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THE CLOSURE OF THE ANALYTIC TYPE TENT SPACES IN THE WEIGHTED–TYPE SPACE AND GENERALIZED WEIGHTED COMPOSITION OPERATORS

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Abstract. In this paper, the closure of analytic type tent spaces within the weighted-type space is characterized. The boundedness and compactness of generalized weighted composition operators on the closure of analytic type tent spaces in the weighted-type space are studied. Additionally, the boundedness, compactness, and essential norm of generalized weighted composition operators from the weighted-type space to analytic type tent spaces are also investigated.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Define $H(\mathbb{D})$ as the set of all analytic functions defined on $\mathbb{D}$. For $\beta > 0$, the weighted-type space H_β^∞ consists of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{H_\beta^\infty} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |f(z)| < \infty.$$

The weighted-type space H_β^∞ is a Banach space with respect to the norm $\|\cdot\|_{H_\beta^\infty}$. The little weighted-type space $H_{\beta,0}^\infty$, which is a subspace of H_β^∞, includes all $f \in H(\mathbb{D})$ that satisfy $\lim_{|z|\to 1}(1-|z|^2)^\beta |f(z)| = 0$. If $f' \in H_\beta^\infty$, we say that f belongs to the Bloch type space $\mathscr{B}^\beta$. In particular, when $\beta = 1$, $\mathscr{B}^\beta$ is the classical Bloch space, denoted by $\mathscr{B}$. As usual, H^∞ represents the space of bounded analytic functions in $\mathbb{D}$.

Let $\eta \in \mathbb{T}$ and $\zeta > \frac{1}{2}$. The non-tangential approach region $\Gamma_\zeta(\eta)$ is defined as

$$\Gamma(\eta) = \Gamma_\zeta(\eta) = \{z \in \mathbb{D} : |z-\eta| < \zeta(1-|z|^2)\}.$$

For $0 < p < \infty$ and $\alpha > -2$, the tent space $T_p^\infty(\alpha)$, introduced by Coifman, Meyer and Stein [9], consists of all measurable functions f on $\mathbb{D}$ for which

$$\|f\|_{T_p^\infty(\alpha)}^p = \operatorname{esssup}_{\eta\in\mathbb{T}} \sup_{u\in\Gamma(\eta)} \frac{1}{1-|u|^2} \int_{S(u)} |f(z)|^p (1-|z|^2)^{\alpha+1} dA(z) < \infty,$$

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where

$$S(re^{i\theta}) = \left\{ \lambda e^{it} : |t-\theta| \leqslant \frac{1-r}{2}, 1-\lambda \leqslant 1-r \right\}$$

for $re^{i\theta} \in \mathbb{D}\backslash\{0\}$, $S(0) = \mathbb{D}$ and $dA(z) = \frac{1}{\pi} dxdy$ is the normalized Lebesgue area measure on $\mathbb{D}$. In the above definition, the aperture ζ of the non-tangential region $\Gamma_\zeta(\eta)$ is suppressed because any two apertures yield function spaces with equivalent quasi-norms. We denote the intersection $T_p^\infty(\alpha) \cap H(\mathbb{D})$ by $AT_p^\infty(\alpha)$, called the analytic type tent space.

Let X be a subspace of the space Y. Denote $C_Y(X)$ as the closure of X in the Y-norm. Anderson, Clunie and Pommerenke [1] posed the following question: What is the closure of H^∞ in the Bloch norm? This problem remains unresolved. Ghatage and Zheng [12] characterized $C_{\mathscr{B}}(BMOA)$ and presented a detailed proof of Jones's Theorem. Specifically, a Bloch function f belongs to $C_{\mathscr{B}}(BMOA)$ if and only if for every $\varepsilon > 0$,

$$\sup_{a \in \mathbb{D}} \int_{\Omega_\varepsilon(f)} \frac{1-|\sigma_a(z)|^2}{(1-|z|^2)^2} dA(z) < \infty,$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ and

$$\Omega_\varepsilon(f) = \left\{ z \in \mathbb{D} : (1-|z|^2)|f'(z)| \geqslant \varepsilon \right\}.$$

When $1 < p < \infty$, Monreal Galán and Nicolau investigated $C_{\mathscr{B}}(H^p \cap \mathscr{B})$ in [18]. Zhao [34] studied the closure of a certain Möbius-invariant space in the Bloch space. Bao and Göğüş [3] characterized $C_{\mathscr{B}}(\mathscr{D}^2_\alpha \cap \mathscr{B})$ in the Bloch space. Liu and Rättyä described the closure of the weighted Bergman and Dirichlet spaces in the Bloch norm in [17]. For more results, see [4, 7, 11, 21] and the references therein. Inspired by these, the first purpose of this paper is to investigate the closure of the analytic type tent space in the weighted-type space H^∞_β.

Let $\mathbb{N}$ be the set of all positive integers. Let φ be an analytic self-map of $\mathbb{D}$, and $u \in H(\mathbb{D})$. In order to extend the weighted composition operators and the products of composition and differentiation operators (see, e.g., [14, 23, 26]), in [37] the first author introduced the following operator

$$(D^n_{\varphi,u} f)(z) = u(z) f^{(n)}(\varphi(z)), \quad z \in \mathbb{D},$$

which is called the generalized weighted composition operator or the weighted differentiation composition operator [25]. When $n = 0$, the operator $D^n_{\varphi,u}$ is precisely the weighted composition operator uC_φ. Specifically, when $u = 1$, the operator uC_φ is exactly the composition operator C_φ. The theory of generalized weighted composition operators on analytic function spaces has been extensively investigated. For further information and results on generalized weighted composition operators, their sums and extensions, one may consult the references [15, 16, 25, 27–31, 37–41].

In [13], He and Jiang studied the composition operator C_φ on weighted-type spaces and little weighted-type spaces. In [33], Wang and Liu gave some necessary and sufficient conditions for the operator uC_φ to be bounded, compact, and weakly compact on weighted-type spaces. See [10] for additional results of composition operators on weighted-type spaces. The second purpose of this paper is to provide characterizations of the boundedness, compactness, and essential norm of the operator $D^n_{\varphi,u}: H^\infty_\beta \to AT^\infty_p(\alpha)$.

Aulaskari and Zhao [2] studied composition operators from the Bloch space to the closure of certain Möbius invariant spaces in the Bloch space. Qian and Li [21] investigated composition operators from the logarithmic Bloch space to the closure of Dirichlet type spaces $\mathscr{D}^2_\alpha$ in the logarithmic Bloch space. However, to the best of our knowledge, no prior research has investigated generalized weighted composition operators or even weighted composition operators acting on the closure of certain function spaces. The final purpose of this paper is to study the boundedness and compactness of the operators $D^n_{\varphi,u}: H^\infty_\beta(H^\infty_{\beta,0}) \to C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)$ and $D^n_{\varphi,u}: C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta) \to C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)$.

