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PROPER VERSION OF CUSA–HUYGENS' INEQUALITY AND DJOKVIE'S TYPE INEQUALITY FOR PARABOLIC TRIGONOMETRIC FUNCTIONS

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Abstract. This paper aims to investigate several inequalities for parabolic trigonometric functions (PTF). We solve an open problem of proving Cusa-Huygens' inequality for PTF. Furthermore, we find indefinite integrals for PTF; consequently, we derive a tight approximation for a particular integral. Additionally, we develop Djokvie's type inequality for PTF and discover a precise upper bound on the function $\operatorname{sinp}(x)+\operatorname{cosp}(x)$, where $\operatorname{sinp}(x)$ denotes parabolic sine and $\operatorname{cosp}(x)$ parabolic cosine.

1. Introduction

Various generalizations of classical trigonometric functions have been investigated in the past, for example Gauss's lemniscate functions [5,10,29,34,36] or Jacobi's elliptic functions [20,28,32]. However, research into these generalized functions continues. Properties of these functions are still being developed and there are many applications for these functions. Yet new ways how to generalize trigonometric functions are still being introduced. Recently, generalized trigonometric functions that also generalize further Gauss's lemniscate function have attracted much interest [11,12,22,24,25,30,33] (see also [9, 13]). Possible applications of these functions are also explored, see recent development in [37]. This development highlights that new and not yet considered variants of trigonometric functions also need attention.

In the article [8], the authors introduce an interesting generalization of trigonometric functions as a link between a parabolic circle and its area. These functions are called parabolic trigonometric functions (PTF), and some of their properties were recently discovered in articles [16,17,31]. Article [16] finds both upper and lower bounds on hypothetical version of Cusa-Huygens' inequality for PTF (see also other generalizations in [18,26,35] and further developments in [27,38]). Nevertheless, the proof of Cusa-Huygens' inequality for PTF remained an open problem until it was proved in this article. We show that, indeed it holds

$$\frac{\operatorname{cosp}(x)+\frac{x}{2}+2}{3} > \frac{\operatorname{sinp}(x)}{x}, \quad x \in \left(0, \frac{p\pi}{2}\right), \tag{1}$$

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where $\operatorname{sinp}(x)$ denotes parabolic sine, $\operatorname{cosp}(x)$ parabolic cosine, and ${}_p\pi$ parabolic pi, see the forthcoming definitions in (5), (6), and (7).

In the fourth section, we investigate Djokvie's type inequality for PTF. Djokvie's inequality shows (see [6, 7, 15]) that tangent can be tightly bounded for $x \in \left(0, \frac{\pi}{6}\right)$ as

$$x + \frac{x^3}{3} < \tan x < x + \frac{4x^3}{9}. \tag{2}$$

Nevertheless, it is a simple matter to show that Eq. (2) does not hold in the same manner for parabolic tangent. In this article, we prove a similar bound for parabolic tangent in the form

$$x + \frac{x^2}{2} + \frac{x^3}{3} < \operatorname{tanp}(x) < x + \frac{x^2}{2} + \frac{2}{3}x^3, \quad x \in \left(0, \frac{{}_p\pi}{6}\right], \tag{3}$$

where $\operatorname{tanp}(x)$ denotes parabolic tangent.

In the fifth section, we develop an upper bound

$$\operatorname{sinp}(x) + \operatorname{cosp}(x) \leqslant \frac{5}{4}, \quad x \in \left[0, {}_p\pi\right]. \tag{4}$$

Eq. (4) is a PTF version of clasical inequality

$$\sin x + \cos x \leqslant \sqrt{2}$$

and it provides further tools for subsequent research and potential applications. With the ongoing investigation of mathematical inequalities across various fields, see also [2, 23, 39], we find that this might interest a wider mathematical community.

We predominantly use basic methods in the spirit of Paul Erdős. Nevertheless, the obtained results provide strong tools for future analysis of PTF, for example, in applications. Moreover, let us emphasize that we lack some tools that are, on the other hand, available for classical trigonometric functions. Among other open problems, it is still unknown whether we can find a general formula for PTF's Taylor series.

