1-\lambda)K_2, C) \leqslant \lambda D_1 + (1-\lambda) D_2.$$

If the diameters are defined by the same a*-breadth or* a*-length, we have equality.*

Proof.

i) Obviously, $\lambda c^1 + (1-\lambda)c^2 + (\lambda r_1 + (1-\lambda) r_2) C \subset \lambda K_1 + (1-\lambda) K_2$. Let $c^1 + r_1 p^i$, $c^2 + r_2 p^i$, with $p^i \in \text{bd}(C)$, be the touching points and a^i the corresponding outer normals with $i \in \{1, \dots, k\}$ for some $k \in \{1, \dots, n+1\}$ as in Proposition 2.6. Then,

$$\begin{aligned} h_{\lambda K_1 + (1-\lambda) K_2}(a^i) &= \lambda (c^1)^\top a^i + \lambda r_1 (p^i)^\top a^i + (1-\lambda)(c^2)^\top a^i + (1-\lambda) r_2 (p^i)^\top a^i \\ &= h_{\lambda c^1 + (1-\lambda)c^2 + (\lambda r_1 + (1-\lambda) r_2) C}(a_i). \end{aligned}$$

Thus, we have touching points in the intersection of $\text{bd}(\lambda c^1 + (1-\lambda)c^2 + (\lambda r_1 + (1-\lambda) r_2) C)$ and $\text{bd}(\lambda K_1 + (1-\lambda) K_2)$ with corresponding outer normals a^i, $i \in \{1, \dots, k\}$, with 0 in their convex hull. This shows that we have optimal containment.

ii) The statement for the circumradius follows analogously. Here, the outer body is C in both cases, so we automatically have the same supporting hyperplanes.

iii) Since $D_{\mathrm{M}}(K, C) = 2R(K_{\mathrm{AM}}, C_{\mathrm{M}})$, the inequality follows from part *ii)*. If the diameters are attained by the same a-length, we have the same touching points in the containments $\frac{K_1 - K_1}{2} \subset^{\text{opt}} \frac{D_{\mathrm{M}}(K_1, C)}{2} C_{\mathrm{M}}$ and $\frac{K_2 - K_2}{2} \subset^{\text{opt}} \frac{D_{\mathrm{M}}(K_2, C)}{2} C_{\mathrm{M}}$ and thus the equality case follows again from part *ii*). If the diameter is attained by the same a-breadth, we have

$$\begin{aligned}&\lambda D_{\mathrm{M}}(K_1,C)+(1-\lambda)D_{\mathrm{M}}(K_2,C)=\lambda b_{a,\mathrm{M}}(K_1,C)+(1-\lambda)b_{a,\mathrm{M}}(K_2,C)\\ =&\lambda\frac{h_{K_1-K_1}(a)}{h_{C_{\mathrm{M}}}(a)}+(1-\lambda)\frac{h_{K_2-K_2}(a)}{h_{C_{\mathrm{M}}}(a)}=\frac{h_{(\lambda K_1+(1-\lambda)K_2)-(\lambda K_1+(1-\lambda)K_2)}(a)}{h_{C_{\mathrm{M}}}(a)}\\ =&b_{a,\mathrm{M}}(\lambda K_1+(1-\lambda)K_2,C)\leqslant D_{\mathrm{M}}(\lambda K_1+(1-\lambda)K_2,C)\end{aligned}$$

and equality follows. □

LEMMA 2.9. *Let $K\in\mathscr{C}^n$, $C\in\mathscr{C}_0^n$, and $a\in\mathbb{R}^n\setminus\{0\}$, and let $A:\mathbb{R}^n\to\mathbb{R}^n$ be a non-singular affine transformation and L_A its corresponding linear transformation. Then*

$$\begin{aligned}&R(A(K),A(C))=R(K,C)\\ &r(A(K),A(C))=r(K,C)\\ &D_{\mathrm{M}}(A(K),L_A(C))=D_{\mathrm{M}}(K,C).\end{aligned}$$

Proof. It follows from Proposition 2.6 that the in- and circumradius are invariant under affine transformations. All symmetrizations interchange with linear transformations: $(L_A(C))_{\mathrm{M}}=L_A(C_{\mathrm{M}})$ (see [6], Lemma 4). However, we need to restrict the transformation in the second argument of the diameter to be linear, since the symmetrizations are not invariant under translations if $\mathrm{M}\neq\mathrm{AM}$. Since $D_{\mathrm{M}}(K,C)=2R(K_{\mathrm{AM}},C_{\mathrm{M}})$, the invariance of the diameter then follows from that of the circumradius. The position of K does not change the diameter, and therefore we can apply a corresponding affine transformation to K. □

To analyze properties such as constant width and completeness we extend their definitions to the different diameters.

DEFINITION 2.10. Let $K,K^*\in\mathscr{C}^n$, $K^*\supset K$, $C\in\mathscr{C}_0^n$.

i) K is of *constant width* if $w_{\mathrm{M}}(K,C)=D_{\mathrm{M}}(K,C)$.

ii) K is *complete* if $D_{\mathrm{M}}(K',C)>D_{\mathrm{M}}(K,C)$ for all $K'\in\mathscr{C}^n$ such that $K'\supsetneq K$.

iii) K^* is a *completion* of K if K^* is complete and $D_{\mathrm{M}}(K^*,C)=D_{\mathrm{M}}(K,C)$.

REMARK 2.11. Since $D_{\mathrm{M}}(K,C)=D(K,C_{\mathrm{M}})$, we obtain the following from [17]:

i) K has constant width if and only if $K_{\mathrm{AM}}=\lambda C_{\mathrm{M}}$ for some $\lambda\in\mathbb{R}$.

ii) If K is of constant width, it is complete.

iii) In the planar case, K is complete if and only if K is of constant width.

LEMMA 2.12. *Let $K \in \bar{\mathscr{C}}^n$. If K^* is a completion of K, then $K \subset^{\mathrm{opt}} K^*$.*

Proof. By definition $K \subset K^*$ and $D_{\mathrm{M}}(K^*,C) = D_{\mathrm{M}}(K,C)$. Now, assume there exist $c \in \mathbb{R}^n$ and $0 \leqslant \rho < 1$ such that $K \subset^{\mathrm{opt}} c + \rho K^*$. Then, $D_{\mathrm{M}}(K,C) \leqslant D_{\mathrm{M}}(c+\rho K^*,C) < D_{\mathrm{M}}(K^*,C)$, a contradiction. □

REMARK 2.13. Let us observe two things:

i) Whenever the gauge is symmetric, the only (up to translation and dilatation) complete and symmetric set is the gauge itself. Thus, when considering D_{M} with respect to a possibly non-symmetric gauge C, the only complete and symmetric set is always C_{M}.

ii) In the case that $\mathrm{M} = \mathrm{AM}$, the gauge itself is always complete. This is not necessarily true when considering the other diameter definitions. In the following sections, we will characterize when the gauge is complete and what the completion looks like otherwise.

In the following the optimal containment factors between C_{AM} and C_{M} will prove helpful in the analysis of the different diameters.

NOTATION. Let $C \in \mathscr{C}_0^n$ be Minkowski-centered. By $\delta_{\mathrm{M}} := \delta_{\mathrm{M}}(C)$ and $\rho_{\mathrm{M}} := \rho_{\mathrm{M}}(C)$ we denote the dilatation factors needed, such that

$$\rho_{\mathrm{M}} C_{\mathrm{M}} \subset^{\mathrm{opt}} C_{\mathrm{AM}} \subset^{\mathrm{opt}} \delta_{\mathrm{M}} C_{\mathrm{M}}.$$

Since all symmetrizations are 0-symmetric and full-dimensional, these factors always exist. For better readability, we omit the argument C whenever it is clear from the context.

While some of the values ρ_{M} and δ_{M} depend at most on the asymmetry $s(C)$, some do not. The following proposition collects results from [6, 7, 19].

PROPOSITION 2.14. *Let $C \in \mathscr{C}_0^n$ be Minkowski-centered. Then,*

$$\rho_{\mathrm{MIN}} \in \left[1, \frac{s(C)+1}{2}\right], \qquad \rho_{\mathrm{HM}} \in \left[1, \frac{(s(C)+1)^2}{4s(C)}\right], \qquad \rho_{\mathrm{MAX}} = \frac{2s(C)}{s(C)+1},$$
$$\delta_{\mathrm{MIN}} = \frac{s(C)+1}{2}, \qquad \delta_{\mathrm{HM}} \in \left[\frac{(s(C)+1)^2}{4s(C)}, \frac{s(C)+1}{2}\right], \qquad \delta_{\mathrm{MAX}} = 1.$$

The factors also appear as the width and diameter of the gauge.

LEMMA 2.15. *Let $C \in \mathscr{C}_0^n$ be Minkowski-centered.*

i) $w_{\mathrm{M}}(C,C) = 2\rho_{\mathrm{M}}$ *and* $D_{\mathrm{M}}(C,C) = 2\delta_{\mathrm{M}}$.

ii) For any segment $L \subset^{\mathrm{opt}} C$

$$2\rho_{\mathrm{M}} \leqslant D_{\mathrm{M}}(L,C) \leqslant 2\delta_{\mathrm{M}}$$

with equality on the right-hand side if and only if L *is the convex hull of a diametric pair of points of* C *and equality on the left-hand side if and only if* L *is a width chord of* C*. All values in between are attained.*

Proof. Part i): By the definition of δ_M as well as the diameter properties collected in Proposition 2.2 and Lemma 2.4, we have

$$w_{\mathrm{M}}(C,C) = w(C_{\mathrm{AM}},C_{\mathrm{M}}) = 2r(C_{\mathrm{AM}},C_{\mathrm{M}}) = 2\rho_{\mathrm{M}}$$

as well as

$$D_{\mathrm{M}}(C,C) = D(C_{\mathrm{AM}},C_{\mathrm{M}}) = 2R(C_{\mathrm{AM}},C_{\mathrm{M}}) = 2\delta_{\mathrm{M}}.$$

Now, for part ii), let $L \subset^{\mathrm{opt}} C$ be a segment. From i), we see $D_{\mathrm{M}}(L,C) \leqslant D_{\mathrm{M}}(C,C) = 2\delta_{\mathrm{M}}$ and obviously equality is attained if and only if L is the convex hull of a diametric pair of points. Since $R(L,C) = R(L,C_{\mathrm{AM}}) = \frac{1}{2}D(L,C_{\mathrm{AM}})$ for segments L [12], we obtain for segments $L \subset^{\mathrm{opt}} C$

$$\begin{aligned} D_{\mathrm{M}}(L,C) = D(L,C_{\mathrm{M}}) &\leqslant D\left(L, \frac{1}{\rho_{\mathrm{M}}} C_{\mathrm{AM}}\right) \\ &= \rho_{\mathrm{M}} D(L,C_{\mathrm{AM}}) = 2\rho_{\mathrm{M}} R(L,C) = 2\rho_{\mathrm{M}}. \end{aligned}$$

If L is a width chord, we obtain $D_{\mathrm{M}}(L,C) = w_{\mathrm{M}}(C,C) = 2\rho_{\mathrm{M}}$ from part i). Finally, all values in between are attained, since the a-length and a-breadth are continuous as functions of a on $\mathbb{R}^n \setminus \{0\}$. $\square$

In the following, a segment $L \subset^{\mathrm{opt}} C$ is denoted by L_w if $D_{\mathrm{M}}(L,C) = 2\rho_{\mathrm{M}}$ and by L_D if $D_{\mathrm{M}}(L,C) = 2\delta_{\mathrm{M}}$.

The containment factors between the symmetrizations of the gauge can be used to improve this chain and to formulate new inequalities.

LEMMA 2.16. *Let* $K \in \mathscr{C}^n$ *and* $C \in \mathscr{C}^n_0$ *be Minkowski-centered.*

i) $\rho_{\mathrm{M}} D_{\mathrm{AM}}(K,C) \leqslant D_{\mathrm{M}}(K,C) \leqslant \delta_{\mathrm{M}} D_{\mathrm{AM}}(K,C)$,

ii) $\delta_{\mathrm{M}} r(K,C) \leqslant \dfrac{D_{\mathrm{M}}(K,C)}{2}$,

iii) $\dfrac{D_{\mathrm{M}}(K,C)}{2} \leqslant \delta_{\mathrm{M}} R(K,C)$,

iv) $\rho_{\mathrm{M}}(s(C)r(K,C) + R(K,C)) \leqslant (s(C)+1)\dfrac{D_{\mathrm{M}}(K,C)}{2}$, *and*

v) $r(K,C) + R(K,C) \leqslant R(C_{\mathrm{M}},C) D_{\mathrm{M}}(K,C)$.

Proof.

i) This is a direct consequence of Remark 2.7 and Proposition 1.1.

ii) Since $r(K,C)C$ is contained in some translation of K, we obtain from Lemma 2.15 *i)*
$$\delta_{\mathrm{M}} r(K,C) = \frac{1}{2} D_{\mathrm{M}}(C,C) r(K,C) = \frac{1}{2} D_{\mathrm{M}}(r(K,C)C,C) \leqslant \frac{D_{\mathrm{M}}(K,C)}{2}.$$

iii) This follows completely analogously to *ii)*.

iv) By [10, Theorem 1.1] we have
$$s(C)r(K,C) + R(K,C) \leqslant \frac{s(C)+1}{2} D_{\mathrm{AM}}(K,C)$$
and
$$(s(C)+1)\frac{D_{\mathrm{AM}}(K,C)}{2} \leqslant (s(C)+1)\frac{D_{\mathrm{M}}(K,C)}{2\rho_{\mathrm{M}}}$$
follows directly from part *i)*.

v) For the symmetrization C_{M} it holds $r(K,C_{\mathrm{M}}) + R(K,C_{\mathrm{M}}) \leqslant D(K,C_{\mathrm{M}})$. Thus,
$$D_{\mathrm{M}}(K,C) \geqslant r(K,C_{\mathrm{M}}) + R(K,C_{\mathrm{M}}) \geqslant \frac{1}{R(C_{\mathrm{M}},C)}\left(r(K,C) + R(K,C)\right). \qquad \square$$

DEFINITION 2.17. Let $K \in \mathscr{C}^n$ and $C \in \mathscr{C}^n_0$. The set
$$K^{\sup} = \bigcap_{x \in K} \left(x + D_{\mathrm{M}}(K,C)C_{\mathrm{M}}\right)$$
is called the *super-completion* of K.

Let us remark that Moreno and Schneider [37] call $K^{\sup}$ the *wide spherical hull*. It is shown in [17] for arbitrary Minkowski spaces (i.e. for 0-symmetric C) that a set K is complete with respect to a symmetric gauge C if and only if $K^{\sup} = K$ and in [36] that $K^{\sup}$ is the union of all completions of K. All this has so far only been considered for 0-symmetric C, but it is easy to see that it all directly transfers to the general case.

DEFINITION 2.18. Let $K \in \mathscr{C}^n$ be full-dimensional.

i) A boundary point of K is called *smooth* if the supporting half-space of K in this point is unique.

ii) A *supporting slab* of K is the intersection of two antipodal parallel supporting half-spaces of K.

iii) A supporting slab is *regular* if at least one of the bounding hyperplanes contains a smooth boundary point of K.

iv) We say that $a \in \mathbb{R}^n \setminus \{0\}$ defines a regular slab if there exists a supporting slab such that the defining half-spaces have outer normals $\pm a$.