The paper is organized as follows. In Section 2, we describe $C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)$, the closure of $AT^\infty_p(\alpha)$ in the weighted-type space. In Section 3, we characterize the boundedness, compactness, and essential norm of the operator $D^n_{\varphi,u}: H^\infty_\beta \to AT^\infty_p(\alpha)$. In Section 3, we study the operators $D^n_{\varphi,u}: H^\infty_\beta(H^\infty_{\beta,0}) \to C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)$ and $D^n_{\varphi,u}: C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta) \to C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)$.

Throughout this paper, we assert that $E \lesssim F$ if there exists a constant C such that $E \leqslant CF$. The notation $E \asymp F$ signifies that both $E \lesssim F$ and $F \lesssim E$.

2. $C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)$

First, we state some necessary notations and lemmas, which will be used in the main results of this section.

A positive measure μ on $\mathbb{D}$ is said to be an s-Carleson measure for $0 < s < \infty$ if

$$\sup_{I\subset\mathbb{T}} \frac{\mu(S(I))}{|I|^s} < \infty,$$

where $|I|$ is the arc length of subarc I of $\mathbb{T}$, and $S(I)$ denotes the Carleson box

$$S(I) = \left\{ z\in\mathbb{D} : \frac{z}{|z|} \in I, 1-|I| \leqslant |z| < 1 \right\}.$$

When $s=1$, a 1-Carleson measure coincides with the standard Carleson measure (see [5,6]). It is well known that μ is a Carleson measure if and only if for each (or some) $t>0$,

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}} d\mu(z) < \infty. \tag{1}$$

The following lemma is repeatedly used in the proof of main results in this paper.

LEMMA 1. [19, Lemma 2.5] *Let $s>-1$, $r,t>0$. If $r<s+2<t$, then*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\bar{a}z|^r|1-\bar{b}z|^t}dA(z) \lesssim \frac{1}{|1-\bar{a}b|^r(1-|b|^2)^{t-s-2}}$$

for all $a,b\in\mathbb{D}$.

We are now in a position to state and prove our main result in this section.

THEOREM 1. *Let $0<p,\beta<\infty$, $\alpha>-2$. The following assertions hold.*

(i) If $\beta<\frac{\alpha+2}{p}$, then $C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)=H^\infty_\beta$.

(ii) If $\beta>\frac{\alpha+2}{p}$, then $C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)=H^\infty_{\beta,0}$.

(iii) If $\beta=\frac{\alpha+2}{p}$, $p\geqslant 1$ and $f\in H^\infty_\beta$, then $f\in C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)$ if and only if for any $\varepsilon>0$ and $t>0$,

$$\sup_{a\in\mathbb{D}}\int_{\Omega^\beta_\varepsilon(f)} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)}dA(z)<\infty, \tag{2}$$

where

$$\Omega^\beta_\varepsilon(f)=\left\{z\in\mathbb{D}:(1-|z|^2)^\beta|f(z)|\geqslant\varepsilon\right\}.$$

Proof. First we observe that a function $f\in AT^\infty_p(\alpha)$ if and only if the measure $|f(z)|^p(1-|z|^2)^{\alpha+1}\,dA(z)$ is a Carleson measure. By (1), we see that $f\in AT^\infty_p(\alpha)$ if and only if

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}|f(z)|^p(1-|z|^2)^{\alpha+1}dA(z)<\infty \tag{3}$$

for any $t>0$.

(i) Let $\beta<\frac{\alpha+2}{p}$ and $f\in H^\infty_\beta$. Choose $t>\alpha+2-p\beta$. Using [36, Lemma 3.10], we obtain

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}|f(z)|^p(1-|z|^2)^{\alpha+1}dA(z)\\
\lesssim&\|f\|^p_{H^\infty_\beta}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|a|^2)^t(1-|z|^2)^{\alpha+1-p\beta}}{|1-\overline{a}z|^{t+1}}dA(z)\\
\lesssim&\|f\|^p_{H^\infty_\beta}\sup_{a\in\mathbb{D}}(1-|a|^2)^{\alpha+2-p\beta}\lesssim\|f\|^p_{H^\infty_\beta}<\infty,
\end{aligned}$$

which implies that $H^\infty_\beta\subseteq AT^\infty_p(\alpha)$. Consequently, $C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)=H^\infty_\beta$.

(ii) Let $\beta>\frac{\alpha+2}{p}$. Applying [8, Lemma 3.1], for any $f\in AT^\infty_p(\alpha)$, we get

$$|f(z)|\lesssim\frac{\|f\|_{AT^\infty_p(\alpha)}}{(1-|z|^2)^{\frac{\alpha+2}{p}}},\quad z\in\mathbb{D},$$

which implies that $\lim_{|z|\to 1}(1-|z|^2)^{\beta}|f(z)|=0$. Thus, $AT_p^{\infty}(\alpha)\subset H_{\beta,0}^{\infty}$. Since polynomials lie in $AT_p^{\infty}(\alpha)$, we get that $C_{H_\beta^\infty}(AT_p^{\infty}(\alpha)\cap H_\beta^{\infty})=H_{\beta,0}^{\infty}$.

(iii) *Necessity.* Let $\beta=\frac{\alpha+2}{p}$ and $f\in H_\beta^\infty$. Suppose that $f\in C_{H_\beta^\infty}(AT_p^{\infty}(\alpha)\cap H_\beta^{\infty})$. Then for any $\varepsilon>0$, there exists $g\in AT_p^{\infty}(\alpha)\cap H_\beta^{\infty}$ such that

$$\|f-g\|_{H_\beta^\infty}<\frac{\varepsilon}{2},$$

which implies that $\Omega_\varepsilon^\beta(f)\subset\Omega_{\frac{\varepsilon}{2}}^\beta(g)$. Hence, for any $t>0$, we obtain

$$\begin{aligned}
\infty &>\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}|g(z)|^p(1-|z|^2)^{\alpha+1}dA(z)\\
&\geqslant\sup_{a\in\mathbb{D}}\int_{\Omega_{\frac{\varepsilon}{2}}^\beta(g)}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}|g(z)|^p(1-|z|^2)^{\alpha+1}dA(z)\\
&\geqslant\left(\frac{\varepsilon}{2}\right)^p\sup_{a\in\mathbb{D}}\int_{\Omega_\varepsilon^\beta(f)}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)}dA(z),
\end{aligned}$$

as desired.