Finally, let us recall for reader's convenience the notation we use throughout the article. For a real function f and an interval $[a,b]$ we denote by $f([a,b])$ the image under f of an interval $[a,b]$, in other words

$$f([a,b]) = \{f(x) | x \in [a,b]\}.$$

2. Preliminaries

In this section, we recall the definition of PTF and review some of the standard facts about them. For more details, see [8, 16]. Additionally, we have proven a few lemmas that will benefit us later.

It is a well-established fact that clasic functions $\sin x$ and $\cos x$ describe moving point on the unit circle that moves in response to the changing angle x. Authors of [8] proposed a new (as far as we know) definition of generalized trigonometric functions $\operatorname{sinp}(x)$, $\operatorname{cosp}(x)$ that describe moving point on the unit parabolic circle in response to the changing area $\frac{x}{2}$, see [8] for further details. Functions $\operatorname{sinp}(x)$, $\operatorname{cosp}(x)$ are therefore defined as the functions which for initial conditions $\operatorname{sinp}(0)=0$, $\operatorname{cosp}(0)=1$ solve the following system

$$\operatorname{cosp}^2(x)+|\operatorname{sinp}(x)|=1, \tag{5}$$

$$\frac{\operatorname{cosp}(x)\operatorname{sinp}(x)}{2}+\int_{\operatorname{cosp}(x)}^{1} 1-t^2\,\mathrm{d}t=\frac{x}{2} \tag{6}$$

for $x\in\left[0,{}_p\pi\right]$, where $\operatorname{cosp}^2(x)$ denotes function $(\operatorname{cosp}(x))^2$. Furthermore, odd and even periodical extensions of solutions to (5), (6) extend functions $\operatorname{sinp}(x)$, $\operatorname{cosp}(x)$ onto $\mathbb{R}$ which completes the definition. Nevertheless, let us emphasize that throughout the paper we limit our investigation solely to the interval $\left[0,{}_p\pi\right]$. Parabolic ${}_p\pi$ is defined via the area of unit parabolic circle, namely

$$\frac{{}_p\pi}{2}=\int_{-1}^{1} 1-x^2\,\mathrm{d}x=\frac{4}{3}. \tag{7}$$

Furthermore, we define parabolic tangent $\operatorname{tanp}(x)$ in the usual manner as

$$\operatorname{tanp}(x)=\frac{\operatorname{sinp}(x)}{\operatorname{cosp}(x)}.$$

For further details and more precise statements, we refer the reader to [8].

One consequence of (5), (6), see [8], are the formulas for derivatives given as

$$(\operatorname{cosp}(x))'=-\frac{1}{1+\operatorname{cosp}^2(x)}, \tag{8}$$

$$(\operatorname{sinp}(x))'=\frac{2\operatorname{cosp}(x)}{1+\operatorname{cosp}^2(x)}, \tag{9}$$

$$(\operatorname{tanp}(x))'=\frac{1}{\operatorname{cosp}^2(x)}.$$

Here and subsequently, we utilize the following notation for function powers $\operatorname{sinp}^M(x)=(\operatorname{sinp}(x))^M$, $\operatorname{cosp}^M(x)=(\operatorname{cosp}(x))^M$ for whichever $M\in\mathbb{R}$, where the power on the right-hand side is defined.

Equations (5), (6) can be solved directly, see [8], and we know that the solutions have the form

$$\operatorname{cosp}(x)=-2\sinh\left(\frac{1}{3}\operatorname{arcsinh}\frac{3x-4}{2}\right), \tag{10}$$

$$\operatorname{sinp}(x)=3-2\cosh\left(\frac{2}{3}\operatorname{arcsinh}\frac{3x-4}{2}\right). \tag{11}$$

Function $\operatorname{sinp}(x)$ behaves similarly to $\sin x$ in the way that $\operatorname{sinp}(x)$ in positive on $\left(0,{}_p\pi\right)$ and increasing on $\left(0,\frac{{}_p\pi}{2}\right)$, respectively decreasing on $\left(\frac{{}_p\pi}{2},{}_p\pi\right)$ with its function values being $\operatorname{sinp}(0) = \operatorname{sinp}({}_p\pi) = 0$ and $\operatorname{sinp}\left(\frac{p\pi}{2}\right) = 1$. Similarly, $\operatorname{cosp}(x)$ has analogical properties to $\cos x$.