It is easy to argue that a lower-dimensional convex body $K \in \bar{\mathscr{C}}^n$ is never complete. On the other hand, every full-dimensional convex body is the intersection of its regular slabs and completeness can be characterized by using these slabs [37, Theorem 1].

PROPOSITION 2.19. *Let $K \in \mathscr{C}^n$ be full-dimensional. Then the following are equivalent:*

i) K is complete.

ii) For every outer normal a, defining a regular supporting slab of K, we have

$$\frac{h_K(a)+h_{-K}(a)}{h_{C_{\mathrm{M}}}(a)} = D_{\mathrm{M}}(K,C).$$

REMARK 2.20. Defining $K_X := \bigcap_{x\in X}(x+D_{\mathrm{M}}(K,C)C_{\mathrm{M}})$ for some $X \subset K$, we obviously have $K^* \subset K^{\mathrm{sup}} \subset K_X$. As mentioned after the definition of the super-completion, $K^* = K^{\mathrm{sup}}$ implies uniqueness for the completion K^* of K. This means that describing properties for such subsets X that imply $K^* = K_X$ in the following implicitly guarantee uniqueness of the completion K^*.

LEMMA 2.21. *Let $C \in \mathscr{C}^n_0$, X be a closed subset of $K \in \mathscr{C}^n$, full-dimensional, and $K_X := \bigcap_{x\in X}(x+D_{\mathrm{M}}(K,C)C_{\mathrm{M}})$. Then the following are equivalent:*

i) K_X is a completion of K.

ii) For every $a \in \mathbb{R}^n \setminus \{0\}$ that defines a regular slab of C_{M} there exist $\tilde{a} \in \{a,-a\}$ and $p \in X$ such that $h_{K_X}(-\tilde{a}) = p^\top(-\tilde{a})$ and $h_{K_X}(\tilde{a}) = h_{p+D_{\mathrm{M}}(K,C)C_{\mathrm{M}}}(\tilde{a})$.

Proof. Let us abbreviate $D := D_{\mathrm{M}}(K,C)$ for the proof. "*ii*) $\Rightarrow$ *i*)": In [37] it is shown that $D(K_X,C_{\mathrm{M}}) = \max\{b_a(K_X,C_{\mathrm{M}}) : a \text{ defines a regular slab of } C_{\mathrm{M}}\}$. Using $\tilde{a}$ and p as described in *ii*), we have for any such a

$$\begin{aligned} b_a(K_X,C_{\mathrm{M}}) &= \frac{h_{K_X}(a)+h_{K_X}(-a)}{h_{C_{\mathrm{M}}}(a)} = \frac{h_{p+DC_{\mathrm{M}}}(\tilde{a})+p^\top(-\tilde{a})}{h_{C_{\mathrm{M}}}(\tilde{a})} \\ &= \frac{p^\top\tilde{a}+Dh_{C_{\mathrm{M}}}(\tilde{a})+p^\top(-\tilde{a})}{h_{C_{\mathrm{M}}}(\tilde{a})} = D. \end{aligned}$$

Thus, $D_{\mathrm{M}}(K_X,C) = D(K_X,C_{\mathrm{M}}) = D$.

To show completeness of K_X using Proposition 2.19, we need that all the regular slabs of K_X are of diametric breadth. However, by the construction of K_X, every a that defines a regular slab of K_X also defines a regular slab of C_{M}.

"*i*) $\Rightarrow$ *ii*)": Assume K_X is a completion of K and there exists some $a \in \mathbb{R}^n \setminus \{0\}$ that defines a regular slab such that there is no p as defined in *ii*). By the construction of K_X there exist $p^1, p^2 \in X$ such that $h_{K_X}(a) = (p^1)^\top a + h_{DC_M}(a)$ and $h_{K_X}(-a) = (p^2)^\top(-a) + h_{DC_M}(-a)$. By our assumption $(p^2)^\top a < (p^1)^\top a + h_{DC_M}(a)$, otherwise we could choose $p = p^2$. Then,

$$
\begin{aligned}
b_a(K_X, C_M) = \frac{h_{K_X}(a) + h_{K_X}(-a)}{h_{C_M}(a)} &= \frac{(p^1)^\top a + h_{DC_M}(a) + (p^2)^\top(-a) + h_{DC_M}(-a)}{h_{C_M}(a)} \\
&> \frac{(p^2)^\top a + (p^2)^\top(-a) + h_{DC_M}(-a)}{h_{C_M}(a)} = \frac{h_{DC_M}(-a)}{h_{C_M}(a)} = D = D_M(K, C),
\end{aligned}
$$

which contradicts K_X being a completion of K. □

Now, we consider the special case where the set X in Lemma 2.21 is a simplex.

DEFINITION 2.22. Let $C \in \mathscr{C}_0^n$. We say that a subset $X \subset K \in \mathscr{C}^n$ is a *diametric simplex* of K if

i) X is a simplex, and

ii) $D_M([x,y], C) = D_M(K, C)$ for all pairs of vertices x, y of X.

LEMMA 2.23. *i) Let X be a diametric simplex of $K \in \mathscr{C}^2$. Then, K_X is the unique completion of K.*

ii) Any triangle T for which $X = T$ is a diametric simplex of K has a unique completion.

Proof.

i) We show that property *ii*) of Lemma 2.21 is fulfilled. Assume w.l.o.g. that $D_M(X, C) = D_M(K, C) = 1$ and let $X = \operatorname{conv}(\{p^1, p^2, p^3\})$. Then, the translations $-p^i + X$ with $i \in \{1,2,3\}$ are subsets of C_M, all with one vertex in the

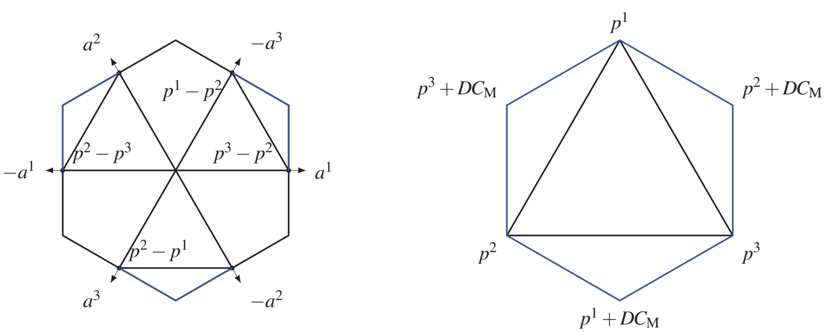

Figure 2: *If K contains a diametric simplex, its completion is constructed similar to the Reuleaux triangle in the Euclidean case. It suffices to consider the extreme points, i. e. the vertices of a diametric simplex.*

origin and the other two on the boundary of C_{M} (cf. Figure 2). We have three pairs of points $\pm(p^i - p^j)$ on the boundary of C_{M}. For each, we choose an outer normal a^k, $k \in \{1,2,3\}$ ordered as given in Figure 2. Now, the boundary of $K_X = K_{\mathrm{ext}(X)}$ consists of three parts which are built by parts of the boundary of C_{M} (colored in blue in the left part of Figure 2). Then, if $a \in \mathrm{pos}(\{a^i, -a^j\})$, $i \neq j$, property *ii*) of Lemma 2.21 is fulfilled for K_X with $p = p^k$, $k \notin \{i,j\}$. Hence, for all $a \in \mathbb{R}^n \setminus \{0\}$, property *ii*) is fulfilled, and it follows that K_X is a completion of K. Using Remark 2.20 we obtain the uniqueness.

ii) This follows directly from part *i*). □

In higher dimensions, diametric simplices do not necessarily have unique completions. This is well understood for the three-dimensional Euclidean case, where the Reuleaux tetrahedron is not of constant width. Examples of completions of the tetrahedron are, e. g. the two so-called Meissner bodies [34].

From the containment chain in Proposition 1.1 we know

$$D_{\mathrm{MAX}}(K,C) \leqslant D_{\mathrm{AM}}(K,C) \leqslant D_{\mathrm{HM}}(K,C) \leqslant D_{\mathrm{MIN}}(K,C).$$

3. Blaschke-Santaló diagrams

In this section, we show some general properties of the Blaschke-Santaló diagrams. It is shown in [9] that $f_{\mathrm{AM}}(\bar{\mathscr{C}}^n, C)$, independently of the choice of C, is star-shaped with respect to the vertex $f_{\mathrm{AM}}(C,C) = (1,1)$. This means that these diagrams can be fully described by characterizing the boundaries of the set. In the following, we prove similar (slightly weaker, but sufficient for our purposes) results for the other diameters. One may note that all diagrams with respect to triangles that are described in the following chapters are still star-shaped w.r.t. $f_{\mathrm{M}}(C,C)$.

LEMMA 3.1. *The diagram $f_{\mathrm{M}}(\bar{\mathscr{C}}^n, C)$ with respect to $C \in \mathscr{C}^n_0$ is closed and if there is a continuous description of the outer boundary, it is simply connected.*

Proof. Assume there is a sequence $(K_n)_{n\in\mathbb{N}} \subset \bar{\mathscr{C}}^n$ such that $K_n \subset^{\mathrm{opt}} C$ for all $n \in \mathbb{N}$ and $r(K_n, C) \to r^*$ and $D_{\mathrm{M}}(K_n, C) \to D^*$ for $n \to \infty$. The sequence $(K_n)_{n\in\mathbb{N}}$ is bounded as all sets are contained in C. Thus, by the Blaschke-Selection-Theorem there exists a converging subsequence $K_{n_k} \to K^*$ for $k \to \infty$. The inradius and diameter are continuous and therefore $r(K^*, C) = r^*$ and $D_{\mathrm{M}}(K^*, C) = D^*$. Hence, $f_{\mathrm{M}}(\bar{\mathscr{C}}^n, C)$ is closed.

As a consequence, we know that $f_{\mathrm{M}}(\bar{\mathscr{C}}^n, C)$ could at most have open holes and therefore only full-dimensional holes. For $K \in \bar{\mathscr{C}}^n$ such that $K \subset^{\mathrm{opt}} C$, define $K_t := (1-t)K + tC$ for $t \in [0,1]$.

Then by Lemma 2.8,

$$r(K_t, C) = (1-t)r(K,C) + t,$$

$$R(K_t,C) = (1-t)R(K,C) + t = 1,$$

and

$$D_{\mathrm{M}}(K_t,C) \leqslant (1-t)D_{\mathrm{M}}(K,C) + tD_{\mathrm{M}}(C,C).$$

In the case $\mathrm{M} = \mathrm{AM}$, we also have equality for the diameter, but this does not necessarily hold for the other variants. Since $R(\cdot,C), r(\cdot,C)$, and $D_{\mathrm{M}}(\cdot,C)$ are continuous with respect to the Hausdorff distance and $t \in [0,1] \mapsto (1-t)K + tC$ is continuous in t, the composition $\Gamma_K : [0,1] \to \mathbb{R}^2$, $t \mapsto \left(r(K_t,C), \frac{D_{\mathrm{M}}(K_t,C)}{2}\right)$ is continuous as well. Thus, for every such K there is a continuous curve Γ_K in the diagram from $f_{\mathrm{M}}(K,C)$ to $f_{\mathrm{M}}(C,C)$.

Let $(K^n)_{n\in\mathbb{N}}$ be a sequence of bodies mapped to the boundary of the diagram, converging to K, also mapped to the boundary. We show that the functions Γ_{K^n} converge uniformly to Γ_K. We can consider the components separately. For the inradius, we get

$$\begin{aligned}
|r(K_t,C) - r(K_t^n,C)| &= |(1-t)r(K,C) + t - (1-t)r(K^n,C) - t| \\
&= (1-t)|r(K,C) - r(K^n,C)| \\
&\leqslant |r(K,C) - r(K^n,C)|.
\end{aligned}$$

Let $\varepsilon > 0$. Since $|r(K,C) - r(K^n,C)| \to 0$ for $n \to \infty$, there exists an N such that for all $n \geqslant N$, $|r(K_t,C) - r(K_t^n,C)| < \varepsilon$ for all $t \in [0,1]$. It is known that when convex, continuous functions $f_n : [a,b] \to \mathbb{R}$ converge pointwise to a convex and continuous function f, the convergence is uniform [31, Lemma 21]. The functions $g_n : [0,1] \to \mathbb{R}$, $t \mapsto D_{\mathrm{M}}(K_t^n,C)$ are convex and continuous in t, and they converge pointwise to the convex and continuous function $g : [0,1] \to \mathbb{R}$, $t \mapsto D_{\mathrm{M}}(K_t,C)$. Thus, this convergence is also uniform, and the curves converge uniformly.

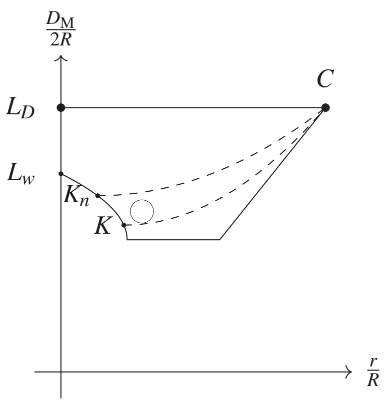

Figure 3: *Proof of Lemma 3.1: $K_n \to K$ but Γ_K lies below the hole and Γ_{K_n} above.*

Now, assume the diagram has a hole. We call the boundary of the smallest closed, simply connected set containing the diagram, the outer boundary of the diagram. For K

on the outer boundary of the diagram we say that the curve Γ_K lies *above* the hole, if the set enclosed by $[f_{\mathrm{M}}(L_D,C), f_{\mathrm{M}}(C,C)]$, Γ_K and the outer boundary between $f_{\mathrm{M}}(K,C)$ and $f_{\mathrm{M}}(L_D,C)$, which does not contain the segment $[f_{\mathrm{M}}(L_D,C), f_{\mathrm{M}}(C,C)]$ does not contain the hole. Analogously, we say that Γ_K lies *below* the hole if that set contains the hole. Thus, Γ_{L_D} lies above the hole and Γ_C below. Hence, there exists a converging sequence $(K_n)_{n\in\mathbb{N}}$ with $K_n \to K$ of bodies on the outer boundary such that all Γ_{K^n} lie above the hole and Γ_K below or vice versa (cf. Figure 3). This contradicts the fact that the curves converge uniformly. $\square$

In [9], $f_{\mathrm{AM}}(\overline{\mathscr{C}^2}, S)$, with S being a triangle, is described, and it is shown that this diagram is equal to the union of the diagrams over all possible gauges.

PROPOSITION 3.2. *For every triangle $S \in \mathscr{C}^2$, the diagram $f_{\mathrm{AM}}(\overline{\mathscr{C}^2}, S)$ is fully described by the inequalities*

$$D_{\mathrm{AM}}(K,S) \leqslant 2R(K,S)$$
$$4r(K,S) + 2R(K,S) \leqslant 3D_{\mathrm{AM}}(K,S)$$
$$\frac{D_{\mathrm{AM}}(K,S)}{2R(K,S)}\left(1 - \frac{D_{\mathrm{AM}}(K,S)}{2R(K,S)}\right) \leqslant \frac{r(K,S)}{R(K,S)}.$$

Moreover, $f_{\mathrm{AM}}(\overline{\mathscr{C}^2}, S) = f_{\mathrm{AM}}(\overline{\mathscr{C}^2}, \mathscr{C}_0^2)$.