Sufficiency. Let $\varepsilon>0$ and $f\in H_\beta^\infty$. Assume that (2) holds. Choose $\gamma>\beta$ large enough. For any $z\in\mathbb{D}$, by [36, Theorem 4.24] we have

$$f(z)=(\gamma+1)\int_{\mathbb{D}}\frac{f(w)(1-|w|^2)^\gamma}{(1-z\overline{w})^{2+\gamma}}dA(w).$$

Write $f(z)=f_1(z)+f_2(z)$, where

$$f_1(z)=(\gamma+1)\int_{\Omega_\varepsilon^\beta(f)}\frac{f(w)(1-|w|^2)^\gamma}{(1-z\overline{w})^{2+\gamma}}dA(w)$$

and

$$f_2(z)=(\gamma+1)\int_{\mathbb{D}\setminus\Omega_\varepsilon^\beta(f)}\frac{f(w)(1-|w|^2)^\gamma}{(1-z\overline{w})^{2+\gamma}}dA(w).$$

Using [36, Lemma 3.10], we get

$$\begin{aligned}
&\|f-f_1\|_{H_\beta^\infty}\asymp\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f_2(z)|\\
&\lesssim\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\int_{\mathbb{D}\setminus\Omega_\varepsilon^\beta(f)}\frac{|f(w)|(1-|w|^2)^\gamma}{|1-z\overline{w}|^{2+\gamma}}dA(w)\\
&=\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\int_{\mathbb{D}\setminus\Omega_\varepsilon^\beta(f)}\frac{|f(w)|(1-|w|^2)^\beta(1-|w|^2)^{\gamma-\beta}}{|1-z\overline{w}|^{2+\gamma}}dA(w)\\
&\lesssim\varepsilon\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\int_{\mathbb{D}}\frac{(1-|w|^2)^{\gamma-\beta}}{|1-z\overline{w}|^{2+\gamma}}dA(w)\lesssim\varepsilon.
\end{aligned}$$

Hence, $f_1 \in H_\beta^\infty$.

To complete the proof, it suffices to show that $f_1 \in AT_p^\infty(\alpha)$. Consider $0 < t < \beta$. Using Fubini's theorem, we deduce that

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}|f_1(z)|^p(1-|z|^2)^{\alpha+1}dA(z)\\
=&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}|f_1(z)|^{p-1}|f_1(z)|(1-|z|^2)^{\alpha+1}dA(z)\\
\lesssim&\|f_1\|_{H_\beta^\infty}^{p-1}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|a|^2)^t(1-|z|^2)^{\alpha-p\beta+1+\beta}}{|1-\overline{a}z|^{t+1}}|f_1(z)|dA(z)\\
\lesssim&\|f_1\|_{H_\beta^\infty}^{p-1}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|a|^2)^t(1-|z|^2)^{\alpha-p\beta+1+\beta}}{|1-\overline{a}z|^{t+1}}\int_{\Omega_\varepsilon^\beta(f)}\frac{|f(w)|(1-|w|^2)^\gamma}{|1-z\overline{w}|^{2+\gamma}}dA(w)dA(z)\\
\lesssim&\|f_1\|_{H_\beta^\infty}^{p-1}\|f\|_{H_\beta^\infty}\\
&\times\sup_{a\in\mathbb{D}}\int_{\Omega_\varepsilon^\beta(f)}(1-|a|^2)^t(1-|w|^2)^{\gamma-\beta}\int_{\mathbb{D}}\frac{(1-|z|^2)^{\alpha-p\beta+1+\beta}dA(z)}{|1-\overline{a}z|^{t+1}|1-z\overline{w}|^{2+\gamma}}dA(w).
\end{aligned}$$

Recalling that $\gamma > \beta = \frac{\alpha+2}{p}$, and applying Lemma 1, we obtain

$$\int_{\mathbb{D}}\frac{(1-|z|^2)^{\alpha-p\beta+1+\beta}}{|1-\overline{a}z|^{t+1}|1-z\overline{w}|^{2+\gamma}}dA(z) \lesssim \frac{1}{|1-\overline{a}w|^{t+1}(1-|w|^2)^{\gamma-\beta+1}}.$$

Therefore,

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}|f_1(z)|^p(1-|z|^2)^{\alpha+1}dA(z)\\
\lesssim&\|f_1\|_{H_\beta^\infty}^{p-1}\|f\|_{H_\beta^\infty}\sup_{a\in\mathbb{D}}\int_{\Omega_\varepsilon^\beta(f)}\frac{(1-|a|^2)^t}{|1-\overline{a}w|^{t+1}(1-|w|^2)}dA(w) < \infty.
\end{aligned}$$

It follows that $f_1 \in AT_p^\infty(\alpha)$. The proof is complete. □

3. $D_{\varphi,u}^n : H_\beta^\infty \to AT_p^\infty(\alpha)$

In this section, we study the boundedness, compactness and essential norm of the operator $D_{\varphi,u}^n : H_\beta^\infty \to AT_p^\infty(\alpha)$. Based on the well-established result that $H_\beta^\infty = \mathscr{B}^{\beta+1}$ and the higher-order derivatives characterization of the Bloch type space (see [35]), the following lemma is obtained.

LEMMA 2. *Let $\beta > 0$, $n \in \mathbb{N}$ and $f \in H(\mathbb{D})$. Then $f \in H_\beta^\infty$ if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta+n}|f^{(n)}(z)| < \infty.$$

Moreover, $\|f\|_{H_\beta^\infty}$ *is equivalent to* $\|f\|_{H_\beta^{\infty,n}}$*, where*

$$\|f\|_{H_\beta^{\infty,n}} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^{\beta+n} |f^{(n)}(z)|.$$

The following lemma will play a crucial role in the proof of our main theorem. It can be found in [20, Lemma 3.1].

LEMMA 3. *Let* $0<\tau<\infty$, $1<\lambda<\infty$ *and* $e^{-\tau/\lambda} \leqslant r_0 < 1$. *Then there is a positive constant* C*, depending only on* λ, τ *and* r_0*, such that*

$$\sum_{j=1}^{\infty} \lambda^{j\lambda\tau} r^{\lambda^{j+1}} \geqslant \frac{C}{(1-r^2)^{\lambda\tau}}$$

for all $r_0 \leqslant r < 1$.

THEOREM 2. *Let* $0<p,\beta<\infty$, $n\in\mathbb{N}\cup\{0\}$, $\alpha>-2$, $u\in H(\mathbb{D})$ *and* φ *be an analytic self-map of* $\mathbb{D}$. *Then* $D_{\varphi,u}^n : H_\beta^\infty \to AT_p^\infty(\alpha)$ *is bounded if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\bar{a}z|^{t+1}} \frac{|u(z)|^p (1-|z|^2)^{\alpha+1}}{(1-|\varphi(z)|^2)^{p(\beta+n)}} dA(z) < \infty$$

for any $t>0$.

Proof. Sufficiency. Let $f\in H_\beta^\infty$. Using Lemma 2 we obtain

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\bar{a}z|^{t+1}} |(D_{\varphi,u}^n f)(z)|^p (1-|z|^2)^{\alpha+1} dA(z)\\
=&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\bar{a}z|^{t+1}} |u(z)|^p |f^{(n)}(\varphi(z))|^p (1-|z|^2)^{\alpha+1} dA(z)\\
\lesssim&\|f\|_{H_\beta^{\infty,n}}^p \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\bar{a}z|^{t+1}} \frac{|u(z)|^p (1-|z|^2)^{\alpha+1}}{(1-|\varphi(z)|^2)^{p(\beta+n)}} dA(z)\\
<&\infty
\end{aligned}$$

for any $t>0$. Hence, $D_{\varphi,u}^n : H_\beta^\infty \to AT_p^\infty(\alpha)$ is bounded.