Let us also recall a special version of L'Hopital's rule that is sometimes utilized when proving inequalities. It appears, for example, in book [1, Theorem 1.25] (see also [4, 18, 35, 36] and many others).

LEMMA 1. *For $-\infty, a < b < \infty$ let functions $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b) and let $g'(x) \neq 0$ on (a,b). If $\frac{f'(x)}{g'(x)}$ is increasing (decreasing) on (a,b) then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $\frac{f'(x)}{g'(x)}$ is strictly monotone, then the monotonicity in the conclusion is also strict.

LEMMA 2. *It holds*

$$\frac{1}{\operatorname{cosp}(x)} > x + \operatorname{cosp}(x), \quad x \in \left(0, \frac{p\pi}{2}\right).$$

Proof. We first observe that from clasical L'hopital's rule follows

$$\lim_{x\to 0^+} \frac{\ln \frac{1}{\operatorname{cosp}(x)}}{\ln(x+\operatorname{cosp}(x))} = \lim_{x\to 0^+} \frac{\frac{1}{\operatorname{cosp}(x)(1+\operatorname{cosp}^2(x))}}{\frac{1}{x+\operatorname{cosp}(x)}\left(1-\frac{1}{1+\operatorname{cosp}^2(x)}\right)} \lim_{x\to 0^+} \frac{x+\operatorname{cosp}(x)}{\operatorname{cosp}^3(x)} = 1.$$

Moreover, function $\operatorname{cosp}(x)$ is decreasing for $x \in \left(0, \frac{4}{3}\right)$ and accordingly $\frac{1}{\operatorname{cosp}(x)}$ is increasing function. Hence, the following function

$$\frac{x+\operatorname{cosp}(x)}{\operatorname{cosp}^3(x)} = \frac{x}{\operatorname{cosp}^3(x)} + \frac{1}{\operatorname{cosp}^2(x)}$$

is also increasing. Lemma 1 now finishes the proof. □

Let us mention that combining Lemma 2 together with a result from [16] yields further refinement of the result

$$\frac{1}{\operatorname{cosp}(x)} > x + \operatorname{cosp}(x) > 1.$$

A potent tool that will be subsequently applied is the formula for inverse parabolic cosine $\operatorname{arccosp}(x)$ that appears in the following lemma.

LEMMA 3. *It holds*

$$\operatorname{arccosp}(x) = \frac{4}{3} - x - \frac{x^3}{3}, \quad x \in [0,1].$$

Proof. It is a consequence of inverse function rule and (8) (see also [8]) that $(\operatorname{arccosp}(x))' = -1 - x^2$. Accordingly, we solve

$$\operatorname{arccosp}(x) = -x - \frac{x^3}{3} + C,$$

where $C = \operatorname{arccosp}(0) = \frac{4}{3}$. $\square$

3. Cusa-Huygens type inequality for parabolic trigonometric functions

In this section, we prove Cusa-Huygens inequality for PTF in the form

$$\frac{\operatorname{cosp}(x) + \frac{x}{2} + 2}{3} > \frac{\operatorname{sinp}(x)}{x}, \quad x \in \left(0, \frac{p\pi}{2}\right).$$

LEMMA 4. *Function*

$$\operatorname{cosp}^2(x) + \frac{2}{\operatorname{cosp}(x)} + (x+2)\operatorname{cosp}(x)$$

is increasing for $x \in \left(0, \frac{p\pi}{2}\right)$.

Proof. Let us start by considering the function $f(x) = \operatorname{cosp}^2(x) + \frac{2}{\operatorname{cosp}(x)} + (x+2)\operatorname{cosp}(x)$. We find its derivative which is

$$\begin{aligned} f'(x) &= -\frac{2(1+\operatorname{cosp}(x))}{1+\operatorname{cosp}^2(x)} + \frac{2}{\operatorname{cosp}^2(x)(1+\operatorname{cosp}^2(x))} + \operatorname{cosp}(x) - \frac{x}{1+\operatorname{cosp}^2(x)} \\ &\overset{\text{Lemma } 2}{>} -\frac{2(1+\operatorname{cosp}(x))}{1+\operatorname{cosp}^2(x)} + \frac{2}{\operatorname{cosp}^2(x)(1+\operatorname{cosp}^2(x))} + \operatorname{cosp}(x) \\ &\quad + \frac{\operatorname{cosp}^2(x) - 1}{\operatorname{cosp}(x)(1+\operatorname{cosp}^2(x))}. \end{aligned} \tag{12}$$