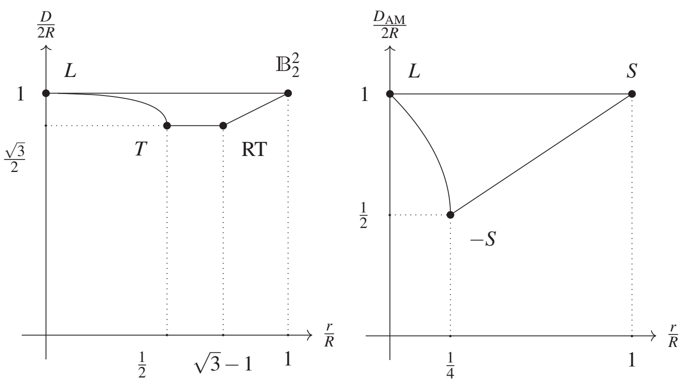

Figure 4: *The* (r,D,R)*-diagram w. r. t. the Euclidean disc* $\mathbb{B}_2^2$ [38] *(left) and the* (r,D_{AM},R)*-diagram w. r. t. a triangle* S [9] *(right).*

In the case of the standard diameter it was sufficient to describe the Blaschke-Santaló diagram with respect to a triangle to obtain $f_{\mathrm{AM}}(\overline{\mathscr{C}^2}, \mathscr{C}_0^2)$. Thus, it seems reasonable to look at the diagrams for the three other diameters D_{MIN}, D_{MAX} and D_{HM} in terms of triangular gauges first, which we do in the remaining sections.

4. The maximum diameter

When we use the notions "equilateral", "regular", and "isosceles" in the following, it is meant in the Euclidean sense. Unless otherwise specified, we fix the *equilateral triangle* to be $T := \operatorname{conv}\left(\{p^1, p^2, p^3\}\right) \subset \mathbb{R}^2$ with $p^1 = (0,1)^\top$, $p^2 = (-\sqrt{3}/2, -1/2)^\top$ and $p^3 = (\sqrt{3}/2, -1/2)^\top$. It is Minkowski-centered with $s(T) = 2$ and T_{MAX} is the regular hexagon $\operatorname{conv}\left(\{p^1, p^2, p^3, -p^1, -p^2, -p^3\}\right)$.

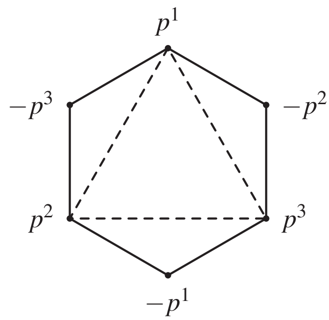

Figure 5: *The equilateral triangle T and its maximum T_{MAX}.*

We know from Proposition 2.14, that for the maximum, the factors ρ_{MAX} and δ_{MAX} depend only on the Minkowski asymmetry $s(C)$, namely, $\rho_{\mathrm{MAX}} = \frac{s(C)+1}{2s(C)}$ and $\delta_{\mathrm{MAX}} = 1$. Thus, for $K \in \mathscr{C}^n$ and $C \in \mathscr{C}^n_0$ Minkowski-centered, the inequalities from Lemma 2.16 have the form:

$$\frac{D_{\mathrm{MAX}}(K,C)}{2} \leqslant R(K,C), \tag{1}$$

$$r(K,C) \leqslant \frac{D_{\mathrm{MAX}}(K,C)}{2}, \tag{2}$$

$$s(C)r(K,C) + R(K,C) \leqslant s(C)D_{\mathrm{MAX}}(K,C), \tag{3}$$

and

$$0 \leqslant r(K,C). \tag{4}$$

We obtain $R(C_{\mathrm{MAX}},C) = s(C)$, $r(C_{\mathrm{MAX}},C) = 1$, and $D_{\mathrm{MAX}}(C_{\mathrm{MAX}},C) = 2$ from [7]. Inequalities (1) and (2) become equalities in the case of $K = C$ while (2) and (3) become equalities if $K = C_{\mathrm{MAX}}$.

It turns out that in the case of D_{MAX}, asymmetric gauges are not complete, but a completion is easy to find.

DEFINITION 4.1. Let $C \in \mathscr{C}^n_0$ and $A^{\mathrm{oss}}_C := \operatorname{bd}(C^\circ) \cap \operatorname{bd}(-C^\circ)$.
We define the *outer symmetric support*:

$$C^{\mathrm{oss}} := \bigcap_{a \in A^{\mathrm{oss}}_C} H^{\leqslant}_{(a,1)}.$$

LEMMA 4.2. *Let $C \in \mathscr{C}_0^n$ be Minkowski-centered.*

i) C_{MAX} *is always a completion of* C. *For every completion* C^* *of* C *we have* $R(C^*,C) \leqslant R(C_{\mathrm{MAX}},C) = s(C)$.

ii) $C_{\mathrm{MAX}} \subset C^{\mathrm{sup}} \subset C^{\mathrm{oss}}$.

iii) $C_{\mathrm{MAX}} = C^{\mathrm{sup}}$ *if and only if* $C_{\mathrm{MAX}} = C^{\mathrm{oss}}$.

iv) C_{MAX} *is always the unique 0-symmetric completion of* C.

One should recognize that we need a scaling factor of up to n to cover the completion C_{MAX} by C here, while with the arithmetic diameter C is always already complete itself.

Proof.

i) C_{MAX} is a completion of C, since $C \subset C_{\mathrm{MAX}}$ and

$$D_{\mathrm{MAX}}(C,C) = 2\delta_{\mathrm{MAX}} = 2 = D(C_{\mathrm{MAX}},C_{\mathrm{MAX}}) = D_{\mathrm{MAX}}(C_{\mathrm{MAX}},C),$$

while for all $K \supsetneq C_{\mathrm{MAX}}$ we have $K_{\mathrm{AM}} \supsetneq C_{\mathrm{MAX}}$ and

$$D_{\mathrm{MAX}}(K,C) = D(K_{\mathrm{AM}},C_{\mathrm{MAX}}) = 2R(K_{\mathrm{AM}},C_{\mathrm{MAX}}) > 2.$$

The maximality of the circumradius can be seen as follows: Let C^* be any completion of C. Then, from (3) and Lemma 2.12 we obtain

$$\begin{aligned} R(C^*,C) &\leqslant s(C)(D_{\mathrm{MAX}}(C^*,C) - r(C^*,C)) \\ &= s(C)(D_{\mathrm{MAX}}(C_{\mathrm{MAX}},C) - r(C_{\mathrm{MAX}},C)) = s(C) = R(C_{\mathrm{MAX}},C). \end{aligned}$$

ii) Next, we show that $C_{\mathrm{MAX}} \subset C^{\mathrm{sup}} \subset C^{\mathrm{oss}}$. The first containment follows from the fact that C^{sup} is the union of all completions of C. For the second containment it suffices to show that $h_{C^{\mathrm{sup}}}(a) \leqslant h_{C^{\mathrm{oss}}}(a)$ for all $a \in A_C^{\mathrm{oss}}$. For any $a \in A_C^{\mathrm{oss}}$ there exists $p \in -C \cap H_{(a,1)}$. Since $-p \in C$, we obtain $C^{\mathrm{sup}} \subset -p + 2C_{\mathrm{MAX}}$, and since $a \in \mathrm{bd}(C^\circ) \cap \mathrm{bd}(-C^\circ)$,

$$h_{C^{\mathrm{sup}}}(a) \leqslant h_{-p+2C_{\mathrm{MAX}}}(a) = -p^\top a + 2h_{C_{\mathrm{MAX}}}(a) = 1 = h_{C^{\mathrm{oss}}}(a).$$

iii) The backward direction directly follows from part *ii*). To show the forward direction, assume $C_{\mathrm{MAX}} \neq C^{\mathrm{oss}}$. Since C_{MAX} is the intersection of its regular slabs, there must exist some $a \in \mathrm{bd}(C^\circ)$ which defines a regular slab of C_{MAX} but $h_C(a) > h_C(-a)$. There exists a smooth boundary point x of C_{MAX} supported by the hyperplane $H_{(a,1)}$. Since $h_C(a) > h_C(-a)$, the point x must belong to $\mathrm{bd}(C) \cap H_{(a,1)}$. Now, let us assume that $x \in \mathrm{bd}(C^{\mathrm{sup}})$ as well. Then, there exist an outer normal $a_x \in \mathrm{bd}((C_{\mathrm{MAX}})^\circ)$ and a point $p_x \in C$ such that $a_x^\top x = h_{C^{\mathrm{sup}}}(a_x)$ and $(a_x)^\top(x - p_x) = 2$. This means that $H_{(a_x, h_{C^{\mathrm{sup}}}(a_x))}$ also supports C_{MAX}, which implies $a_X = a$, since x is a smooth boundary point of C_{MAX}. This contradicts $p_x \in C$, since $h_C(-a) < h_C(a)$. Thus, x is not contained in the boundary of C^{sup} and therefore, $C_{\mathrm{MAX}} \neq C^{\mathrm{sup}}$.

iv) Finally, any 0-symmetric completion of C must contain C and $-C$ and therefore C_{MAX}. □

EXAMPLE 4.3. Trapezoids within the following family have completions besides their maximum:

$$Z_\lambda := \operatorname{conv}\left(\left\{\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)^\top,\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)^\top,\left(\lambda\frac{\sqrt{3}}{2},1-\frac{\lambda}{2}\right)^\top,\left(-\lambda\frac{\sqrt{3}}{2},1-\frac{\lambda}{2}\right)^\top\right\}\right)$$

with $\lambda \in (0,1)$. Z_λ is Minkowski-centered with Minkowski asymmetry $2-\lambda$ and $(Z_\lambda)^{\mathrm{oss}} \neq (Z_\lambda)_{\mathrm{MAX}}$, which because of Lemma 4.2 means that $(Z_\lambda)_{\mathrm{MAX}}$ is not the unique completion of Z_λ. In the extreme cases $\lambda \in \{0,1\}$, Z_λ is a triangle or a rectangle and $(Z_\lambda)_{\mathrm{MAX}} = (Z_\lambda)^{\mathrm{sup}} = (Z_\lambda)^{\mathrm{oss}}$.

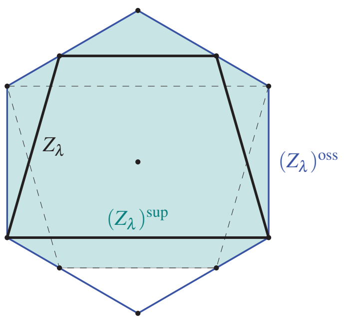

Figure 6: *For trapezoids* Z_λ, $\lambda \in (0,1)$, *the maximum* $(Z_\lambda)_{\mathrm{MAX}}$ *is not their unique completion, since* $(Z_\lambda)_{\mathrm{MAX}} \neq (Z_\lambda)^{\mathrm{oss}}$.

The first new inequality we provide is the one in Theorem 1.3 part i), a lower bound for the diameter-circumradius ratio for $C \in \mathscr{C}_0^2$ Minkowski-centered and $K \in \mathscr{C}^2$:

$$D_{\mathrm{MAX}}(K,C) \geqslant R(K,C).$$

Proof of Theorem 1.3 *i).* If K is a single point, $R(K,C) = D_{\mathrm{MAX}}(K,C) = 0$. Thus, we may assume $K \in \overline{\mathscr{C}}^2$ and $K \subset^{\mathrm{opt}} C$, which implies $R(K,C) = 1$. Then, there exist touching points $q^1,\dots,q^k \in \mathrm{bd}(K) \cap \mathrm{bd}(C)$ with $k \in \{2,3\}$ and corresponding outer normals a^i as described in Proposition 2.6.

If $k = 2$ is possible, there exists a segment $L \subset K$ with the same circumradius as K and by Lemma 2.15 $D_{\mathrm{MAX}}(K,C) \geqslant D_{\mathrm{MAX}}(L,C) \geqslant 2\rho_{\mathrm{MAX}}(C) = \frac{s(C)+1}{s(C)} > 1 = R(K,C)$.

For $k = 3$, the triangle $\operatorname{conv}(\{q^1,q^2,q^3\})$ has the same circumradius as K and $D_{\mathrm{MAX}}(K,C) \geqslant D_{\mathrm{MAX}}(\operatorname{conv}(\{q^1,q^2,q^3\}),C)$. Thus, it suffices to prove the claim in the case that $K = \operatorname{conv}(\{q^1,q^2,q^3\})$ is a proper triangle.

Let $S := \bigcap_{k=1}^3 H^{\leqslant}_{(a^k,1)}$ be the intersection of the three supporting half-spaces of C s. t. $q^k \in H_{(a^k,1)}$. Denote the vertex opposite to the edge defined by a^k by $\tilde{p}^k$. Then,

$R(K,S) = R(K,C) = 1$ and $D_{\mathrm{MAX}}(K,S) \leqslant D_{\mathrm{MAX}}(K,C)$. Due to invariance under linear transformations we may assume that $S = c + T$ where T is the Minkowski-centered equilateral triangle as defined at the beginning of the section. Hence, $\tilde{p}^k = c + p^k$, $k = 1,2,3$. In the following, indices are to be understood modulo 3. Let $\alpha_k \in [0,1]$ be s. t. $q^k = \alpha_k \tilde{p}^{k+1} + (1-\alpha_k)\tilde{p}^{k+2}$. Since C is Minkowski-centered, 0 lies in the interior of C.

We split the proof into two parts. First, we consider the case where the origin is close to the center c, i. e. $0 \in \operatorname{int}\left(\operatorname{conv}\left(\left\{c - \frac{1}{2}p^k,\ k = 1,2,3\right\}\right)\right)$. Afterward, we care about the case, where c is further apart from the origin.

Let us start with the case where $0 \in \operatorname{int}\left(\operatorname{conv}\left(\left\{c - \frac{1}{2}p^k, k = 1,2,3\right\}\right)\right)$, which is equivalent to $c \in \operatorname{int}\left(\operatorname{conv}\left(\left\{\frac{1}{2}p^k, k = 1,2,3\right\}\right)\right)$. Define $\lambda_1, \lambda_2, \lambda_3 > 0$ with $\sum_{k=1}^3 \lambda_k = \frac{1}{2}$ such that $c = \sum_{k=1}^3 \lambda_k p^k$. Let $z^i \in \mathbb{R}^2$ be defined as the direction such that

$$(z^i)^\top \tilde{p}^{i+1} = -1 \quad \text{and} \quad (z^i)^\top \tilde{p}^{i+2} = 1 \quad \text{and} \quad i \in \{1,2,3\}, \tag{5}$$

see Figure 7. This is possible as $0 \in \operatorname{int}(S)$. Since $c = \frac{1}{3}\sum_{k=1}^3 \tilde{p}^k$, we have

$$(z^i)^\top \tilde{p}^i = 3(z^i)^\top c. \tag{6}$$

Inserting $c = \sum_{k=1}^3 \lambda_k p^k$ yields

$$(z^i)^\top c = (z^i)^\top \sum_{k=1}^3 \lambda_k(\tilde{p}^k - c) = 3\lambda_i (z^i)^\top c - \lambda_{i+1} + \lambda_{i+2} - \frac{1}{2}(z^i)^\top c,$$

which implies

$$3(z^i)^\top c = \frac{\lambda_{i+2} - \lambda_{i+1}}{\frac{1}{2} - \lambda_i} = \frac{\lambda_{i+2} - \lambda_{i+1}}{\lambda_{i+2} + \lambda_{i+1}}.$$

Thus,

$$1 + 3(z^i)^\top c = \frac{2\lambda_{i+2}}{\lambda_{i+2} + \lambda_{i+1}} \geqslant 0 \quad \text{and} \quad 1 - 3(z^i)^\top c = \frac{2\lambda_{i+1}}{\lambda_{i+2} + \lambda_{i+1}} \geqslant 0 \tag{7}$$

and therefore by (6)

$$(\pm z^i)^\top \tilde{p}^i \leqslant 1,$$

which shows that z^i is an outer normal of $\operatorname{conv}(S \cup (-S))$ with $h_{\operatorname{conv}(S\cup(-S))}(z^i) = 1$.

