

## PROPER VERSION OF CUSA–HUYGENS’ INEQUALITY AND DJOKVIE’S TYPE INEQUALITY FOR PARABOLIC TRIGONOMETRIC FUNCTIONS

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**Abstract.** This paper aims to investigate several inequalities for parabolic trigonometric functions (PTF). We solve an open problem of proving Cusa-Huygens’ inequality for PTF. Furthermore, we find indefinite integrals for PTF; consequently, we derive a tight approximation for a particular integral. Additionally, we develop Djokvie’s type inequality for PTF and discover a precise upper bound on the function  $\text{sinp}(x) + \text{cosp}(x)$ , where  $\text{sinp}(x)$  denotes parabolic sine and  $\text{cosp}(x)$  parabolic cosine.

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