Now substituting $y = \operatorname{cosp}(x)$ into (12) we get

$$f'(y) = \frac{y^5 - 2y^2 - y + 2}{y^2(1+y^2)} = \frac{(y+1)(y-1)^2(y^2+y+2)}{y^2(1+y^2)} > 0$$

for all y and thus also for $y \in (0,1)$. $\square$

LEMMA 5. *It holds*

$$x\left(\operatorname{cosp}(x) + \frac{x}{2} + 2\right) > 3\operatorname{sinp}(x), \quad x \in \left(0, \frac{p\pi}{2}\right).$$

Proof. We first observe that from clasical L'hopital's rule follows

$$\lim_{x\to 0^+}\frac{x\left(\operatorname{cosp}(x)+\frac{x}{2}+2\right)}{3\operatorname{sinp}(x)}=\lim_{x\to 0^+}\frac{\operatorname{cosp}(x)-\frac{x}{1+\operatorname{cosp}^2(x)}+x+2}{\frac{6\operatorname{cosp}(x)}{1+\operatorname{cosp}^2(x)}}=1.$$

Furthermore, we can apply Lemma 1 because

$$\begin{aligned}\frac{\operatorname{cosp}(x)-\frac{x}{1+\operatorname{cosp}^2(x)}+x+2}{\frac{\operatorname{cosp}(x)}{1+\operatorname{cosp}^2(x)}}&=1+\operatorname{cosp}^2(x)-\frac{x}{\operatorname{cosp}(x)}+(x+2)\left(\frac{1}{\operatorname{cosp}(x)}+\operatorname{cosp}(x)\right)\\&=1+\operatorname{cosp}^2(x)+\frac{2}{\operatorname{cosp}(x)}+(x+2)\operatorname{cosp}(x)\end{aligned}$$

is an increasing function by Lemma 4. □

THEOREM 1. *It holds*

$$1>\frac{\operatorname{cosp}(x)+\frac{x}{2}+2}{3}>\frac{\operatorname{sinp}(x)}{x},\quad x\in\left(0,\frac{\pi_p}{2}\right). \tag{13}$$

Proof. This is an immediate consequence of Lemma 5 together with the fact that

$$\left(\operatorname{cosp}(x)+\frac{x}{2}\right)'=-\frac{1}{1+\operatorname{cosp}^2(x)}+\frac{1}{2}<-\frac{1}{2}+\frac{1}{2}=0.\quad\square$$

COROLLARY 1. *Let there be real parameters α, β, r, s such that $\alpha>0$, $\beta>0$ and $r\leqslant\frac{s\beta}{\alpha}$. Then for $s\geqslant\max\left\{\frac{\alpha}{\beta},1\right\}$ holds*

$$\frac{\alpha}{\alpha+\beta}\left(\frac{\operatorname{cosp}(x)+\frac{x}{2}+2}{3}\right)^r+\frac{\beta}{\alpha+\beta}\left(2-\frac{\operatorname{sinp}(x)}{x}\right)^s>1,\quad x\in\left(0,\frac{\pi_p}{2}\right).$$

Proof. By Theorem 1 we conclude

$$\begin{aligned}&\frac{\alpha}{\alpha+\beta}\left(\frac{\operatorname{cosp}(x)+\frac{x}{2}+2}{3}\right)^r+\frac{\beta}{\alpha+\beta}\left(2-\frac{\operatorname{sinp}(x)}{x}\right)^s\\&>\frac{\alpha}{\alpha+\beta}\left(\frac{\operatorname{cosp}(x)+\frac{x}{2}+2}{3}\right)^{s\frac{\beta}{\alpha}}+\frac{\beta}{\alpha+\beta}\left(2-\frac{\operatorname{sinp}(x)}{x}\right)^s\\&>\frac{\alpha}{\alpha+\beta}\left(\frac{\operatorname{sinp}(x)}{x}\right)^{s\frac{\beta}{\alpha}}+\frac{\beta}{\alpha+\beta}\left(2-\frac{\operatorname{sinp}(x)}{x}\right)^s=:f\left(\frac{\operatorname{sinp}(x)}{x}\right).\end{aligned}$$