Necessity. For each $0<\beta<\infty$, $0\leqslant\theta<2\pi$ and $0<s\leqslant 1$, set

$$f_{\theta,s}(z) = \sum_{j=0}^{\infty} 2^{j\beta} (se^{i\theta})^{2^j} z^{2^j}, \quad z\in\mathbb{D}. \tag{4}$$

From [32, Lemma 2], we see that $f_{\theta,s}\in H_\beta^\infty$ and $\|f_{\theta,s}\|_{H_\beta^\infty} \lesssim 1$, which is independent of θ and s. For simplicity, denote $f_{\theta,1}$ by f_θ. Using Fubini's theorem, we get

$$\begin{aligned}
1 &\gtrsim \int_0^{2\pi} \|D_{\varphi,u}^n f_\theta\|_{AT_p^\infty(\alpha)}^p \frac{d\theta}{2\pi}\\
&\asymp \sup_{a\in\mathbb{D}} \int_0^{2\pi} \left(\int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\bar{a}z|^{t+1}} |(D_{\varphi,u}^n f_\theta)(z)|^p (1-|z|^2)^{\alpha+1} dA(z) \right) \frac{d\theta}{2\pi}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{a\in\mathbb{D}} \int_0^{2\pi} \left(\int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}} |u(z)|^p |f_\theta^{(n)}(\varphi(z))|^p (1-|z|^2)^{\alpha+1} dA(z) \right) \frac{d\theta}{2\pi} \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}} |u(z)|^p (1-|z|^2)^{\alpha+1} \left(\int_0^{2\pi} |f_\theta^{(n)}(\varphi(z))|^p \frac{d\theta}{2\pi} \right) dA(z)
\end{aligned}
$$

for any $t>0$. Using the following inequality (see [42] or [20, Lemma F])

$$
\prod_{k=0}^{n-1}(2^j-k) \geqslant \frac{2^{nj}}{n!}, \qquad n\in\mathbb{N},\ j\in\mathbb{N},\ 2^j\geqslant n,
$$

we have

$$
\begin{aligned}
\int_0^{2\pi} |f_\theta^{(n)}(\varphi(z))|^p \frac{d\theta}{2\pi} &\geqslant \int_0^{2\pi} \left| \sum_{j\geqslant[\log_2 n]+1}^{\infty} \prod_{k=0}^{n-1}(2^j-k) 2^{j\beta} \varphi(z)^{2^j-n} e^{i\theta 2^j} \right|^p \frac{d\theta}{2\pi} \\
&\gtrsim \left(\sum_{j\geqslant[\log_2 n]+1}^{\infty} \left(\prod_{k=0}^{n-1}(2^j-k) \right)^2 2^{2j\beta} |\varphi(z)|^{2(2^j-n)} \right)^{\frac{p}{2}} \\
&\gtrsim \left(\sum_{j\geqslant[\log_2 n]+1}^{\infty} 2^{2j(\beta+n)} |\varphi(z)|^{2(2^j-n)} \right)^{\frac{p}{2}}.
\end{aligned}
$$

Using Lemma 3, with $r_0=e^{-\tau/2}$, there is a positive constant C, depending only on τ and n, such that

$$
\sum_{j=1}^{\infty} 2^{2j(\beta+n)} |\varphi(z)|^{2^{j+1}} \geqslant \frac{C}{(1-|\varphi(z)|^2)^{2(\beta+n)}}
$$

for all $e^{-\tau/2}\leqslant|\varphi(z)|<1$. Hence,

$$
\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\geqslant e^{-\tau/2}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}} \frac{|u(z)|^p(1-|z|^2)^{\alpha+1}}{(1-|\varphi(z)|^2)^{p(\beta+n)}} dA(z) \lesssim 1
$$

for any $t>0$. Since $D_{\varphi,u}^n : H_\beta^\infty \to AT_p^\infty(\alpha)$ is bounded, we get that $u\in AT_p^\infty(\alpha)$. Hence,

$$
\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|<e^{-\tau/2}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}} \frac{|u(z)|^p(1-|z|^2)^{\alpha+1}}{(1-|\varphi(z)|^2)^{p(\beta+n)}} dA(z) \lesssim \|u\|_{AT_p^\infty(\alpha)}^p
$$

for any $t>0$. Therefore,

$$
\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}} \frac{|u(z)|^p(1-|z|^2)^{\alpha+1}}{(1-|\varphi(z)|^2)^{p(\beta+n)}} dA(z) < \infty
$$

for any $t>0$. The proof is complete. $\square$

Next, we consider the compactness and essential norm of $D_{\varphi,u}^n : H_\beta^\infty \to AT_p^\infty(\alpha)$. Recall that the essential norm of $T:X\to Y$ is defined by

$$
\|T\|_{e,X\to Y} = \inf_K \{\|T-K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\},
$$

where $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ are Banach spaces and $T:X\to Y$ is a bounded linear operator. It is clear that $T:X\to Y$ is compact if and only if $\|T\|_{e,X\to Y}=0$.

For each $m\in\mathbb{N}$, set

$$g^m_{\theta,s}(z)=z^m f_{\theta,s}(z).$$

Here $f_{\theta,s}$ represents the test function defined in (4). Given that

$$z^m H^\infty_{\beta,0}\subseteq H^\infty_{\beta,0},$$

it is evident that $\{g^m_{\theta,s}\}_{m\in\mathbb{N}}\subseteq H^\infty_{\beta,0}$, and the norm $\|g^m_{\theta,s}\|_{H^\infty_\beta}$ is uniformly bounded with respect to θ, s and m. By [32, Lemma 4], for every $\Lambda\in(H^\infty_\beta)^*$,

$$\sup_{\theta,s}|\Lambda(g^m_{\theta,s})|\to 0$$

as $m\to\infty$. This shows that $\{g^m_{\theta,s}\}_{m\in\mathbb{N}}$ converges weakly to 0 in H^∞_β. By the complete continuity of compact operators, we get the following lemma. We omit the detail of the proof.

LEMMA 4. *Let $0<p,\beta<\infty$, $m\in\mathbb{N}\cup\{0\}$ and $\alpha>-2$. For any compact operator $T:H^\infty_\beta\to AT^\infty_p(\alpha)$, it holds that*

$$\lim_{m\to\infty}\sup_{\theta,s}\|Tg^m_{\theta,s}\|_{AT^\infty_p(\alpha)}=0,$$

where the supremum is taken over all $0\leqslant\theta<2\pi$ and $0<s<1$.

The following compactness criterion for the operator $D^n_{\varphi,u}:H^\infty_\beta\to AT^\infty_p(\alpha)$ is available (see [10, Proposition 3.11]).