It follows, using (5), (6) and (7), that

$$\begin{aligned}
D_{\mathrm{MAX}}(K,S) &\geqslant \max_{i=1,2,3} b_{z^i}(K, S_{\mathrm{MAX}}) \\
&\geqslant \max_{i=1,2,3} \frac{(z^i)^\top q^{i+1} - (z^i)^\top q^{i+2}}{h_{\operatorname{conv}(S\cup(-S))}(z^i)} \\
&= \max_{i=1,2,3} (z^i)^\top(\alpha_{i+1}\tilde{p}^{i+2} + (1-\alpha_{i+1})\tilde{p}^i) - (z^i)^\top(\alpha_{i+2}\tilde{p}^i + (1-\alpha_{i+2})\tilde{p}^{i+1}) \\
&= \max_{i=1,2,3} \alpha_{i+1}(1 - 3(z^i)^\top c) + (1-\alpha_{i+2})(1 + 3(z^i)^\top c) \\
&= \max_{i=1,2,3} \frac{2\alpha_{i+1}\lambda_{i+1} + 2(1-\alpha_{i+2})\lambda_{i+2}}{\lambda_{i+2} + \lambda_{i+1}}.
\end{aligned} \tag{8}$$

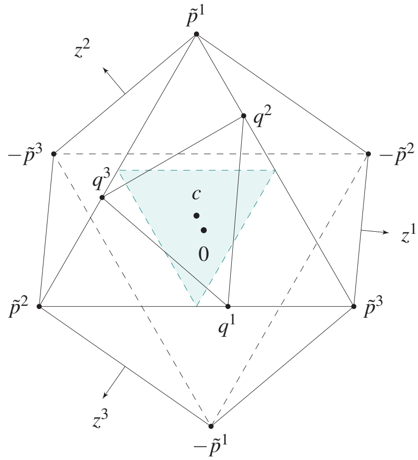

Figure 7: *Proof of Theorem* 1.3 *i*). *If* $0 \in \operatorname{int}\left(\operatorname{conv}\left(\left\{c - \frac{1}{2}p^i,\ i = 1,2,3\right\}\right)\right)$, *then* $\max_{i=1,2,3} b_{z^i}(K, S \cup (-S)) \geqslant 1$.

We show that if the last expression is smaller than 1 for $i = 1,2$, then its is at least 1 for $i = 3$. Thus, let us assume

$$\begin{aligned} &2\alpha_2\lambda_2 + 2(1-\alpha_3)\lambda_3 < \lambda_2 + \lambda_3 \quad \text{and} \\ &2\alpha_3\lambda_3 + 2(1-\alpha_1)\lambda_1 < \lambda_3 + \lambda_1. \end{aligned}$$

By adding these inequalities we obtain

$$2\lambda_3 + 2\alpha_2\lambda_2 + 2(1-\alpha_1)\lambda_1 < \lambda_2 + 2\lambda_3 + \lambda_1$$

or equivalently

$$2\alpha_1\lambda_1 + 2(1-\alpha_2)\lambda_2 > \lambda_1 + \lambda_2,$$

which proves $\max_{i=1,2,3} \frac{2\alpha_{i+1}\lambda_{i+1}+2(1-\alpha_{i+2})\lambda_{i+2}}{\lambda_{i+2}+\lambda_{i+1}} \geqslant 1$. Thus, $D_{\mathrm{MAX}}(K,C) \geqslant 1$.

Now, consider the case that $0 \notin \operatorname{int}\left(\operatorname{conv}\left(\left\{c - \frac{1}{2}p^k, k = 1,2,3\right\}\right)\right)$. Due to the symmetries of T we may assume that $0 = \sum_{k=1}^{3} \beta_i \tilde{p}^k$ with $\sum_{k=1}^{3} \beta_k = 1$, $\beta_3 \geqslant \frac{1}{2}$, and $\beta_2 \geqslant \beta_1 > 0$ (cf. Figure 8). Then,

$$(-a^k)^\top \tilde{p}^k = \frac{\beta_{k+1} + \beta_{k+2}}{\beta_k} \quad \text{for } k \in \{1,2,3\} \tag{9}$$

and therefore

$$h_C(-a^3) \leqslant h_S(-a^3) = (-a^3)^\top \tilde{p}^3 = \frac{\beta_1 + \beta_2}{\beta_3} \leqslant 1 = h_C(a^3). \tag{10}$$

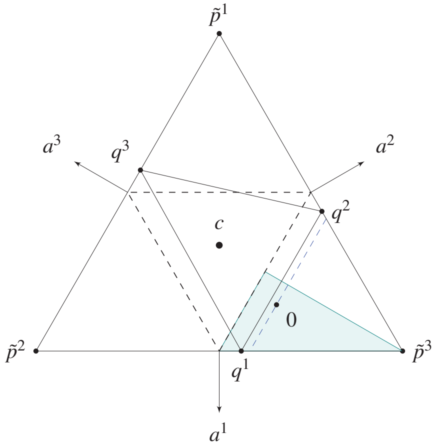

Figure 8: *Proof of Theorem 1.3 i). If* $0 \notin \operatorname{int}\left(\operatorname{conv}\left(\left\{c - \frac{1}{2}p^k, k = 1,2,3\right\}\right)\right)$*, we may assume, w. l. o. g., that the origin is within the colored area. Then,* $\max_{l=1,3} b_{a^k}(K, \operatorname{conv}(C \cup (-C))) \geqslant 1$.

Thus, using $\alpha_1, \alpha_2, \alpha_3$ as above, we know, if $\alpha_1 < 1 - \beta_3$ or $\alpha_2 > \beta_3$, then

$$\begin{aligned} h_K(-a^3) &\geqslant \max\left\{(-a^3)^\top q^1, (-a^3)^\top q^2\right\} \\ &= \max\left\{(-a^3)^\top(\alpha_1 \tilde{p}^2 + (1-\alpha_1)\tilde{p}^3), (-a^3)^\top(\alpha_2 \tilde{p}^3 + (1-\alpha_2)\tilde{p}^1)\right\} \\ &= \max\left\{-\alpha_1 + (1-\alpha_1)\frac{1-\beta_3}{\beta_3}, \alpha_2 \frac{1-\beta_3}{\beta_3} - (1-\alpha_2)\right\} > 0 \end{aligned}$$

and therefore by (10),

$$D_{\mathrm{MAX}}(K,C) \geqslant b_{a^3}(K, C_{\mathrm{MAX}}) \geqslant \frac{h_K(a^3) + h_K(-a^3)}{h_C(a^3)} > 1.$$

Now, let $\alpha_1 \geqslant 1 - \beta_3$ and $\alpha_2 \leqslant \beta_3$. Since $\beta_1 \leqslant \beta_2$, we have $\beta_1 \leqslant \frac{1-\beta_3}{2}$. From (9), we obtain

$$\begin{aligned} (-a^1)^\top q^2 &= \alpha_2((-a^1)^\top \tilde{p}^3) + (1-\alpha_2)((-a^1)^\top \tilde{p}^1) = -\alpha_2 + (1-\alpha_2)\left(\frac{\beta_2+\beta_3}{\beta_1}\right) \\ &\geqslant -\beta_3 + (1-\beta_3)\left(1 + \frac{2\beta_3}{1-\beta_3}\right) = 1. \end{aligned}$$

Using that C is Minkowski-centered, we see that $h_C(-a^1) \leqslant s(C) h_C(a^1) = s(C)$ and

$$\begin{aligned} D_{\mathrm{MAX}}(K,C) \geqslant b_{a^1}(K, C_{\mathrm{MAX}}) &\geqslant \frac{(a^1)^\top q^1 - (a^1)^\top q^2}{\max\{h_C(a^1), h_C(-a^1)\}} \\ &\geqslant \frac{1+1}{s(C)} = \frac{2}{s(C)} \geqslant 1. \quad \square \end{aligned}$$

Since D_{MAX} is the smallest diameter, Theorem 1.3 $i)$ also provides a lower bound for all four diameters.

COROLLARY 4.4. *Let $K,C \in \mathscr{C}^2$, C Minkowski-centered. Then*

$$D_{\mathrm{M}}(K,C) \geqslant R(K,C).$$

If $\mathrm{M} \neq \mathrm{MAX}$, it follows directly from Proposition 1.1 and the translation invariance in the arithmetic case that Corollary 4.4 stays true for non-Minkowski-centered $C \in \mathscr{C}^2$. However, $D_{\mathrm{M}}(K,C) = R(K,C)$ for $\mathrm{M} = \mathrm{AM}$ is obtained if and only if C is a triangle and K is a homothet of $-C$ [10, 12], and we will see in the next section that this bound cannot be reached if $\mathrm{M} \in \{\mathrm{MIN}, \mathrm{HM}\}$.

If we omit the restriction of C being Minkowski-centered, Theorem 1.3 $i)$ is not necessarily true. However, for planar gauges containing the origin, we still obtain a weaker Jung-type inequality:

$$D_{\mathrm{MAX}}(K,C) \geqslant \frac{2}{3} R(K,C).$$

Proof of Theorem 1.3 ii). We use the same notation as in the proof of Theorem 1.3 part $i)$ and assume again that K is a triangle with $K \subset^{\mathrm{opt}} C$. Since $D_{\mathrm{MAX}}(K,C) = D(K,C_{\mathrm{MAX}}) \geqslant D(K,S_{\mathrm{MAX}})$ for S as given in the proof of part $i)$, it suffices to show $D(K,S_{\mathrm{MAX}}) \geqslant \frac{2}{3}$.

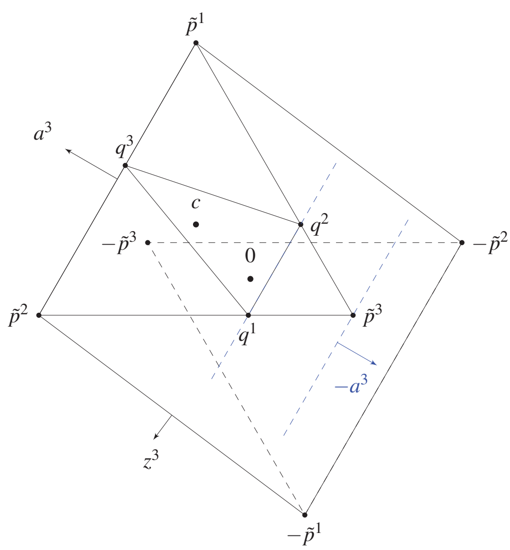

Figure 9: *Proof of Theorem 1.3. If $\alpha_1 < \frac{1}{3}$ or $\alpha_2 > \frac{2}{3}$, we have $b_{a^3}(K, \mathrm{conv}(S \cup (-S))) > \frac{2}{3}$.*

If there exists $\alpha_i \notin \left[\frac{1}{3}, \frac{2}{3}\right]$, there is an a^j with $\frac{a^{j\top} q^j - a^{j\top} q^i}{h_C(a^j) + h_C(-a^j)} > \frac{2}{3}$ (cf. Figure 9) and therefore,

$$
\begin{aligned}
D(K,S_{\mathrm{MAX}}) \geqslant b_{a^j}(K,S_{\mathrm{MAX}}) &\geqslant \frac{{a^j}^\top q^j - {a^j}^\top q^i}{\max\{h_C(a^j),h_C(-a^j)\}} \\
&\geqslant \frac{{a^j}^\top q^j - {a^j}^\top q^i}{h_C(a^j)+h_C(-a^j)} > \frac{2}{3}.
\end{aligned} \tag{11}
$$

Now, consider the case where $\alpha_i \in \left[\frac{1}{3},\frac{2}{3}\right]$ for all $i \in \{1,2,3\}$. There exists $i \in \{1,2,3\}$ such that z^i, defined as in the previous proof, defines a supporting hyperplane of S_{MAX} with $h_{S_{\mathrm{MAX}}}(z^i) = 1$, w.l.o.g. $i = 3$. Thus, $1 = h_{S_{\mathrm{MAX}}}(z^3) \geqslant (z^3)^\top(\pm p^3) = \pm 3(z^3)^\top c$ and using (8) we obtain

$$
\begin{aligned}
D(K,\operatorname{conv}(S\cup(-S))) &\geqslant b_{z^3}(K,S_{\mathrm{MAX}}) \\
&\geqslant \alpha_1(1-3(z^3)^\top c) + (1-\alpha_2)(1+3(z^3)^\top c) \\
&\geqslant \frac{1}{3}(1-3(z^3)^\top c) + \frac{1}{3}(1+3(z^3)^\top c) = \frac{2}{3}. \quad \square
\end{aligned} \tag{12}
$$

REMARK 4.5. Equality can be attained for some K if and only if C is a triangle with one vertex at the origin. To obtain equality in the last inequality chain (12) we need $\alpha_1 = 1-\alpha_2 = \frac{1}{3}$. Now, we see from (11) that $b_{a^3}(K,S_{\mathrm{MAX}}) = \frac{2}{3}$ if and only if $0 \in \left\{\tilde{p}^3\right\} \cup [\tilde{p}^1,\tilde{p}^2]$. However, z^3 cannot define a supporting hyperplane as described above if $0 \in [\tilde{p}^1,\tilde{p}^2]$. Thus, 0 remains to be a vertex of S to reach equality. Moreover, for $D_{\mathrm{MAX}}(K,C) = \frac{2}{3}$ to be true,

$$
b_{z^3}(K,C_{\mathrm{MAX}}) = b_{z^3}(K,S_{\mathrm{MAX}}) = \frac{2}{3}
$$

is necessary which is only possible if $\tilde{p}^1 \in C$ or $\tilde{p}^2 \in C$. Assuming only one of these vertices to be contained in C, it would follow that $\frac{3}{2}(q^2-q^1) \notin C_{\mathrm{MAX}}$ implying $D_{\mathrm{MAX}}(K,C) > \frac{2}{3}$. Thus, $\tilde{p}^1,\tilde{p}^2,\tilde{p}^3 \in C$, which means $C = S$.

In total, we see that equality can be obtained if and only if C is a triangle with a vertex (say $\tilde{p}^3$) at the origin and $K = \operatorname{conv}\left(\left\{q^1,q^2,q^3\right\}\right)$ with $\alpha_1 = (1-\alpha_2) = \frac{1}{3}$ and $\alpha_3 \in \left[\frac{1}{3},\frac{2}{3}\right]$, e.g. $K = \operatorname{conv}\left(\left\{\frac{1}{3}\tilde{p}^1, \frac{1}{3}\tilde{p}^2, \frac{1}{2}(\tilde{p}^1+\tilde{p}^2)\right\}\right)$. In that case

$$
D_{\mathrm{MAX}}(K,C) = b_{a^3}(K,C_{\mathrm{MAX}}) = b_{z^3}(K,C_{\mathrm{MAX}}) = \frac{2}{3}
$$

(cf. Figure 10).