Therefore, by fixing $t=\frac{\operatorname{sinp}(x)}{x}$ we gain the function $f(t)=\frac{\alpha}{\alpha+\beta}t^{s\frac{\beta}{\alpha}}+\frac{\beta}{\alpha+\beta}(2-t)^s$ which derivative is

$$f'(t)=\frac{s\beta}{\alpha+\beta}\left(t^{s\frac{\beta}{\alpha}-1}-(2-t)^{s-1}\right).$$

Moreover, the substitution satisfies $t = \frac{\operatorname{sinp}(x)}{x} \in (0,1)$. Which allows us to find a lower bound

$$(2-t)^{s-1} > 2-t > 1 > t^{s\frac{\beta}{\alpha}-1}.$$

Hence, $f(t)$ is decreasing and it holds

$$f(t) > f(1) = \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} = 1. \quad \square$$

In article [16], authors discuss a potential application of inequality (13) that was at the time still unproven. If we find $\int \operatorname{cosp}(x)\,\mathrm{d}x$ then we can approximate for small x

$$\int_0^x \frac{\operatorname{sinp}(t)}{t}\,\mathrm{d}t \approx \frac{1}{3}\int_0^x \operatorname{cosp}(t)\,\mathrm{d}t + \frac{x^2}{12} + \frac{2x}{3}.$$

Nevertheless, the question of finding $\int \operatorname{cosp}(t)\,\mathrm{d}t$ is still unanswered. The following theorem resolves this open problem.

THEOREM 2. *It holds*

$$\int \operatorname{cosp}(x)\,\mathrm{d}x = \operatorname{sinp}(x) - \frac{\operatorname{sinp}^2(x)}{4} + C,$$
$$\int \operatorname{sinp}(x)\,\mathrm{d}x = x + \frac{\operatorname{cosp}^3(x)}{3} + \frac{\operatorname{cosp}^5(x)}{5} + C$$

for $x \in \left(0, \frac{8}{3}\right)$.

Proof. Differentiating the right-hand sides via (8), (9) yields

$$\begin{aligned}\left(\operatorname{sinp}(x) - \frac{\operatorname{sinp}^2 x}{4}\right)' &= \frac{2\operatorname{cosp}(x)}{1+\operatorname{cosp}^2(x)} - \frac{\operatorname{sinp}(x)\operatorname{cosp}(x)}{1+\operatorname{cosp}^2(x)} = \frac{2-\operatorname{sinp}(x)}{1+\operatorname{cosp}^2(x)}\operatorname{cosp}(x)\\ &= \frac{1+\operatorname{cosp}^2(x)+\operatorname{sinp}(x)-\operatorname{sinp}(x)}{1+\operatorname{cosp}^2(x)}\operatorname{cosp}(x) = \operatorname{cosp}(x),\\ \left(x + \frac{\operatorname{cosp}^3(x)}{3} + \frac{\operatorname{cosp}^5 x}{5}\right)' &= 1 - \frac{\operatorname{cosp}^2(x)}{1+\operatorname{cosp}^2(x)} - \frac{\operatorname{cosp}^4 x}{1+\operatorname{cosp}^2(x)}\\ &= 1 - \operatorname{cosp}^2(x) = \operatorname{sinp}(x). \quad \square\end{aligned}$$

As a consequence, we conclude that for small $x > 0$ it holds

$$\int_0^x \frac{\operatorname{sinp}(t)}{t}\,\mathrm{d}t \approx \frac{x^2 - \operatorname{sinp}^2(x)}{12} + \frac{2x + \operatorname{sinp}(x)}{3}.$$

4. Djokvie's type inequality

In this section, we prove Djokvie's type inequality for PTF in the form

$$x+\frac{x^2}{2}+\frac{x^3}{3}<\operatorname{tanp}(x)<x+\frac{x^2}{2}+\frac{2}{3}x^3, \quad x\in\left(0,\frac{p\pi}{6}\right].$$

LEMMA 6. *Function*

$$f(x)=\operatorname{cosp}^2(x)(1+x+x^2)$$

is decreasing for $x\in\left[0,\frac{p\pi}{2}\right]$.