LEMMA 5. *Let $0<p,\beta<\infty$, $n\in\mathbb{N}\cup\{0\}$, $\alpha>-2$, $u\in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$ such that $D^n_{\varphi,u}:H^\infty_\beta\to AT^\infty_p(\alpha)$ is bounded. Then $D^n_{\varphi,u}:H^\infty_\beta\to AT^\infty_p(\alpha)$ is compact if and only if for every bounded sequence $\{f_j\}$ in H^∞_β which converges to 0 uniformly on compact subsets of $\mathbb{D}$, we have*

$$\lim_{j\to\infty}\|D^n_{\varphi,u}f_j\|_{AT^\infty_p(\alpha)}=0.$$

THEOREM 3. *Let $0<p,\beta<\infty$, $n\in\mathbb{N}\cup\{0\}$, $\alpha>-2$, $u\in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Suppose that $D^n_{\varphi,u}:H^\infty_\beta\to AT^\infty_p(\alpha)$ is bounded. Then it holds that*

$$\|D^n_{\varphi,u}\|^p_{e,H^\infty_\beta\to AT^\infty_p(\alpha)}\asymp\limsup_{r\to1}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}\frac{|u(z)|^p(1-|z|^2)^{\alpha+1}}{(1-|\varphi(z)|^2)^{p(\beta+n)}}dA(z)\quad(5)$$

for any $t>0$.

Proof. First, we prove the lower estimates in (5). It is clear that the norm $\|g_{\theta,s}^m\|_{H_\beta^\infty}$ is uniformly bounded on θ, s and m. For any compact operators $K : H_\beta^\infty \to AT_p^\infty(\alpha)$, we deduce that

$$\|D_{\varphi,u}^n - K\| \gtrsim \|(D_{\varphi,u}^n - K)g_{\theta,s}^m\|_{AT_p^\infty(\alpha)} \geqslant \|D_{\varphi,u}^n g_{\theta,s}^m\|_{AT_p^\infty(\alpha)} - \|Kg_{\theta,s}^m\|_{AT_p^\infty(\alpha)}$$

for any θ, s and m. By Fatou's lemma we have that

$$\begin{aligned}
&\sup_{\theta,s} \|D_{\varphi,u}^n g_{\theta,s}^m\|_{AT_p^\infty(\alpha)}^p \\
&\geqslant \liminf_{s\to 1} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}} |u(z)|^p |(g_{\theta,s}^m)^{(n)}(\varphi(z))|^p (1-|z|^2)^{\alpha+1} dA(z) \\
&\geqslant \int_{|\varphi(z)|>r} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}} |u(z)|^p |\varphi(z)|^{pm} |f_\theta^{(n)}(\varphi(z))|^p (1-|z|^2)^{\alpha+1} dA(z)
\end{aligned}$$

for any $r \in (0,1)$ and $t > 0$. By integrating these inequalities with respect to θ from 0 to 2π and using Fubini's theorem, we get

$$\begin{aligned}
&\int_0^{2\pi} \sup_{\theta,s} \|D_{\varphi,u}^n g_{\theta,s}^m\|_{AT_p^\infty(\alpha)}^p \frac{d\theta}{2\pi} \\
&\geqslant \int_{|\varphi(z)|>r} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}} |u(z)|^p |\varphi(z)|^{pm} (1-|z|^2)^{\alpha+1} \left(\int_0^{2\pi} |f_\theta^{(n)}(\varphi(z))|^p \frac{d\theta}{2\pi} \right) dA(z)
\end{aligned}$$

for any $t > 0$. From the proof of Theorem 2, it follows that

$$\int_0^{2\pi} |f_\theta^{(n)}(\varphi(z))|^p \frac{d\theta}{2\pi} \gtrsim \frac{1}{(1-|\varphi(z)|^2)^{p(\beta+n)}}$$

for any $z \in \mathbb{D}$ with $|\varphi(z)| \geqslant e^{-\tau/2}$. Hence,

$$\sup_{\theta,s} \|D_{\varphi,u}^n g_{\theta,s}^m\|_{AT_p^\infty(\alpha)}^p \gtrsim \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}} \frac{|u(z)|^p (1-|z|^2)^{\alpha+1}}{(1-|\varphi(z)|^2)^{p(\beta+n)}} dA(z)$$

for any $r \in (e^{-\tau/2}, 1)$ and $t > 0$. Letting $r \to 1$, we have

$$\sup_{\theta,s} \|D_{\varphi,u}^n g_{\theta,s}^m\|_{AT_p^\infty(\alpha)}^p \gtrsim \limsup_{r\to 1} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}} \frac{|u(z)|^p (1-|z|^2)^{\alpha+1}}{(1-|\varphi(z)|^2)^{p(\beta+n)}} dA(z)$$

for any $t > 0$. Note that this estimate does not depend on m. Given that

$$\lim_{m\to\infty} \sup_{\theta,s} \|Kg_{\theta,s}^m\|_{AT_p^\infty(\alpha)} = 0$$

for any compact operator $K : H_\beta^\infty \to AT_p^\infty(\alpha)$ by Lemma 4, it follows that

$$\|D_{\varphi,u}^n\|_{e,H_\beta^\infty \to AT_p^\infty(\alpha)}^p \gtrsim \limsup_{r\to 1} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}} \frac{|u(z)|^p (1-|z|^2)^{\alpha+1}}{(1-|\varphi(z)|^2)^{p(\beta+n)}} dA(z)$$

for any $t>0$.

Now, we prove the upper estimates in (5). For each positive integer j and $f\in H(\mathbb{D})$, set

$$C_jf(z)=f\left(\frac{jz}{j+1}\right).$$

It is easy to see that C_j is bounded on H^∞_β. Using Lemma 5 to the case $u\equiv 1$ and $\varphi(z)=\frac{jz}{j+1}$, we get that C_j is compact on H^∞_β. So,

$$\begin{aligned}\|D^n_{\varphi,u}\|_{e,H^\infty_\beta\to AT^\infty_p(\alpha)}\leqslant&\liminf_{j\to\infty}\|D^n_{\varphi,u}-D^n_{\varphi,u}C_j\|\\ =&\liminf_{j\to\infty}\sup_{\|f\|_{H^\infty_\beta}\leqslant 1}\|D^n_{\varphi,u}(id-C_j)f\|_{AT^\infty_p(\alpha)},\end{aligned}$$

where id denotes the identity operator on H^∞_β. Fix a positive integer j and an $f\in H^\infty_\beta$ with $\|f\|_{H^\infty_\beta}\leqslant 1$, we have

$$\begin{aligned}&\|D^n_{\varphi,u}(id-C_j)f\|^p_{AT^\infty_p(\alpha)}\\ \lesssim&\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}|u(z)|^p\left|f^{(n)}(\varphi(z))-f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right|^p(1-|z|^2)^{\alpha+1}dA(z)\\ &+\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|\leqslant r}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}|u(z)|^p\left|f^{(n)}(\varphi(z))-f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right|^p(1-|z|^2)^{\alpha+1}dA(z)\end{aligned}$$

for any $r\in(0,1)$ and $t>0$. Using Lemma 2 we get

$$\left|f^{(n)}(\varphi(z))-f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right|\lesssim\frac{\|f\|_{H^\infty_\beta}}{(1-|\varphi(z)|^2)^{\beta+n}}.$$

This implies that

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}|u(z)|^p\left|f^{(n)}(\varphi(z))-f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right|^p(1-|z|^2)^{\alpha+1}dA(z)\\ \lesssim&\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}\frac{|u(z)|^p(1-|z|^2)^{\alpha+1}}{(1-|\varphi(z)|^2)^{p(\beta+n)}}dA(z)\end{aligned}$$

for any $r\in(0,1)$ and $t>0$. It should be noted that this estimate is independent of j.