For fixed C, we refer to any $\tilde{K}$ that fulfills $\frac{D_{\mathrm{M}}(\tilde{K},C)}{R(\tilde{K},C)} = \min_{K\in\mathscr{C}^n} \frac{D_{\mathrm{M}}(K,C)}{R(K,C)}$ as *Jung-extremal*. If $C = T$, we have equality in Theorem 1.3 *ii)* for the following family of triangles.

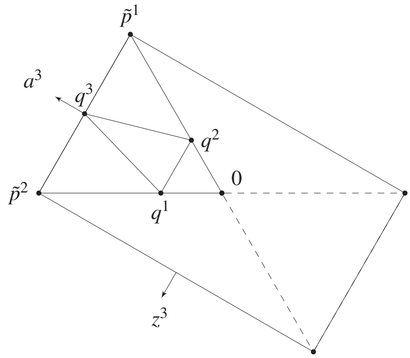

Figure 10: *The equality case in Theorem 1.3:* $C = \operatorname{conv}(\{0, \tilde{p}^1, \tilde{p}^2\})$ *and* $K = \operatorname{conv}\left(\left\{\frac{1}{3}\tilde{p}^1, \frac{1}{3}\tilde{p}^2, \frac{1}{2}(\tilde{p}^1 + \tilde{p}^2)\right\}\right)$.

EXAMPLE 4.6. Let T be the equilateral triangle as defined at the top of the section and

$$T_\alpha := \operatorname{conv}(\{\alpha p^1 + (1-\alpha)p^2, \alpha p^2 + (1-\alpha)p^3, \alpha p^3 + (1-\alpha)p^1\})$$

for $\alpha \in \left[\frac{1}{3}, \frac{2}{3}\right]$ (cf. Figure 11). Then, all T_α are dilated rotations of T (and therefore equilaterals) with

$$D_{\mathrm{MAX}}(T_\alpha, T) = R(T_\alpha, T) \quad \text{and} \quad r(T_\alpha, T) = (1 - 3\alpha + 3\alpha^2)R(T_\alpha, T).$$

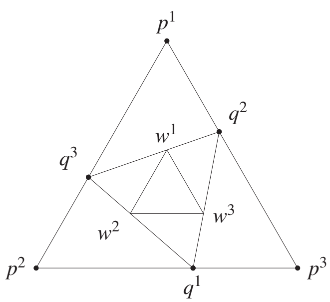

Figure 11: *All the equilateral triangles* T_α *are Jung-extremal w. r. t.* T.

By construction $T_\alpha \subset^{\mathrm{opt}} T$. The diameter of T_α is attained between two of its vertices and since T_{MAX} is a regular hexagon which has rotational symmetry of order six, it does not matter which of the edges of T_α we consider. Let $q^i = \alpha p^{i+1} + (1-$

$\alpha)p^{i+2}$. By using $p^3 = -p^2 - p^1$ we obtain

$$\begin{aligned} q^2 - q^1 &= \alpha p^3 + (1-\alpha)p^1 - \alpha p^2 - (1-\alpha)p^3 \\ &= (2-3\alpha)p^1 + (3\alpha-1)(-p^2) \\ &\subset [p^1, -p^2] \subset \mathrm{bd}\,(T_{\mathrm{MAX}}). \end{aligned}$$

Hence, $D_{\mathrm{MAX}}(T_\alpha, T) = \|q^2 - q^1\|_{T_{\mathrm{MAX}}} = 1$.

The triangle $\mathrm{conv}\left(\left\{w^1, w^2, w^3\right\}\right)$ with $w^i := \alpha q^{i+2} + (1-\alpha)q^{i+1}$ is optimally contained in T_α by Proposition 2.6. Furthermore,

$$\begin{aligned} w^i &= \alpha q^{i+2} + (1-\alpha)q^{i+1} \\ &= \alpha\left(\alpha p^i + (1-\alpha)p^{i+1}\right) + (1-\alpha)\left(\alpha p^{i+2} + (1-\alpha)p^i\right) \\ &= (1-2\alpha+2\alpha^2)p^i + \alpha(1-\alpha)p^{i+1} + \alpha(1-\alpha)p^{i+2} \end{aligned}$$

and therefore

$$\begin{aligned} w^{i+1} - w^i &= (1-2\alpha+2\alpha^2)p^{i+1} + \alpha(1-\alpha)p^i - (1-2\alpha+2\alpha^2)p^i - \alpha(1-\alpha)p^{i+1} \\ &= (1-3\alpha+3\alpha^2)(p^{i+1} - p^i). \end{aligned}$$

Thus, $\mathrm{conv}\left(\left\{w^1, w^2, w^3\right\}\right)$ is a translation of $(1-3\alpha+3\alpha^2)T$ implying $r(T_\alpha, T) = 1 - 3\alpha + 3\alpha^2$.

Special cases are $T_{\frac{1}{2}} = -\frac{1}{2}T$ (rotated by $\frac{\pi}{3}$), and $T_{\frac{1}{3}}, T_{\frac{2}{3}}$, rotations of dilated T by $\frac{\pi}{6}$ and $\frac{\pi}{2}$, respectively. The latter are especially interesting: they are complete since their arithmetic mean is a dilatation of T_{MAX}. Since the symmetrization T_{MAX} of T is also Jung-extremal, there are at least two complete Jung-extremal bodies, T_{MAX} and $T_{\frac{2}{3}}$, with different inradius-circumradius ratios.

Now, we compute the values of the functionals for a second family of triangles for which we will later see that they are mapped to the boundary of the (r, D_{MAX}, R)-diagram with respect to triangular gauges.

LEMMA 4.7. *Let T be the equilateral triangle as defined at the beginning of the section and $S_\lambda := \mathrm{conv}\left(\left\{q^1, q^2, q^3\right\}\right)$ with $q^1 := \frac{1}{2}(p^2 + p^3)$, $q^2 := \lambda p^1 + (1-\lambda)p^3$, and $q^3 := \lambda p^1 + (1-\lambda)p^2$ for some $\lambda \in [\frac{1}{2}, 1]$. Then, $R(S_\lambda, T) = 1$, $D_{\mathrm{MAX}}(S_\lambda, T) = \lambda + \frac{1}{2}$ and $r(S_\lambda, T) = \lambda(1-\lambda)$.*

Proof. $S_\lambda \subset^{\mathrm{opt}} T$ and therefore $R(S_\lambda, T) = 1$ is obvious by construction. Moreover, $\left\|q^2 - q^3\right\|_{T_{\mathrm{MAX}}} \leqslant 1$ and $\left\|q^2 - q^1\right\|_{T_{\mathrm{MAX}}} = \left\|q^3 - q^1\right\|_{T_{\mathrm{MAX}}}$. Using $p^3 = -(p^1 + p^2)$ we obtain

$$\begin{aligned} q^2 - q^1 &= \lambda p^1 + (1-\lambda)p^3 - \frac{1}{2}(p^2 + p^3) = \lambda p^1 + \frac{1}{2}(-p^2) + \left(\frac{1}{2} - \lambda\right)(-p^1 - p^2) \\ &= \left(2\lambda - \frac{1}{2}\right)p^1 + (1-\lambda)(-p^2) \in \left(\lambda + \frac{1}{2}\right)\mathrm{conv}\left(\{p^1, -p^2\}\right) \\ &\subset \left(\lambda + \frac{1}{2}\right)\mathrm{bd}(T_{\mathrm{MAX}}). \end{aligned}$$

Since $\lambda+\frac{1}{2}\geqslant 1$ for $\lambda\in[\frac{1}{2},1]$, it follows $D_{\mathrm{MAX}}(S_\lambda,T)=\lambda+\frac{1}{2}$.

Now, we show the formula for the inradius. For $K\in\mathscr{C}^2$, an inradius (circumradius) triangle of K is a triangle T' of the form $c+\rho T$ with $c\in\mathbb{R}^2$ and $\rho=r(K,T)$ ($\rho=R(K,T)$) such that $T'\subset^{\mathrm{opt}}K$ ($K\subset^{\mathrm{opt}}T'$). Let $\mathrm{conv}\left(\{w^1,w^2,w^3\}\right)$ be the inradius triangle of S_λ (cf. Figure 12). By axial symmetry of T and S_λ we know that $w^1=\frac{1}{2}(q^2+q^3)$. Denote the Euclidean edge length of the inradius triangle by a. Since T has edge length $\sqrt{3}$, we obtain $r(S_\lambda,T)=\frac{a}{\sqrt{3}}$. The segments $[q^3,q^2]$ and $[p^2,p^3]$ are parallel and the corresponding edges of the inradius and circumradius triangle are parallel as well. Thus, the triangles $\mathrm{conv}\left(\{w^1,w^3,q^2\}\right)$ and $\mathrm{conv}\left(\{p^3,q^2,q^1\}\right)$ are similar, implying $\frac{a}{\|q^2-p^3\|_2}=\frac{q_1^2}{p_1^3}$. We obtain

$$a=\frac{q_1^2}{p_1^3}\cdot\left\|q^2-p^3\right\|_2=(1-\lambda)\left\|q^2-p^3\right\|_2=(1-\lambda)\lambda\left\|p^1-p^3\right\|_2=\sqrt{3}\lambda(1-\lambda),$$

and therefore $r(S_\lambda,T)=\frac{a}{\sqrt{3}}=\lambda(1-\lambda)$.

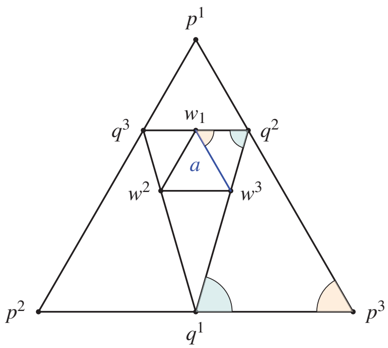

Figure 12: *Calculating the inradius of S_λ.*

□

One may observe that $S_{\frac{1}{2}}=T_{\frac{1}{2}}=-\frac{1}{2}T$ is Jung-extremal and S_1 is a segment.

LEMMA 4.8. *Let T be the equilateral Minkowski-centered triangle as given at the beginning of the section and $K=\mathrm{conv}\left(\{q^1,q^2,q^3\}\right)$ be a triangle that is optimally contained in T, s. t. q^i belongs to the edge of T opposite to p^i. We say that (q^1,q^3) is steep if $\left\|q^3-p^1\right\|_2\leqslant\left\|q^1-p^3\right\|_2$. Define $\tilde{K}:=\mathrm{conv}\left(\{q^1,q^3,\tilde{q}^2\}\right)\subset^{\mathrm{opt}}T$, a triangle such that $\tilde{q}^2$ is on the same edge of T as q^2 and $\left\|q^2-p^1\right\|_2\leqslant\left\|\tilde{q}^2-p^1\right\|_2$. If (q^1,q^3) is steep, $r(\tilde{K},T)\geqslant r(K,T)$.*

We call this property “steep” since it implies that the segment $[q^1,q^3]$ is steeper than $[p^3,p^1]$ (cf. Figure 13). By symmetry of T we can generalize this result to all choices $i,j\in\{1,2,3\}$: (q^i,q^j), $i,j\in\{1,2,3\}$, $i\neq j$ is steep if $\left\|q^j-p^i\right\|_2\leqslant\left\|q^i-p^j\right\|_2$. At least one of the ordered pairs (q^i,q^j) or (q^j,q^i) is always steep.

Proof of Lemma 4.8. Let H_1 and H_2 be the two lines parallel to $[p^1,p^3]$ supporting the inradius triangle $\mathrm{conv}\left(\{w^1,w^2,w^3\}\right)$ of K (that necessarily touches all three edges of K), s.t. H_1 contains the edge $[w^1,w^3]$ and H_2 the opposite to vertex w^2 (cf. Figure 13). Since (q^1,q^3) is steep, the part of H_2 below w^2 intersects $\tilde{K}$. By the Thales intercept theorem with the two parallel lines H_1 and H_2 and the points q^2 or $\tilde{q}^2$, the segment of H_1 contained in $\tilde{K}$ is greater than or equal to the one contained in K. Thus, we can move the inradius triangle of K within the slab between H_1 and H_2 until it touches $[q^1,\tilde{q}^2]$. The resulting translations of w^1,w^2,w^3 are all contained in $\tilde{K}$, the translation of w^2 due to the steepness of (q^1,q^3). Thus, $r(K,T) \leqslant r(\tilde{K},T)$. $\square$

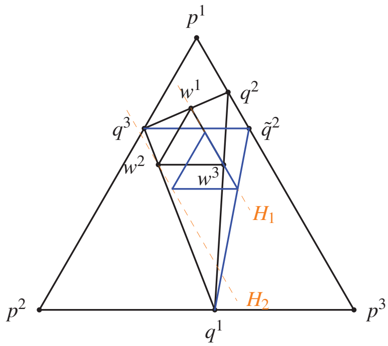

Figure 13: *Proof of Lemma* 4.8*. Since* (q^1,q^3) *is steep, the inradius triangle can be translated along the orange hyperplanes.*

THEOREM 4.9. *Let* $K \in \overline{\mathscr{C}}^2$ *and* T *be the equilateral Minkowski-centered triangle as given at the beginning of the section. Then,*

$$\left(\frac{D_{\mathrm{MAX}}(K,T)}{R(K,T)}-\frac{1}{2}\right)\left(\frac{3}{2}-\frac{D_{\mathrm{MAX}}(K,T)}{R(K,T)}\right) \leqslant \frac{r(K,T)}{R(K,T)}$$

with equality for the triangles S_λ, $\lambda \in [\frac{1}{2},1]$, *as described in Lemma* 4.7.

Proof of Theorem 4.9. The equality case follows directly from Lemma 4.7. So it suffices to prove the correctness of the inequality.

Let $K \subset^{\mathrm{opt}} T$. As already shown in the proof of Theorem 1.3 i), we either have $D_{\mathrm{MAX}}(K,T) \geqslant \frac{3}{2}$ or three touching points $\{q^1,q^2,q^3\}$ of K and the boundary of T, each one situated on a different edge of T. In the first case, the left-hand side of the inequality in Theorem 4.9 is non-positive while the right-hand side is always non-negative. Hence, in this case the inequality is fulfilled.