Proof. In the next step, we will apply substitution $y=\operatorname{cosp}(x)$. However, let us emphasize beforehand that $x=\operatorname{arccosp}(y)$ and by Lemma 3 then holds

$$x=\frac{4}{3}-y-\frac{y^3}{3}.$$

From this we may conclude that

$$\begin{aligned}f(y)&=y^2\left(1+\frac{4}{3}-y-\frac{y^3}{3}+\left(\frac{4}{3}-y-\frac{y^3}{3}\right)^2\right)\\&=\frac{y^8}{9}+\frac{2y^6}{3}-\frac{11y^5}{9}+y^4-\frac{11}{3}y^3+\frac{37}{9}y^2.\end{aligned}$$

Consequently, the derivative of $f(y)$ is

$$\begin{aligned}f'(y)&=\frac{8y^7}{9}+4y^5-\frac{55y^5}{9}+4y^3-11y^2+\frac{74}{9}y\\&=\frac{1}{9}y(y-1)^2(8y^4+16y^3+60y^2+49y+74)>0\end{aligned}$$

for $y\in[0,1]$. On account of the derivative $f'(y)$ we see that $f(y)$ is increasing. On the other hand, $y(x)=\operatorname{cosp}(x)$ is decreasing function and thus $f(x)$ is decreasing. $\square$

LEMMA 7. *It holds*

$$\operatorname{cosp}^2(x)>1-x, \quad x\in\left(0,\frac{p\pi}{2}\right).$$

Proof. It suffices to show that function $f(x)=x+\operatorname{cosp}^2(x)$ is increasing and additionally satisfies $f(0)=1$. Indeed, its derivative is

$$f'(x)=1-\frac{2\operatorname{cosp}(x)}{1+\operatorname{cosp}^2(x)}=\frac{(1+\operatorname{cosp}(x))^2}{1+\operatorname{cosp}^2(x)}>0$$

and $f(x)$ is increasing. $\square$

THEOREM 3. *It holds*

$$x+\frac{x^2}{2}+\frac{x^3}{3}<\operatorname{tanp}(x), \quad x\in\left(0,\frac{p\pi}{2}\right].$$

Proof. Considering the limit

$$\lim_{x\to 0^+}\frac{x+\frac{x^2}{2}+\frac{x^3}{3}}{\operatorname{tanp}(x)}=\lim_{x\to 0^+}\operatorname{cosp}^2(x)\left(1+x+x^2\right)=1$$

we have by Lemma 6 that the convergence is monotone which proves by Lemma 1 that

$$\frac{x+\frac{x^2}{2}+\frac{x^3}{3}}{\operatorname{tanp}(x)}<1$$

as $x\to 0^+$. □

THEOREM 4. *It holds*

$$\operatorname{tanp}(x)<x+\frac{x^2}{2}+\frac{2}{3}x^3, \quad x\in\left(0,\frac{p\pi}{6}\right].$$

Proof. Throughout the proof we will utilize the function $f(x)=x+\frac{x^2}{2}+\frac{2}{3}x^3-\operatorname{tanp}(x)$ which derivative is

$$f'(x)=1+x+2x^2-\frac{1}{\operatorname{cosp}^2(x)}.$$

Furthermore, we know by Lemma 7 that $\operatorname{cosp}^2(x)>1-x$ for $x\in\left(0,\frac{p\pi}{2}\right)$. Accordingly,

$$f'(x)>1+x+2x^2-\frac{1}{1-x}=\frac{x^2(1-2x)}{1-x}.$$

Finally, because $\frac{p\pi}{6}=\frac{4}{9}<\frac{1}{2}$ then both $1-2x>0$ and $1-x>0$. Hence, $f(x)$ is increasing and because $f(0)=0$ it is positive on $\left(0,\frac{p\pi}{6}\right]$. □

Note that Wilker's inequality

$$\left(\frac{\operatorname{sinp}(x)}{x}\right)^2+\frac{\operatorname{tanp}(x)}{x}>2$$

holds for clasic trigonometric functions as well as for PTF (see [16]). However, utilizing Theorem 4 yields the following sharp upper bound.