Now let us prove that

$$\begin{aligned}&\lim_{j\to\infty}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|\leqslant r}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}|u(z)|^p\left|f^{(n)}(\varphi(z))-f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right|^p(1-|z|^2)^{\alpha+1}dA(z)\\ =&0\end{aligned}$$

for any $r\in(0,1)$ and $t>0$. Let $\rho=\varphi(z)$ and denote the radial segment by $\left[\frac{j\rho}{j+1},\rho\right]$. By integrating $f^{(n+1)}$ along $\left[\frac{j\rho}{j+1},\rho\right]$, we have

$$\left|f^{(n)}(\rho)-f^{(n)}\left(\frac{j\rho}{j+1}\right)\right|\leqslant\frac{|\rho|}{j+1}|f^{(n+1)}(\xi(\rho))|$$

for some $\xi(\rho)\in\left[\frac{j\rho}{j+1},\rho\right]$. An application of Cauchy's estimate to $f^{(n+1)}$ on the circle with center at $\xi(\rho)$ and radius $R\in(0,1-r)$ shows that

$$|f^{(n+1)}(\xi(\rho))|\leqslant\frac{1}{R}\max_{|\zeta|=R+r}|f^{(n)}(\zeta)|.$$

Hence,

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|\leqslant r}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}|u(z)|^p\left|f^{(n)}(\varphi(z))-f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right|^p(1-|z|^2)^{\alpha+1}dA(z)\\ \lesssim&\frac{r^p}{R^p(j+1)^p}\frac{1}{[1-(R+r)^2]^{p(\beta+n)}}\|u\|^p_{AT_p^\infty(\alpha)}\end{aligned}$$

for any $t>0$. By the assumption, we see that $u\in AT_p^\infty(\alpha)$. Hence,

$$\begin{aligned}&\lim_{j\to\infty}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|\leqslant r}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}|u(z)|^p\left|f^{(n)}(\varphi(z))-f^{(n)}\left(\frac{j\varphi(z)}{j+1}\right)\right|^p(1-|z|^2)^{\alpha+1}dA(z)\\ =&0\end{aligned}$$

for any $t>0$. Therefore,

$$\|D^n_{\varphi,u}\|^p_{e,H^\infty_\beta\to AT_p^\infty(\alpha)}\lesssim\limsup_{r\to1}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}\frac{|u(z)|^p(1-|z|^2)^{\alpha+1}}{(1-|\varphi(z)|^2)^{p(\beta+n)}}dA(z)$$

for any $t>0$. The proof is complete. □

COROLLARY 1. *Let $0<p,\beta<\infty$, $n\in\mathbb{N}\cup\{0\}$, $\alpha>-2$, $u\in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $D^n_{\varphi,u}:H^\infty_\beta\to AT_p^\infty(\alpha)$ is compact if and only if $D^n_{\varphi,u}:H^\infty_\beta\to AT_p^\infty(\alpha)$ is bounded and*

$$\limsup_{r\to1}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}}\frac{|u(z)|^p(1-|z|^2)^{\alpha+1}}{(1-|\varphi(z)|^2)^{p(\beta+n)}}dA(z)=0$$

for any $t>0$.

4. $D^n_{\varphi,u}:H^\infty_\beta\to C_{H^\infty_\beta}(AT_p^\infty(\alpha)\cap H^\infty_\beta)$

In this section, we study the boundedness and compactness of $D^n_{\varphi,u}:H^\infty_\beta(H^\infty_{\beta,0})\to C_{H^\infty_\beta}(AT_p^\infty(\alpha)\cap H^\infty_\beta)$ and $D^n_{\varphi,u}:C_{H^\infty_\beta}(AT_p^\infty(\alpha)\cap H^\infty_\beta)\to C_{H^\infty_\beta}(AT_p^\infty(\alpha)\cap H^\infty_\beta)$. Throughout this section, for the simplicity of the notation, we use $\varphi^{u,n}_\beta(z)$ to denote

$$\frac{u(z)(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\beta+n}}.$$

THEOREM 4. *Let* $1 \leqslant p < \infty$, $n \in \mathbb{N} \cup \{0\}$, $\alpha > -2$, $\beta = \frac{\alpha+2}{p}$, $u \in H(\mathbb{D})$ *and* φ *be an analytic self-map of* $\mathbb{D}$. *Then* $D^n_{\varphi,u} : H^\infty_\beta \to C_{H^\infty_\beta}(AT^\infty_p(\alpha) \cap H^\infty_\beta)$ *is bounded if and only if for any* $\varepsilon > 0$ *and* $t > 0$,

$$\sup_{a\in\mathbb{D}} \int_{|\varphi^{u,n}_\beta(z)|\geqslant\varepsilon} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)} dA(z) < \infty. \tag{6}$$

Proof. Necessity. Suppose that $D^n_{\varphi,u} : H^\infty_\beta \to C_{H^\infty_\beta}(AT^\infty_p(\alpha) \cap H^\infty_\beta)$ is bounded. Since

$$(1-|z|^2)^{\beta+1}|f'(z)| + |f(0)| \asymp (1-|z|^2)^\beta |f(z)|,$$

by Theorem 3 in [22] (see also [24, Lemma 7]) it easily follows that there exist $g_1, g_2 \in H^\infty_\beta$ such that

$$|g_1'(z)| + |g_2'(z)| \geqslant \frac{1}{(1-|z|^2)^{\beta+1}}, \ z \in \mathbb{D}.$$

Following this rule, we see that there exist $h_1, h_2 \in H^\infty_\beta$ such that

$$|h_1^{(n)}(z)| + |h_2^{(n)}(z)| \geqslant \frac{1}{(1-|z|^2)^{\beta+n}}, \ z \in \mathbb{D}.$$