In the other case we consider the triangle $S := \mathrm{conv}\left(\{q^1,q^2,q^3\}\right)$, where $q^i \in K$ belongs to the edge of T opposite to p^i, $i=1,2,3$. Assume w.l.o.g. that the diameter of S is attained between q^1 and one of the other points and that $\left\|q^1-p^3\right\|_2 \leqslant \left\|q^1-p^2\right\|_2$. Our goal is to show that there exists a triangle $S_\lambda \subset^{\mathrm{opt}} T$, $\lambda \in [\frac{1}{2},1]$, as defined in Lemma 4.7 with at most the same diameter and inradius as S. Using the fact that $\left(D_{\mathrm{MAX}}-\frac{1}{2}\right)\left(\frac{3}{2}-D_{\mathrm{MAX}}\right)$ is decreasing in D_{MAX} for $D_{\mathrm{MAX}} \geqslant 1$, which is the case by Theorem 1.3 $i)$, we conclude

$$\begin{aligned}
&\left(\frac{D_{\mathrm{MAX}}(K,T)}{R(K,T)}-\frac{1}{2}\right)\left(\frac{3}{2}-\frac{D_{\mathrm{MAX}}(K,T)}{R(K,T)}\right)\\
&\leqslant \left(\frac{D_{\mathrm{MAX}}(S,T)}{R(S,T)}-\frac{1}{2}\right)\left(\frac{3}{2}-\frac{D_{\mathrm{MAX}}(S,T)}{R(S,T)}\right)\\
&\leqslant \left(\frac{D_{\mathrm{MAX}}(S_\lambda,T)}{R(S_\lambda,T)}-\frac{1}{2}\right)\left(\frac{3}{2}-\frac{D_{\mathrm{MAX}}(S_\lambda,T)}{R(S_\lambda,T)}\right)\\
&= \frac{r(S_\lambda,T)}{R(S_\lambda,T)} \leqslant \frac{r(S,T)}{R(S,T)} \leqslant \frac{r(K,T)}{R(K,T)}.
\end{aligned}$$

In the following, we distinguish between the two cases if the diameter is attained by $[q^1,q^2]$ or by $[q^1,q^3]$ and show that one may always assume that both segments are diametric. In the case of $D_{\mathrm{MAX}}(S,T)=D_{\mathrm{MAX}}([q^1,q^2],T)$ our assumption $\left\|q^1-p^3\right\|_2 \leqslant \left\|q^1-p^2\right\|_2$ implies that $\left\|q^2-p^1\right\|_2 \leqslant \left\|q^2-p^3\right\|_2$ (cf. Figure 14). To see this assume the converse. Then, one would obtain

$$[q^1,q^2] \subset \operatorname{conv}\left(\left\{p^3, \frac{1}{2}(p^2+p^3), \frac{1}{2}(p^1+p^3)\right\}\right) = \frac{1}{2}(p^3+T)$$

with $[q^1,q^2]$ not being an edge of $\frac{1}{2}(T+p^3)$. It would follow

$$D_{\mathrm{MAX}}([q^1,q^2],T) = 2R([q^1,q^2],T_{\mathrm{MAX}}) < 2R\left(\frac{1}{2}T, T_{\mathrm{MAX}}\right) = 1 = R(K,T),$$

contradicting Theorem 1.3 $i)$. Hence, $\left\|q^2-p^1\right\|_2 \leqslant \sqrt{3}/2 \leqslant \left\|q^1-p^2\right\|_2$, which implies the steepness of (q^1,q^2). Since $[q^1,q^2]$ is diametric, q^3 lies inside the hexagon

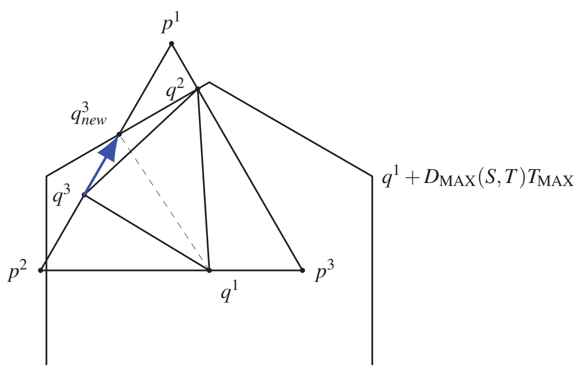

Figure 14: *Proof of the first case of Theorem 4.9: The diameter is attained between q^1 and q^2. In that situation one may replace q^3 by q^3_{new} such that the diameter is attained between q^1 and q^3 as well since (q^1,q^2) is steep.*

$q^1 + D_{\mathrm{MAX}}(S,T)T_{\mathrm{MAX}}$. However, $\|q^1 - p^3\|_2 \leqslant \|q^1 - p^2\|_2$ now implies the existence of an intersection point between $[p^1,p^2]$ and the boundary of $q^1 + D_{\mathrm{MAX}}(S,T)T_{\mathrm{MAX}}$ which is not farther from p^1 than q^3. Choosing this point as our new q^3 neither increases the diameter nor the inradius (the latter because of Lemma 4.8). Doing so, $[q^1,q^3]$ becomes diametric, too.

For the second case $D_{\mathrm{MAX}}(S,T) = D_{\mathrm{MAX}}([q^1,q^3],T)$ we need to consider three subcases:

a) The case $\|q^3 - p^2\|_2 < \|q^3 - p^1\|_2$ corresponds to Case 1 with q^3 in the role of q^1 and q^2 being the vertex that is moved.

b) In the case of $\|q^3 - p^2\|_2 \geqslant \|q^3 - p^1\|_2$ and (q^1,q^3) being steep one can move q^2 in the direction of p^1 such that $[q^1,q^2]$ becomes diametric, too.

c) The case $\|q^3 - p^2\|_2 \geqslant \|q^3 - p^1\|_2$ and (q^3,q^1) being steep again corresponds to b) with the roles of q^1 and q^3 interchanged. Thus, we may move q^2 towards p^3 s.t. $[q^2,q^3]$ becomes diametric.

Altogether, we see that assuming $\|q^2 - q^1\|_{T_{\mathrm{MAX}}} = \|q^3 - q^1\|_{T_{\mathrm{MAX}}}$ is justified and doing so, the points q^2 and q^3 do not only belong to the boundary of $q^1 + D_{\mathrm{MAX}}(S,T)T_{\mathrm{MAX}}$, they essentially belong to the (translated and dilated) edges $[p^1,-p^3]$ or $[p^1,-p^2]$ of T_{MAX}. For q^2 this follows from the fact that it has to be closer to p^1 than to p^3. As described in the proof of the first case, the boundary of $q^1 + D_{\mathrm{MAX}}(S,T)T_{\mathrm{MAX}}$ intersects $[p^1,p^2]$ once or twice. However, it is not possible that it only intersects with the segment $q^1 + D_{\mathrm{MAX}}(S,T)[p^2,-p^3]$ as this would contradict our assumption that $D_{\mathrm{MAX}}(S,T) = D_{\mathrm{MAX}}([q^1,q^2],T)$. If we have two intersection points, we can replace q^3 (if necessary) by the upper one without increasing the inradius since (q^1,q^2) is steep. Thus, we may assume $q^3 \in q^1 + D_{\mathrm{MAX}}(S,T)[p^1,-p^3]$.

Next, we show that we may additionally assume $q^2 \in q^1 + D_{\mathrm{MAX}}(S,T)[p^1,-p^2]$. This is not the case if and only if $q_1^1 > q_1^2$ (cf. Figure 15). But then, we know $q_2^3 < q_2^2$ and therefore (q^2,q^3) is steep. Thus, replacing q^1 by $\hat{q}^1$ such that $\hat{q}_1^1 = q_1^2$ does not increase the diameter or the inradius, achieving this way that $\|q^3 - \hat{q}^1\|_{T_{\mathrm{MAX}}} = \|q^2 - \hat{q}^1\|_{T_{\mathrm{MAX}}}$.

Now, we consider the triangle $S_\lambda = \mathrm{conv}\left(\{\tilde{q}^1,\tilde{q}^2,\tilde{q}^3\}\right)$ as in Lemma 4.7 with λ such that it has the same diameter as S. Due to our assumptions about the positions of q^2 and q^3, we see that the distance between the parallel segments $q^1 + [p^1,-p^3]$ and $\tilde{q}^1 + [p^1,-p^3]$ is equal to the distance between $q^1 + [p^1,-p^2]$ and $\tilde{q}^1 + [p^1,-p^2]$. Furthermore, it is also equal to the distances between $q^3 + [-p^1,p^3]$ and $\tilde{q}^3 + [-p^1,p^3]$ and between $q^2 + [-p^1,p^2]$ and $\tilde{q}^2 + [-p^1,p^2]$. Consequently, $\|q^1 - \tilde{q}^1\|_2 = \|q^2 - \tilde{q}^2\|_2 = \|q^3 - \tilde{q}^3\|_2 =: \kappa$ (cf. Figure 16).

By using the intercept theorem and the law of sines we will show in the following that, possibly after a suitable translation, all vertices of the inradius triangle of S_λ are contained in S, which proves $r(S_\lambda,T) \leqslant r(S,T)$. We denote the vertices of the inradius triangle of S_λ by w^i (cf. Figure 18) and the Euclidean distance in the horizontal direction between w^i and the segment $[q^j,q^k]$, $\{i,j,k\} = \{1,2,3\}$, by l_i. If $\kappa \neq 0$, every

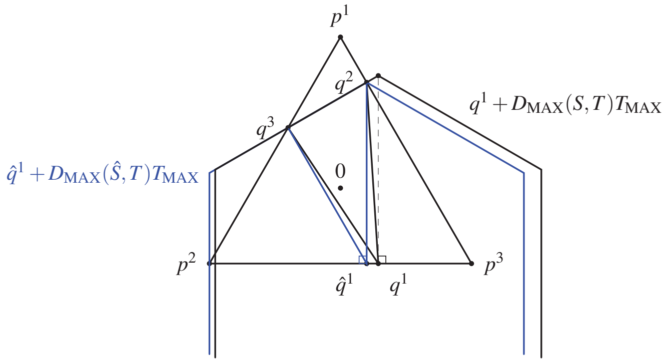

Figure 15: *Proof of Theorem 4.9. We can assume that* q^3 *lies on* $q^1 + D_{\mathrm{MAX}}(S,T)[p^1,-p^3]$ *and* q^2 *lies on* $q^1 + D_{\mathrm{MAX}}(S,T)[p^1,-p^2]$. *In the case the latter is initially not fulfilled, we can consider the triangle* $\hat{S} = \operatorname{conv}\left(\{\hat{q}^1, q^2, q^3\}\right)$ *which has smaller or equal inradius and diameter.*

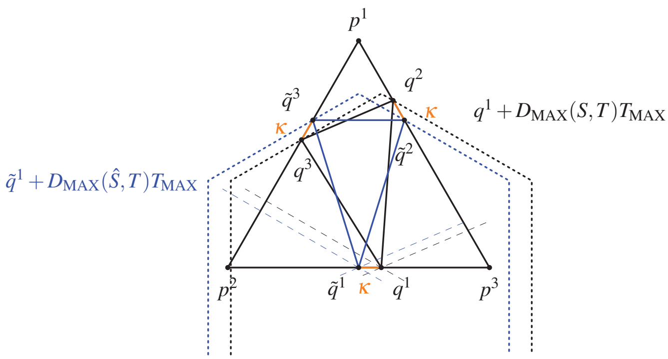

Figure 16: *Proof of Theorem 4.9. Transformation of* S *(black) into* S_λ *(blue) with the same diameter. The distances* $\|q^i - \tilde{q}^i\|_2$ *are all equal.*

edge of S intersects the corresponding edge of S_λ in a unique point v^i (cf. Figure 17). We will show that if we translate the inradius triangle of S_λ by distance l_3 in direction $(-1,0)^\top$, it is completely contained in S. To do so we compute all the values l_i, $i = 1,2,3$ and prove that we have $l_3 \leqslant l_j$, $j = 1,2$.

Computation of l_1 (cf. Figure 17): We use the intercept theorem for the segment $[\tilde{q}^2, \tilde{q}^3]$ and the two lines parallel to it through q^2 and q^3, respectively. The intersection of $[p^1, p^2]$ with the horizontal line through q^2 is called r. Since T is an equilateral triangle of edge length $\sqrt{3}$, $\operatorname{conv}\left(\{p^1, \tilde{q}^3, \tilde{q}^2\}\right)$ is also equilateral with an edge length of $(1-\lambda)\sqrt{3}$ due to the definitions of $\tilde{q}^2, \tilde{q}^3$. Furthermore,

$$\|\tilde{q}^3 - v^1\|_2 = \frac{1}{2}\|r - q^2\|_2 = \frac{1}{2}\|p^1 - q^2\|_2 = \frac{1}{2}(\|p^1 - \tilde{q}^2\|_2 - \kappa).$$

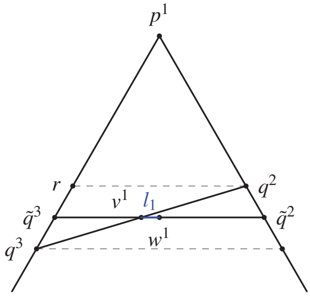

Figure 17: *Proof of Theorem 4.9. Computation of l_1. The triangles defined by p^1 and the intersection points of the parallel lines are all three equilateral.*

It follows that w^1 is always contained in S and

$$l_1 = \frac{1}{2}\|p^1 - \tilde{q}^2\|_2 - \|\tilde{q}^3 - v^1\|_2 = \frac{1}{2}\kappa.$$

Computation of l_2 (cf. Figure 18): Let $\alpha_2 = \angle v^2 q^3 \tilde{q}^3$, $\beta_2 = \angle v^2 q^1 p^2$, and $\gamma_2 = \angle \tilde{q}^1 v^2 q^1$. We compute the fraction $\frac{\|\tilde{q}^3 - v^2\|_2}{\|\tilde{q}^1 - v^2\|_2}$ using the law of sines, first for the triangles $\operatorname{conv}\left(\{v^2, q^1, \tilde{q}^1\}\right)$ and $\operatorname{conv}\left(\{v^2, q^3, \tilde{q}^3\}\right)$, and then for $\operatorname{conv}\left(\{p^2, q^1, q^3\}\right)$.

$$\begin{aligned}\frac{\|\tilde{q}^3 - v^2\|_2}{\|\tilde{q}^1 - v^2\|_2} &= \frac{\sin(\gamma_2)}{\kappa \sin(\beta_2)} \cdot \frac{\kappa \sin(\alpha_2)}{\sin(\gamma_2)} = \frac{\sin(\alpha_2)}{\sin(\beta_2)} \\ &= \frac{\sin(\pi - \alpha_2)}{\sin(\beta_2)} = \frac{\frac{\sqrt{3}}{2} + \kappa}{\sqrt{3}\lambda - \kappa}.\end{aligned}$$

Furthermore, we know from Lemma 4.7 that $r(S_\lambda, T) = \lambda(1-\lambda)$. Together with the intercept theorem we obtain

$$\frac{\|\tilde{q}^3 - w^2\|_2}{\|\tilde{q}^1 - w^2\|_2} = \frac{w_2^1 - w_2^2}{(\tilde{q}_2^3 - \tilde{q}_2^1) - (w_2^1 - w_2^2)} = \frac{1-\lambda}{\lambda}.$$

Since $\frac{\frac{\sqrt{3}}{2} + \kappa}{\sqrt{3}\lambda - \kappa} \geqslant \frac{1-\lambda}{\lambda}$ for $\lambda \in [\frac{1}{2}, 1]$, v^2 is closer to $\tilde{q}^1$ than w^2, which proves $w^2 \in S$.

Using the intercept theorem, the distance of w^2 to $[q^1, q^3]$ in direction $(-1, 0)^\top$ calculates to

$$l_2 = \kappa \cdot \frac{\|v^2 - w^2\|_2}{\|\tilde{q}^1 - v^2\|_2} = \kappa \cdot \frac{\|\tilde{q}^3 - v^2\|_2 - \|\tilde{q}^3 - w^2\|_2}{\|\tilde{q}^1 - v^2\|_2}.$$

Computation of l_3 (cf. Figure 18): If w^3 is also contained in S, we have that the complete inradius triangle of S_λ is contained in S, and we are done.