COROLLARY 2. *It holds*

$$\left(\frac{\operatorname{sinp}(x)}{x}\right)^2+\frac{\operatorname{tanp}(x)}{x}<2+\frac{x}{2}+\frac{2x^2}{3},\quad x\in\left(0,\frac{{}_p\pi}{6}\right].$$

Proof. This is a direct consequence of Theorem 4 and the fact that $\frac{\operatorname{sinp}(x)}{x}<1$ for $x>0$ (see Theorem 1). $\square$

COROLLARY 3. *It holds*

$$\frac{\operatorname{sinp}(x)}{x}<\left(1+\frac{x}{2}+\frac{2}{3}x^2\right)\operatorname{cosp}(x),\quad x\in\left(0,\frac{{}_p\pi}{6}\right]$$

Proof. Direct modification of Theorem 4 yields the result. $\square$

5. Upper bound on $\operatorname{sinp}(x)+\operatorname{cosp}(x)$

It is a well-known fact that clasical trigonometric functions satisfy

$$\sin x+\cos x\leqslant\sqrt{2}. \tag{14}$$

In this section, we prove a version of (14) for PTF in the form

$$\operatorname{sinp}(x)+\operatorname{cosp}(x)\leqslant\frac{5}{4},\quad x\in\left[0,{}_p\pi\right].$$

Proof. Let us consider for purposes of this proof the function $f(x)=\operatorname{sinp}(x)+\operatorname{cosp}(x)$. Subsequently, we will show that $f(x)$ has a global maximum on $\left[0,{}_p\pi\right]$ and that the function value at this maximum is $\frac{5}{4}$. We find that

$$f'(x)=\frac{2\operatorname{cosp}(x)-1}{1+\operatorname{cosp}^2(x)}.$$

Function $\operatorname{cosp}(x)$ is decreasing on $\left[0,{}_p\pi\right]$ and thus there is a unique stationary point x_S solving the equation $f'(x)=0$. If we consider the sign of $f'(x)$, we find that $f'(x)$ is first positive and later negative on $\left[0,{}_p\pi\right]$. Hence, we know that the stationary point x_S is a global maximum by basic tools of Calculus.

What remains now is to find the function value $f(x_S)=\operatorname{sinp}(x_S)+\operatorname{cosp}(x_S)$. Because x_S solves the equation $f'(x)=0$ we know that $2\operatorname{cosp}(x_S)-1=0$ and thus $\operatorname{cosp}(x_S)=\frac{1}{2}$. Furthermore, we can express by Eq. (10) that

$$\begin{aligned}\operatorname{cosp}(x_S)&=\frac{1}{2}\\ -2\sinh\left(\frac{1}{3}\operatorname{arcsinh}\frac{3x_S-4}{2}\right)&=\frac{1}{2}\\ \frac{3x_S-4}{2}&=\sinh\left(3\operatorname{arcsinh}\left(-\frac{1}{4}\right)\right).\end{aligned} \tag{15}$$

Such an advantageous expression then further yields

$$\begin{aligned}
\operatorname{sinp}(x_S) &\overset{(11)}{=} 3-2\cosh\left(\frac{2}{3}\operatorname{arcsinh}\frac{3y-4}{2}\right)\\
&\overset{(15)}{=} 3-2\cosh\left(\frac{2}{3}\operatorname{arcsinh}\sinh\left(3\operatorname{arcsinh}\left(-\frac{1}{4}\right)\right)\right)\\
&= 3-2\cosh\left(2\operatorname{arcsinh}\left(-\frac{1}{4}\right)\right) = 3-2\cosh\left(2\operatorname{arcsinh}\frac{1}{4}\right)\\
&= 3-2\cosh\operatorname{arccosh}\frac{9}{8} = \frac{3}{4}.
\end{aligned}$$

Here we used a well-known identity (see [14, page 60])

$$2\operatorname{arcsinh} x = \operatorname{arccosh}(2x^2+1), \quad x>0.$$

Finally, we conclude that the function value at the point of global maxima is $f(x_S) = \operatorname{sinp}(x_S)+\operatorname{cosp}(x_S) = \frac{3}{4}+\frac{1}{2} = \frac{5}{4}$. $\square$

6. Conclusion and final remarks

Parabolic trigonometric functions generalize trigonometric functions in a broader meaning of the world generalize. Both of these fields describe a movement of a point along a sort of a circle. Because this connection is relatively weak, someone might consider them to be two separate directions of research. However, these theories are related in other ways that particular properties of trigonometric functions hold in an analogical way for PTF. For example, Wilker's inequality [16] holds for both clasic trigonometric functions and for PTF.