By assumption, we have $D^n_{\varphi,u}h_1,\ D^n_{\varphi,u}h_2 \in C_{H^\infty_\beta}(AT^\infty_p(\alpha) \cap H^\infty_\beta)$. Thus, for any $\varepsilon > 0$, by Theorem 1 we get

$$\sup_{a\in\mathbb{D}} \int_{\Omega^\beta_{\frac{\varepsilon}{2}}(D^n_{\varphi,u}h_1)} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)} dA(z) < \infty$$

and

$$\sup_{a\in\mathbb{D}} \int_{\Omega^\beta_{\frac{\varepsilon}{2}}(D^n_{\varphi,u}h_2)} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)} dA(z) < \infty$$

for any $t > 0$. Furthermore, when $|\varphi^{u,n}_\beta(z)| \geqslant \varepsilon$, we obtain

$$\begin{aligned}
&\Big(|(D^n_{\varphi,u}h_1)(z)| + |(D^n_{\varphi,u}h_2)(z)|\Big)(1-|z|^2)^\beta \\
=&\Big(|h_1^{(n)}(\varphi(z))| + |h_2^{(n)}(\varphi(z))|\Big)|u(z)|(1-|z|^2)^\beta \\
\geqslant&\frac{|u(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\beta+n}} \geqslant \varepsilon,
\end{aligned}$$

which implies that either

$$|(D^n_{\varphi,u}h_1)(z)|(1-|z|^2)^\beta \geqslant \frac{\varepsilon}{2}$$

or

$$|(D^n_{\varphi,u}h_2)(z)|(1-|z|^2)^\beta \geqslant \frac{\varepsilon}{2}.$$

Therefore, for any $\varepsilon > 0$,

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{|\varphi^{u,n}_\beta(z)|\geqslant\varepsilon}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)}dA(z)\\
\leqslant&\sup_{a\in\mathbb{D}}\int_{\Omega^\beta_{\frac{\varepsilon}{2}}(D^n_{\varphi,u}h_1)}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)}dA(z)\\
&\quad+\sup_{a\in\mathbb{D}}\int_{\Omega^\beta_{\frac{\varepsilon}{2}}(D^n_{\varphi,u}h_2)}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)}dA(z)\\
<&\infty
\end{aligned}$$

for any $t > 0$. The desired result follows.

Sufficiency. Let $f \in H^\infty_\beta$. Then

$$|(D^n_{\varphi,u}f)(z)|(1-|z|^2)^\beta = |f^{(n)}(\varphi(z))|(1-|\varphi(z)|^2)^{\beta+n}|\varphi^{u,n}_\beta(z)| \leqslant \|f\|_{H^{\infty,n}_\beta}|\varphi^{u,n}_\beta(z)|.$$

Hence, given any $\delta > 0$, it follows that if $|(D^n_{\varphi,u}f)(z)|(1-|z|^2)^\beta > \delta$ then $|\varphi^{u,n}_\beta(z)| \geqslant \varepsilon$, where $\varepsilon = \frac{\delta}{\|f\|_{H^{\infty,n}_\beta}}$. Therefore, for any $t > 0$,

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{\Omega^\beta_\delta(D^n_{\varphi,u}f)}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)}dA(z)\\
\lesssim&\sup_{a\in\mathbb{D}}\int_{|\varphi^{u,n}_\beta(z)|\geqslant\varepsilon}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)}dA(z) < \infty.
\end{aligned}$$

Using Theorem 1, it follows that $D^n_{\varphi,u}f \in C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)$. That is, $D^n_{\varphi,u} : H^\infty_\beta \to C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)$ is bounded. The proof is complete. □

THEOREM 5. *Let* $1 \leqslant p < \infty$, $n \in \mathbb{N}\cup\{0\}$, $\alpha > -2$, $\beta = \frac{\alpha+2}{p}$, $u \in H(\mathbb{D})$ *and* φ *be an analytic self-map of* $\mathbb{D}$. *Then* $D^n_{\varphi,u} : H^\infty_{\beta,0} \to C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)$ *is bounded if and only if* $u \in C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)$ *and*

$$\sup_{z\in\mathbb{D}}|\varphi^{u,n}_\beta(z)| < \infty.$$

Proof. Necessity. Assume that $D^n_{\varphi,u} : H^\infty_{\beta,0} \to C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta)$ is bounded. Observing that $f(z) = z^n \in H^\infty_{\beta,0}$, we get

$$u = D^n_{\varphi,u}f \in C_{H^\infty_\beta}(AT^\infty_p(\alpha)\cap H^\infty_\beta).$$

Since $C_{H_\beta^\infty}(AT_p^\infty(\alpha)\cap H_\beta^\infty)$ is a subspace of H_β^∞ and $D_{\varphi,u}^n : H_{\beta,0}^\infty \to C_{H_\beta^\infty}(AT_p^\infty(\alpha)\cap H_\beta^\infty)$ is bounded, we have that $D_{\varphi,u}^n : H_{\beta,0}^\infty \to H_\beta^\infty$ is bounded. For any $a\in\mathbb{D}$, by taking the test function

$$f_a(z) = \frac{1-|a|^2}{(1-\bar{a}z)^{\beta+1}}, \quad z\in\mathbb{D},$$

and using the fact that $u\in H_\beta^\infty$, after a calculation, we see that $\sup_{z\in\mathbb{D}}|\varphi_\beta^{u,n}(z)|<\infty$.

Sufficiency. Suppose that $u\in C_{H_\beta^\infty}(AT_p^\infty(\alpha)\cap H_\beta^\infty)$ and $\sup_{z\in\mathbb{D}}|\varphi_\beta^{u,n}(z)|\leqslant C<\infty$. Let $f\in H_{\beta,0}^\infty$. For any $\varepsilon>0$, there is a constant r $(0<r<1)$ such that

$$|f^{(n)}(z)|(1-|z|^2)^{\beta+n} < \frac{\varepsilon}{C}$$

whenever $|z|>r$. Let $z\in\Omega_\varepsilon^\beta(D_{\varphi,u}^n f)$. Then,

$$\begin{aligned}\varepsilon \leqslant& |(D_{\varphi,u}^n f)(z)|(1-|z|^2)^\beta = |f^{(n)}(\varphi(z))|(1-|\varphi(z)|^2)^{\beta+n}\frac{|u(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\beta+n}}\\ \leqslant& C|f^{(n)}(\varphi(z))|(1-|\varphi(z)|^2)^{\beta+n},\end{aligned}$$

which implies that $|\varphi(z)|\leqslant r$. Hence,

$$\varepsilon \leqslant |f^{(n)}(\varphi(z))||u(z)|(1-|z|^2)^\beta \leqslant \frac{\|f\|_{H_\beta^{\infty,n}}}{(1-r^2)^{\beta+n}}|u(z)|(1-|z|^2)^\beta.$$

Let $\delta = \frac{\varepsilon(1-r^2)^{\beta+n}}{\|f\|_{H_\beta^{\infty,n}}}$. Then $|u(z)|(1-|z|^2)^\beta \geqslant \delta$. Therefore,

$$\Omega_\varepsilon^\beta(D_{\varphi,u}^n f)\subseteq \Omega_\delta^\beta(u).$$

Since $u\in C_{H_\beta^\infty}(AT_p^\infty(\alpha)\cap H_\beta^\infty)$, using Theorem 1, we have

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{\Omega_\varepsilon^\beta(D_{\varphi,u}^n f)}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)}dA(z)\\ \leqslant& \sup_{a\in\mathbb{D}}\int_{\Omega_\delta^\beta(u)}\frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)}dA(z)<\infty\end{aligned}$$

for any $t>0$. Using Theorem 1 again, we have

$$D_{\varphi,u}^n f\in C_{H_\beta^\infty}(AT_p^\infty(\alpha)\cap H_\beta^\infty).$$

Therefore, $D_{\varphi,u}^n : H_{\beta,0}^\infty \to C_{H_\beta^\infty}(AT_p^\infty(\alpha)\cap H_\beta^\infty)$ is bounded. The proof is complete. □

Similarly to the proof of Theorem 2.3 in [39], we have the following lemma. We omit the details.