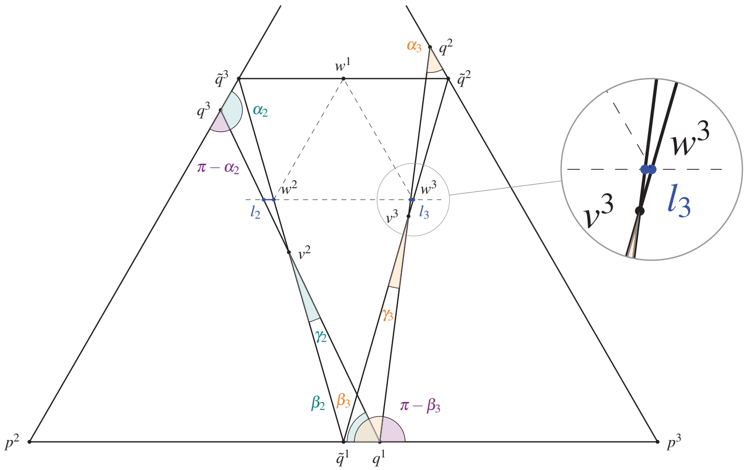

Figure 18: *Proof of Theorem 4.9. Computation of l_2 and l_3.*

Otherwise, $\|\tilde{q}^1 - v^3\|_2 \leqslant \|\tilde{q}^1 - w^3\|_2$ and we need to show that we can translate the inradius triangle to be contained in S. To do so, we compute the distance l_3 of w^3 to $[q^1, q^2]$ in direction $(-1,0)^\top$, which can be done completely analogously to the computation of l_2 above: Let $\alpha_3 = \angle q^1 q^2 p^3$, $\beta_3 = \angle v^3 q^1 p^2$, and $\gamma_3 = \angle \tilde{q}^1 v^3 q^1$. Then,

$$\frac{\|\tilde{q}^2 - v^3\|_2}{\|\tilde{q}^1 - v^3\|_2} = \frac{\sin(\gamma_3)}{\kappa \sin(\beta_3)} \cdot \frac{\kappa \sin(\alpha_3)}{\sin(\gamma_3)} = \frac{\sin(\alpha_3)}{\sin(\pi - \beta_3)} = \frac{\frac{\sqrt{3}}{2} - \kappa}{\sqrt{3}\lambda + \kappa}$$

and $\frac{\|\tilde{q}^2 - w^3\|_2}{\|\tilde{q}^1 - w^3\|_2} = \frac{1-\lambda}{\lambda}$. Thus,

$$\frac{\|\tilde{q}^2 - v^3\|_2}{\|\tilde{q}^1 - v^3\|_2} \leqslant \frac{\|\tilde{q}^3 - v^2\|_2}{\|\tilde{q}^1 - v^2\|_2}.$$

Together with $\|\tilde{q}^2 - \tilde{q}^1\|_2 = \|\tilde{q}^3 - \tilde{q}^1\|_2$ we obtain $\|\tilde{q}^1 - v^2\|_2 \leqslant \|\tilde{q}^1 - v^3\|_2$ as well as $\|\tilde{q}^2 - v^3\|_2 \leqslant \|\tilde{q}^3 - v^2\|_2$. Hence,

$$\begin{aligned} l_3 = \kappa \cdot \frac{\|v^3 - w^3\|_2}{\|\tilde{q}^1 - v^3\|_2} &= \kappa \cdot \left(\frac{\|\tilde{q}^2 - v^3\|_2 - \|\tilde{q}^2 - w^3\|_2}{\|\tilde{q}^1 - v^3\|_2} \right) \\ &\leqslant \kappa \cdot \left(\frac{\|\tilde{q}^3 - v^2\|_2 - \|\tilde{q}^3 - w^2\|_2}{\|\tilde{q}^1 - v^2\|_2} \right) = l_2. \end{aligned}$$

Moreover, since $\kappa \geqslant 0$ and $\lambda \in [\frac{1}{2}, 1]$ it follows

$$\begin{aligned} l_3 &= \kappa \cdot \left(\frac{\|\tilde{q}^2 - v^3\|_2 - \|\tilde{q}^2 - w^3\|_2}{\|\tilde{q}^1 - v^3\|_2} \right) \leqslant \kappa \cdot \left(\frac{\|\tilde{q}^2 - v^3\|_2}{\|\tilde{q}^1 - v^3\|_2} - \frac{\|\tilde{q}^2 - w^3\|_2}{\|\tilde{q}^1 - w^3\|_2} \right) \\ &= \kappa \cdot \left(\frac{\frac{\sqrt{3}}{2} - \kappa}{\sqrt{3}\lambda + \kappa} - \frac{1-\lambda}{\lambda} \right) \leqslant \kappa \cdot \left(\frac{1}{2\lambda} - \frac{1-\lambda}{\lambda} \right) = \kappa \cdot \left(1 - \frac{1}{2\lambda} \right) \leqslant \frac{1}{2} \cdot \kappa = l_1. \end{aligned}$$

Hence, if we translate the inradius triangle of S_λ by $(l_3,0)^\top$, it is contained in S, which proves $r(S_\lambda,T) \leqslant r(S,T)$. $\square$

The collected inequalities are now sufficient to provide a full description of the diagram $f_{\mathrm{MAX}}(\overline{\mathscr{C}^2},S)$ (cf. Figure 19).

Proof of Theorem 1.5. It suffices to show the claim for the equilateral triangle T, since all Minkowski-centered triangles can be linearly transformed into T. We give a continuous description of the boundaries described by the inequalities (1), (2), and (4), as well as those given by Theorem 1.3 $i)$ and Theorem 4.9. First, (1) is attained with equality for $(1-\lambda)L_D+\lambda T$, $\lambda \in [0,1]$ where L_D and T are the extreme cases. Secondly, (2) is attained with equality for $(1-\lambda)T+\lambda T_{\mathrm{MAX}}$, $\lambda \in [0,1]$ by Lemma 2.8 where T and T_{MAX} are the extreme cases. The boundary induced by (4) is filled by segments from L_w to L_D. Next, since T_{MAX} is the completion of $-T$, the boundary from Theorem 1.3 $i)$ is filled by all sets $T_+ \in \mathscr{C}^2$ fulfilling $-T \subset T_+ \subset T_{\mathrm{MAX}}$. Finally, the inequality in Theorem 4.9 is fulfilled with equality by the triangles S_λ as introduced in Lemma 4.7. Since we have presented a continuous description of the boundary, we can apply Lemma 3.1 and infer that the diagram is simply connected. $\square$

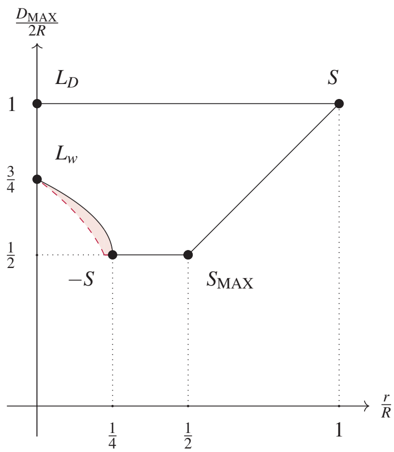

Figure 19: *The diagram* $f_{\mathrm{MAX}}(\overline{\mathscr{C}^2},S)$ *w. r. t. a Minkowski-centered triangle* S *(black) as described in Theorem* 1.5 *and an upper bound for the union over all Minkowski-centered gauges bounded by* (13) *(additional purple region).*

While the inequalities (1), (2), (4) and the boundary corresponding to Theorem 1.3 $i)$ are valid for all choices of planar, Minkowski-centered gauges, the inequality from Theorem 4.9 is only proven for triangles. Using the result from Proposition 3.2 and $D_{\mathrm{AM}}(K,C) \leqslant \frac{4}{3}D_{\mathrm{MAX}}(K,C)$ which follows from $\frac{s(C)+1}{2s(C)}C_{\mathrm{MAX}} \subset^{\mathrm{opt}} C_{\mathrm{AM}}$, see Proposition 2.14, we are able to provide another general inequality

$$\frac{2}{3}\frac{D_{\mathrm{MAX}}(K,C)}{R(K,C)}\left(1-\frac{2}{3}\frac{D_{\mathrm{MAX}}(K,C)}{R(K,C)}\right)\leqslant\frac{r(K,C)}{R(K,C)}, \tag{13}$$

which enables us to give a bound for the union of the diagrams $f_{\mathrm{MAX}}(\overline{\mathscr{C}}^2,C)$ with C Minkowski-centered (depicted in purple within Figure 19).

CONJECTURE 4.10. *For the maximum diameter $f(\mathscr{C}^2,T)$ is dominating, i.e. for every Minkowski-centered $C\in\mathscr{C}_0^2$ we have $f_{\mathrm{MAX}}(\overline{\mathscr{C}}^2,C)\subset f_{\mathrm{MAX}}(\overline{\mathscr{C}}^2,T)$.*

5. The harmonic diameter

In the case of D_{HM}, the inequalities from Lemma 2.16 have the following form. For $K\in\mathscr{C}^n$ and $C\in\mathscr{C}_0^n$ Minkowski-centered,

$$\frac{D_{\mathrm{HM}}(K,C)}{2}\leqslant\delta_{\mathrm{HM}}R(K,C), \tag{14}$$

$$\delta_{\mathrm{HM}}r(K,C)\leqslant\frac{D_{\mathrm{HM}}(K,C)}{2}, \tag{15}$$

$$s(C)r(K,C)+R(K,C)\leqslant\frac{s(C)+1}{2\rho_{\mathrm{HM}}}D_{\mathrm{HM}}(K,C)\leqslant\frac{s(C)+1}{2}D_{\mathrm{HM}}(K,C), \tag{16}$$

$$r(K,C)+R(K,C)\leqslant\frac{2s(C)}{s(C)+1}D_{\mathrm{HM}}(K,C), \tag{17}$$

and

$$0\leqslant r(K,C). \tag{18}$$

Now, we know from Proposition 2.14, that for the harmonic mean, the factors ρ_{HM} and δ_{HM} do not depend solely on the Minkowski asymmetry $s(C)$. However, the following bounds are known.

$$1\leqslant\rho_{\mathrm{HM}}\leqslant\frac{(s(C)+1)^2}{4s(C)}\leqslant\delta_{\mathrm{HM}}\leqslant\frac{s(C)+1}{2}.$$

However, if we take $K=C_{\mathrm{HM}}$, we obtain $R(C_{\mathrm{HM}},C)=\frac{2s(C)}{s(C)+1}$, $r(C_{\mathrm{HM}},C)=\frac{2}{s(C)+1}$, and $D_{\mathrm{HM}}(C_{\mathrm{HM}},C)=2$ from [7]. Thus, whenever we choose C, s.t. $\delta_{\mathrm{HM}}=\frac{s(C)+1}{2}$ then C_{HM} fulfills (15) with equality while for each C with $\rho_{\mathrm{HM}}=\frac{(s(C)+1)^2}{4s(C)}$ C_{HM} fulfills the left inequality in (16) with equality. Unlike with the maximum diameter, (a dilatation of) the C_{HM} is not always a completion of C. Instead, we obtain the following lemma.

LEMMA 5.1. *Let $C \in \mathcal{C}_0^n$ be Minkowski-centered. The following are equivalent:*

i) $\delta_{\mathrm{HM}} = \frac{s(C)+1}{2}$,

ii) $\frac{s(C)+1}{2} C_{\mathrm{HM}}$ *is a completion of* C *w. r. t.* C*, and*

iii) $D(C, C_{\mathrm{HM}}) = 2R(C, C_{\mathrm{HM}})$.

Proof. We know from [7] that $C \subset^{\mathrm{opt}} \frac{s(C)+1}{2} C_{\mathrm{HM}}$ and therefore $\frac{s(C)+1}{2} C_{\mathrm{HM}}$ is a complete set containing C with $R(C, C_{\mathrm{HM}}) = \frac{s(C)+1}{2}$.

$i) \Rightarrow ii)$: If $\delta_{\mathrm{HM}} = \frac{s(C)+1}{2}$, then

$$D_{\mathrm{HM}}\left(\frac{s(C)+1}{2} C_{\mathrm{HM}}, C\right) = s(C)+1 = 2\delta_{\mathrm{HM}} = D_{\mathrm{HM}}(C,C),$$

implying that $\frac{s(C)+1}{2} C_{\mathrm{HM}}$ is a completion of C.

$ii) \Rightarrow iii)$: If $\frac{s(C)+1}{2} C_{\mathrm{HM}}$ is a completion of C, then

$$\begin{aligned} D(C, C_{\mathrm{HM}}) = D_{\mathrm{HM}}(C,C) &= \frac{s(C)+1}{2} D_{\mathrm{HM}}(C_{\mathrm{HM}}, C) \\ &= s(C)+1 = 2R(C, C_{\mathrm{HM}}). \end{aligned}$$

$iii) \Rightarrow i)$: Assuming (iii) it follows

$$\begin{aligned} 2\delta_{\mathrm{HM}} &= 2R(C_{\mathrm{AM}}, C_{\mathrm{HM}}) = D(C, C_{\mathrm{HM}}) \\ &= 2R(C, C_{\mathrm{HM}}) = s(C)+1. \quad \square \end{aligned}$$

Note that if $\frac{s(C)+1}{2} C_{\mathrm{HM}}$ is a completion of C, then it is also a completion of $-C$ and since $R(-C,C) = s(C) = R(\frac{s(C)+1}{2} C_{\mathrm{HM}}, C)$, it is even a Scott-completion of $-C$ (i. e. a completion also preserving the circumradius).

EXAMPLE 5.2. The Reuleaux triangle $\mathrm{RT} := \bigcap_{i=1}^{3} \left(p^i + \sqrt{3}\mathbb{B}_2^2\right)$, where the p_i are the vertices of T and $\mathbb{B}_2^2$ is the Euclidean disc, is the completion of the equilateral triangle T in the Euclidean case. Omitting the detailed calculations, we have $s(\mathrm{RT}) = \frac{1}{\sqrt{3}-1} \approx 1.366$, $\delta_{\mathrm{HM}} = \frac{\sqrt{3}}{\sqrt{11}-\sqrt{3}} \approx 1.093 < \frac{s(\mathrm{RT})+1}{2}$ and $\rho_{\mathrm{HM}} = \frac{(s(\mathrm{RT})+1)^2}{4s(\mathrm{RT})} = \frac{3(\sqrt{3}+1)}{8} \approx 1.025$. Thus, by Lemma 5.1 this is a case where the (dilated) harmonic mean $\mathrm{RT}_{\mathrm{HM}}$ is not a completion of the gauge RT. Since $T \subset \mathrm{RT}$ is diametric, we obtain from Lemma 2.23 that the unique completion of RT is $\mathrm{RT}^* := \bigcap_{i=1}^{3} (p_i + 2\delta_{\mathrm{HM}} \mathrm{RT}_{\mathrm{HM}})$ (cf. Figure 20).

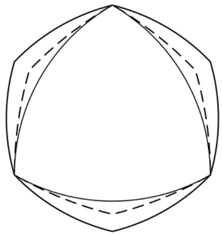

Figure 20: *The Reuleaux Triangle* RT*, its completion* RT^* *(dashed), and its harmonic symmetrization* $\mathrm{RT}_{\mathrm{HM}}$.

Using the symmetries of the Reuleaux triangle we obtain $R(\mathrm{RT}^*,\mathrm{RT}) = (s(\mathrm{RT})+1)\cdot\frac{\delta_{\mathrm{HM}}}{\rho_{\mathrm{HM}}} - s(\mathrm{RT})$ and thus RT^* fulfills the left inequality in (16) with equality, which is also true for $\mathrm{RT}_{\mathrm{HM}}$. See Figure 21 for the diagram $f_{\mathrm{HM}}(\bar{\mathscr{C}}^2,\mathrm{RT})$.