Therefore, it is natural to ask what is the proper formula for $\operatorname{sinp}(2x)$. Is it the same as for clasic trigonometric functions? A rather natural guess is

$$\operatorname{sinp}(2x) \overset{?}{=} 2\operatorname{sinp}(x)\operatorname{cosp}(x).$$

In fact, we can numerically estimate that these two values are pretty close (see Figure 1, part (a)) and we conjecture that it even holds

$$2\operatorname{sinp}(x)\operatorname{cosp}(x) < \operatorname{sinp}(2x), \quad x\in(0,{}_p\pi).$$

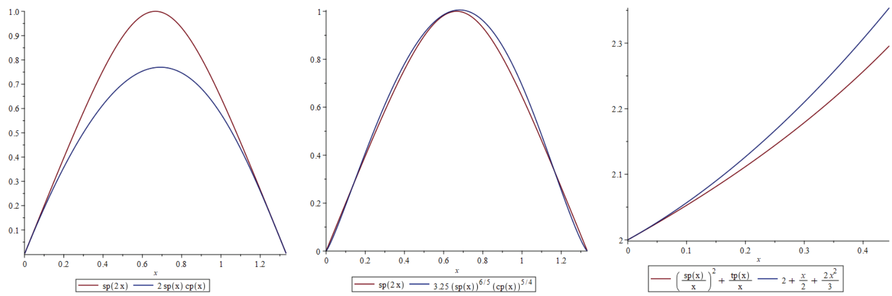
With a further numerical experimentation (see Figure 1, part (b)) we notice that the approximation

$$\operatorname{sinp}(2x) \approx 3.25\operatorname{sinp}^{\frac{6}{5}}(x)\operatorname{cosp}^{\frac{5}{4}}(x)$$

is quite close to the right value for all $x\in\left(0,\frac{p\pi}{2}\right)$. In fact, it appears that

$$\left|\operatorname{sinp}(2x) - 3.25\operatorname{sinp}^{\frac{6}{5}}(x)\operatorname{cosp}^{\frac{5}{4}}(x)\right| < 0.06$$

for all $x\in\left(0,\frac{p\pi}{2}\right)$. We expect that similar approximation could lead to other results. Additionally, if we can find the formula for $\operatorname{sinp}(x+y)$ it could lead to further results, for example, the following conjecture.



(a) (b) (c)

Figure 1: *Representation of studied inequalities. In (a) we see that* $\operatorname{sinp}(2x)$ *is close to* $2\operatorname{sinp}(x)\operatorname{cosp}(x)$. *In (b) we see that* $3.25\operatorname{sinp}^{\frac{6}{5}}(x)\operatorname{cosp}^{\frac{5}{4}}(x)$ *revolves closely around* $\operatorname{sinp}(2x)$. *In (c) we see a tight upper bound on Wilker's inequality.*

CONJECTURE 6.1. (Turán's type inequality) *It holds for all* $n \in \mathbb{N}$ *that*

$$\operatorname{sinp}^2(nx) > \operatorname{sinp}([n-1]x)\operatorname{sinp}([n+1]x), \quad x \in (0, {}_p\pi).$$

Let us also address the name Djokvie's inequality. This name appears in articles [6, 7, 15], which reference a book written in Chinese [21]. However, no other source utilizes the same name, and it is unclear where the name Djokvie comes from. We searched for the primary source for the name, but eventually, we could not verify the name's origin. Furthermore, we were unable to find this inequality under any different name. Articles [3, 19] utilize a lower bound of Djokvie's inequality without naming it.

Competing interests. The author declares that he is also associated with Masaryk University in Brno, Czech Republic.

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