LEMMA 6. *Let $n \in \mathbb{N} \cup \{0\}$, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $D^n_{\varphi,u} : H^\infty_\beta \to H^\infty_\beta$ *is compact.*

(ii) $D^n_{\varphi,u} : H^\infty_{\beta,0} \to H^\infty_\beta$ *is compact.*

(iii) $u \in H^\infty_\beta$ *and* $\lim_{|\varphi(z)|\to 1} |\varphi^{u,n}_\beta(z)| = 0$.

THEOREM 6. *Let $1 \leqslant p < \infty$, $n \in \mathbb{N} \cup \{0\}$, $\alpha > -2$, $\beta = \frac{\alpha+2}{p}$, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $D^n_{\varphi,u} : H^\infty_\beta \to C_{H^\infty_\beta}(AT^\infty_p(\alpha) \cap H^\infty_\beta)$ *is compact;*

(ii) $D^n_{\varphi,u} : H^\infty_{\beta,0} \to C_{H^\infty_\beta}(AT^\infty_p(\alpha) \cap H^\infty_\beta)$ *is compact;*

(iii) $u \in C_{H^\infty_\beta}(AT^\infty_p(\alpha) \cap H^\infty_\beta)$ *and* $\lim_{|\varphi(z)|\to 1} |\varphi^{u,n}_\beta(z)| = 0$.

Proof. $(i) \Rightarrow (ii)$. It is obvious.

$(ii) \Rightarrow (iii)$. Suppose that $D^n_{\varphi,u} : H^\infty_{\beta,0} \to C_{H^\infty_\beta}(AT^\infty_p(\alpha) \cap H^\infty_\beta)$ is compact. Then $D^n_{\varphi,u} : H^\infty_{\beta,0} \to C_{H^\infty_\beta}(AT^\infty_p(\alpha) \cap H^\infty_\beta)$ is bounded. From the proof of Theorem 5 we have that $u \in C_{H^\infty_\beta}(AT^\infty_p(\alpha) \cap H^\infty_\beta)$. Since $C_{H^\infty_\beta}(AT^\infty_p(\alpha) \cap H^\infty_\beta) \subseteq H^\infty_\beta$, we get that $D^n_{\varphi,u} : H^\infty_{\beta,0} \to H^\infty_\beta$ is compact. Using Lemma 6, we get the desired result.

$(iii) \Rightarrow (i)$. By assumption, there exists an r $(0 < r < 1)$, such that

$$|\varphi^{u,n}_\beta(w)| < \frac{\varepsilon}{2}, \quad \text{when} \quad |\varphi(w)| \geqslant r.$$

Let $z \in \mathbb{D}$ be such that $|\varphi^{u,n}_\beta(z)| \geqslant \varepsilon$. Then it follows that $|\varphi(z)| < r$. Hence,

$$\varepsilon \leqslant |\varphi^{u,n}_\beta(z)| \leqslant \frac{|u(z)|(1-|z|^2)^\beta}{(1-r^2)^{\beta+n}},$$

which implies that

$$\varepsilon(1-r^2)^{\beta+n} \leqslant |u(z)|(1-|z|^2)^\beta.$$

Let $\delta = \varepsilon(1-r^2)^{\beta+n}$. Then $z \in \Omega^\beta_\delta(u)$. Since $u \in C_{H^\infty_\beta}(AT^\infty_p(\alpha) \cap H^\infty_\beta)$, by Theorem 1, we obtain

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{|\varphi^{u,n}_\beta(z)|\geqslant\varepsilon} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)} dA(z) \\
\lesssim &\sup_{a\in\mathbb{D}} \int_{\Omega^\beta_\delta(u)} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+1}(1-|z|^2)} dA(z) < \infty
\end{aligned}$$

for any $t > 0$. Using Theorem 4, we have that $D^n_{\varphi,u} : H^\infty_\beta \to C_{H^\infty_\beta}(AT^\infty_p(\alpha) \cap H^\infty_\beta)$ is bounded. By Lemma 6, we get that $D^n_{\varphi,u} : H^\infty_\beta \to H^\infty_\beta$ is compact. Hence, $D^n_{\varphi,u} : H^\infty_\beta \to C_{H^\infty_\beta}(AT^\infty_p(\alpha) \cap H^\infty_\beta)$ is compact. The proof is complete. $\square$

As an immediate consequence of Lemma 6 and Theorem 6, we have the following corollary.

COROLLARY 2. *Let* $1 \leqslant p < \infty$, $\alpha > -2$, $\beta = \frac{\alpha+2}{p}$, $u \in C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty)$, $n \in \mathbb{N} \cup \{0\}$ *and* φ *be an analytic self-map of* $\mathbb{D}$. *Then* $D_{\varphi,u}^n : H_\beta^\infty(H_{\beta,0}^\infty) \to H_\beta^\infty$ *is compact if and only if* $D_{\varphi,u}^n : H_\beta^\infty(H_{\beta,0}^\infty) \to C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty)$ *is compact.*

THEOREM 7. *Let* $1 \leqslant p < \infty$, $n \in \mathbb{N} \cup \{0\}$, $\alpha > -2$, $\beta = \frac{\alpha+2}{p}$, $u \in H(\mathbb{D})$ *and* φ *be an analytic self-map of* $\mathbb{D}$. *Then* $D_{\varphi,u}^n : C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty) \to C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty)$ *is compact if and only if* $u \in C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty)$ *and* $\lim_{|\varphi(z)| \to 1} |\varphi_\beta^{u,n}(z)| = 0$.

Proof. Suppose that $D_{\varphi,u}^n : C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty) \to C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty)$ is compact. It is clear that $u \in C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty)$. Since $H_{\beta,0}^\infty$ is the closure of all polynomials in H_β^∞ and the space $AT_p^\infty(\alpha)$ contains all polynomials, we obtain $D_{\varphi,u}^n : H_{\beta,0}^\infty \to C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty)$ is compact. The desired result follows from Theorem 6.

Conversely, suppose that $u \in C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty)$ and $\lim_{|\varphi(z)| \to 1} |\varphi_\beta^{u,n}(z)| = 0$. By Lemma 6, $D_{\varphi,u}^n : H_\beta^\infty \to H_\beta^\infty$ is compact. Using Corollary 2, we know that $D_{\varphi,u}^n : H_\beta^\infty \to C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty)$ is compact. Since $C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty) \subseteq H_\beta^\infty$, we have that $D_{\varphi,u}^n : C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty) \to C_{H_\beta^\infty}(AT_p^\infty(\alpha) \cap H_\beta^\infty)$ is compact. The proof is complete. $\square$

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