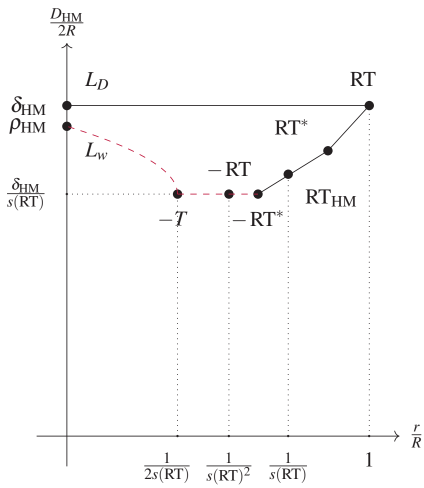

Figure 21: *In addition to the known inequalities* (14)*,* (15)*,* (16)*,* (18) *and the points we have calculated above, the remaining part of the boundary as (approx.) conjectured is depicted (dashed). Keep in mind that the (dilated) harmonic symmetrization of the Reuleaux triangle is not its completion.*

Proof of Corollary 1.6*.* For the equilateral triangle, we have $\delta_{\mathrm{HM}} = \frac{s(T)+1}{2} = \frac{3}{2}$, $\rho_{\mathrm{HM}} = \frac{(s(T)+1)^2}{4s(T)} = \frac{9}{8}$ and $T_{\mathrm{HM}} = \frac{2}{3}T_{\mathrm{MAX}}$ [6]. Thus, $D_{\mathrm{HM}}(K,T) = \frac{3}{2}D_{\mathrm{MAX}}(K,T)$ for all $K \in \mathscr{C}^2$ and we can transfer all inequalities concerning triangles from the previous section. $\square$

REMARK 5.3. The bound $D_{\mathrm{M}}(K,C) \geqslant R(K,C)$, derived from Corollary 4.4 for the planar case, cannot be reached for $\mathrm{M} = \mathrm{HM}$. If $D_{\mathrm{HM}}(K,C) = 2$, we obtain from the containment chain

$$\frac{1}{2}\left(1+\frac{1}{s(K)}\right)K \subset K_{\mathrm{AM}} \subset C_{\mathrm{HM}} \subset \frac{2s(C)}{s(C)+1}C,$$

that

$$R(K,C) \leqslant \frac{4s(K)s(C)}{(s(K)+1)(s(C)+1)} \leqslant \frac{4n^2}{(n+1)^2}. \tag{19}$$

Thus,

$$D_{\mathrm{HM}}(K,C) \geqslant \frac{(n+1)^2}{2n^2} R(K,C),$$

which for $n=2$ gives $D_{\mathrm{HM}}(K,C) \geqslant \frac{9}{8}R(K,C) > R(K,C)$. Furthermore, equality in (19) can only be reached if K and C are simplices, but we have shown that for Minkowski-centered triangles $D_{\mathrm{HM}}(K,S) \geqslant \frac{3}{2}R(K,S)$. Finally, one may observe that if we consider 0-symmetric planar gauges, (19) also yields $D(K,C) \geqslant \frac{3}{2}R(K,C)$.

The following system of inequalities provides an upper bound for the union of the diagrams $f_{\mathrm{HM}}(\overline{\mathscr{C}}^2,C)$ over all Minkowski-centered gauges $C \in \mathscr{C}_0^2$ (cf. Figure 22).

$$\begin{aligned}
0 &\leqslant r(K,C)\\
r(K,C) &\leqslant R(K,C)\\
D_{\mathrm{HM}}(K,C) &\leqslant 3R(K,C)\\
2r(K,C) &\leqslant D_{\mathrm{HM}}(K,C)\\
9R(K,C) &\leqslant 8D_{\mathrm{HM}}(K,C)\\
\frac{D_{\mathrm{HM}}(K,C)}{2R(K,C)}\left(1-\frac{D_{\mathrm{HM}}(K,C)}{2R(K,C)}\right) &\leqslant \frac{r(K,C)}{R(K,C)}.
\end{aligned}$$

Moreover, the following parts of the boundary described by the above inequalities are reached:

i) $r(K,C)=0$ for segments $K=L_D$ and gauges C with all possible values $s(C) \in [1,2]$.

ii) $r(K,C)=R(K,C)$ for $K=C$ and gauges C with all possible values $s(C) \in [1,2]$.

iii) $D_{\mathrm{HM}}(K,C)=3R(K,C)$ with C being a Minkowski-centered triangle and K as described for the diagram $f_{\mathrm{HM}}(\overline{\mathscr{C}}^2,S)$.

The first two inequalities are trivial. The third and fourth follow from Lemma 2.16 and the fifth from Remark 5.3. The last inequality follows from Proposition 3.2 together with Remark 2.7. The equality cases follow from Lemma 2.8, Lemma 2.15 and Corollary 1.6.

REMARK 5.4. Since δ_{HM} is not the same for every C, as it is in the arithmetic case, the diagram for the equilateral triangle cannot be dominating. For every C, $f_{\mathrm{HM}}(C,C)=(1,\delta_{\mathrm{HM}})$ and this is always the only combination where inradius and circumradius coincide. Thus, we cannot find a single gauge C which defines the union of the diagrams.

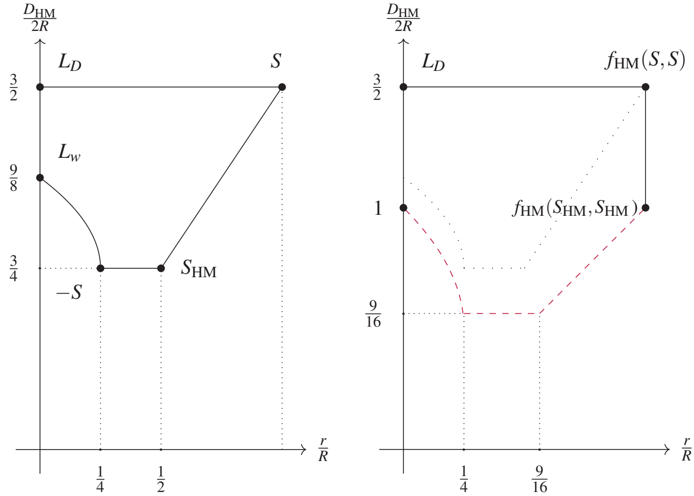

Figure 22: *The diagram $f_{HM}(C^2,S)$ w. r. t. a Minkowski-centered triangle S (left) and an upper bound for the union of the diagrams over all Minkowski-centered gauges $C \in \mathscr{C}_0^2$ (right). The dotted lines correspond to the boundary of $f_{HM}(C^2,S)$ while the dashed, purple ones correspond to the upper bounds given above. The points $f_{HM}(S,S)$ and $f_{HM}(S_{HM},S_{HM})$ for a Minkowski-centered triangle S are vertices of this diagram.*

6. The minimum diameter

The inequalities from Lemma 2.16 have the following form. For $K \in \mathscr{C}^n$ and $C \in \mathscr{C}_0^n$ Minkowski-centered,

$$D_{MIN}(K,C) \leqslant (s(C)+1)R(K,C) \tag{20}$$

$$(s(C)+1)r(K,C) \leqslant D_{MIN}(K,C) \tag{21}$$

$$s(C)r(K,C)+R(K,C) \leqslant \rho_M(s(C)r(K,C)+R(K,C)) \leqslant (s(C)+1)\frac{D_{MIN}(K,C)}{2} \tag{22}$$

$$r(K,C)+R(K,C) \leqslant D_{MIN}(K,C) \tag{23}$$

$$0 \leqslant r(K,C). \tag{24}$$

In the case of the minimum, δ_{MIN} depends solely on the asymmetry of C but ρ_{MIN} not. We obtain $1 \leqslant \rho_{MIN} \leqslant \frac{s(C)+1}{2} = \delta_{MIN}$ from Proposition 2.14.

Moreover, choosing $K = C_{MIN}$, we obtain $R(C_{MIN},C) = 1$, $r(C_{MIN},C) = \frac{1}{s(C)}$, and $D_{MIN}(C_{MIN},C) = 2$ from [7]. In case of D_{MIN} the (dilated) symmetrization only yields a completion of C in the case that C is 0-symmetric.

LEMMA 6.1. *Let $C \in \mathscr{C}_0^n$ be Minkowski-centered. ρC_{MIN} is a completion of C if and only if $\rho = s(C) = 1$.*

Proof. Since C is Minkowski-centered, we have $C \subset^{\mathrm{opt}} s(C)C_{\mathrm{MIN}}$, which shows that ρ must equal $s(C)$ by Lemma 2.12. Thus,

$$D_{\mathrm{MIN}}(\rho C_{\mathrm{MIN}}, C) = s(C)D_{\mathrm{MIN}}(C_{\mathrm{MIN}}, C) = 2s(C) \geqslant s(C) + 1 = 2\delta_{\mathrm{MIN}} = D_{\mathrm{MIN}}(C,C),$$

with equality if and only if $s(C) = 1$. □

Next, we show a Jung-type inequality for D_{MIN} which sharpens the inequality of Leichtweiss [33] by including the Minkowski-asymmetry $s(K)$.

Proof of Theorem 1.2. It follows from the containment chain

$$\left(1 + \frac{1}{s(K)}\right) K \subset K - K \subset D_{\mathrm{MIN}}(K,C)C_{\mathrm{MIN}} \subset D_{\mathrm{MIN}}(K,C)C$$

that $R(K,C) \leqslant \frac{s(K)}{s(K)+1} D_{\mathrm{MIN}}(K,C)$. □

REMARK 6.2. For every $s_K \in [1,n]$ and $s_C \in [1,n]$, there exist $K, C \in \mathscr{C}^n$ with $s(K) = s_K$ and $s(C) = s_C$ such that we have equality in Theorem 1.2. Let S be the Minkowski-centered regular simplex. Choosing $K := (s_K S) \cap -S$ and $C := S \cap (-s_C S)$, we fulfill $s(K) = s_K$ and $s(C) = s_C$. Note that K and C are Minkowski-centered. Since $R(K,C) = s_K$ and $D_{\mathrm{MIN}}(K,C) = s_K + 1$, we obtain equality. Leichtweiss also showed that $D_{\mathrm{MIN}}(K,C) = \frac{n+1}{n} R(K)$ is only possible if K is a simplex [33]. This can be also seen from Theorem 1.2 since it is necessary that $s(K) = n$.

Proof of Corollary 1.7. In the planar case, $D_{\mathrm{MIN}}(K,S) = \frac{3}{2} D_{\mathrm{AM}}(K,S)$ since we have $S_{\mathrm{MIN}} = \frac{2}{3} S_{\mathrm{AM}}$. The inequalities from Proposition 3.2 can be transferred to describe $f_{\mathrm{MIN}}(\mathscr{C}^2, S)$. □

As in the case of the harmonic diameter, since δ_{MIN} is not the same for every gauge, the diagram for the equilateral triangle cannot be dominating. The following system of inequalities provides an upper bound for the union of the diagrams $f_{\mathrm{MIN}}(\overline{\mathscr{C}}^2, C)$ over all Minkowski-centered gauges $C \in \mathscr{C}_0^2$ (cf. Figure 23).

$$\begin{aligned}
0 &\leqslant r(K,C) \\
r(K,C) &\leqslant R(K,C) \\
D_{\mathrm{MIN}}(K,C) &\leqslant 3R(K,C) \\
r(K,C) + R(K,C) &\leqslant D_{\mathrm{MIN}}(K,C) \\
3R(K,C) &\leqslant 2D_{\mathrm{MIN}}(K,C) \\
\frac{D_{\mathrm{MIN}}(K,C)}{2R(K,C)} \left(1 - \frac{D_{\mathrm{MIN}}(K,C)}{2R(K,C)}\right) &\leqslant \frac{r(K,C)}{R(K,C)}.
\end{aligned}$$

Moreover, the following parts of the boundary described by the above inequalities are reached:

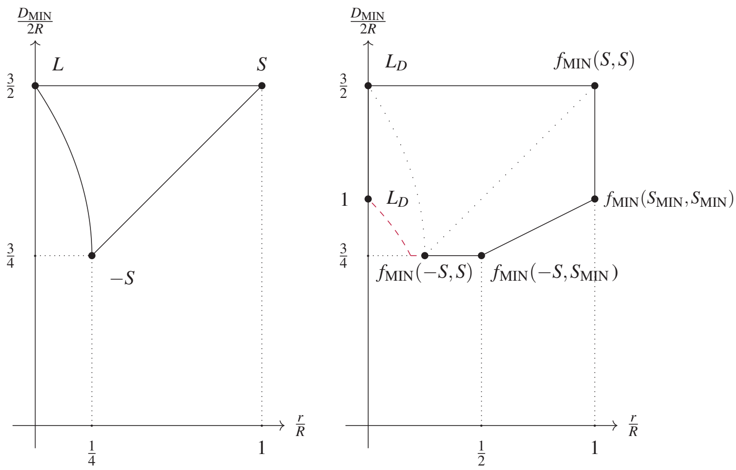

Figure 23: *The diagram $f_{\mathrm{MIN}}(\mathscr{C}^2,S)$ w. r. t. a Minkowski-centered triangle S (left) and an upper bound for the union of the diagrams $f_{\mathrm{MIN}}(\mathscr{C}^2,C)$ over all Minkowski-centered $C \in \mathscr{C}_0^2$ (right). The dotted lines in the right diagram correspond to the boundaries of $f_{\mathrm{MIN}}(\mathscr{C}^2,S)$ and the dashed one to the given upper bound. The points $f_{\mathrm{MIN}}(S,S)$, $f_{\mathrm{MIN}}(-S,S)$, $f_{\mathrm{MIN}}(-S,S_{\mathrm{MIN}})$ and $f_{\mathrm{MIN}}(S_{\mathrm{MIN}},S_{\mathrm{MIN}})$ for a Minkowski-centered triangle S are depicted.*

i) $r(K,C) = 0$ for segments $K = L_D$ and gauges C with all possible values $s(C) \in [1,2]$.

ii) $r(K,C) = R(K,C)$ for $K = C$ and gauges C with all possible values $s(C) \in [1,2]$.

iii) $D_{\mathrm{MIN}}(K,C) = 3R(K,C)$ with C being a triangle as in $f_{\mathrm{MIN}}(\overline{\mathscr{C}^2},S)$.

iv) $3R(K,C) = 2D_{\mathrm{MIN}}(R,C)$ for $K = -S$ and $C = S \cap s(-S)$ with $s \in [1,2]$.

v) $r(K,C) + R(K,C) = D_{\mathrm{MIN}}(K,C)$ for $K = \lambda(-S) + (1-\lambda)S_{\mathrm{MIN}}$ with $\lambda \in [0,1]$ and $C = S_{\mathrm{MIN}}$.

The first two inequalities are trivial. From Lemma 2.16 we obtain the third and the fourth ones and from Theorem 1.2 the fifth one. The last inequality follows from Proposition 3.2, Remark 2.7, and the fact that $R(K,C) \leqslant D_{\mathrm{MIN}}(K,C)$. The equality cases follow from Lemma 2.8, Lemma 2.15, Corollary 1.7, and the fact that $R(-S,S_{\mathrm{MIN}}) = 2$.

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