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ON A SUFFCIENT CONDITION FOR STARLIKENESS FOR MULTIVALENT FUNCTIONS

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Abstract. We consider univalent functions, analytic in the unit disc $|z|<1$ in the complex plane $\mathbb{C}$ which map $|z|<1$ onto a domain with some property. In M. Nunokawa, J. Sokół, E. Trybucka, On some sufficient conditions for a function to be p-valent starlike, Symmetry, 11 (11) (2019), Art. 1417, we have established a sufficient condition for a function analytic in $|z|<1$ to be p-valent starlike in $|z|<1$. In this paper we shall determine the new sufficient conditions for the starlikeness of order α for p-valent functions and some symmetric results.

1. Introduction

Let $\mathscr{H}$ denote the class of analytic functions in the unit disc $\mathbb{D}=\{z: \ |z|<1\}$ on the complex plane $\mathbb{C}$. Endowed with the topology of uniform convergence on compact sets, $\mathscr{H}$ is a linear topological space, moreover it is metrizable and locally convex space.

The book [7] by Goodman almost provides an encyclopedia about analytic functions and describes a very large number of results in the field. Recent results on the subject with the theory of differential subordinations one can find in the books [3] and [21].

Let $\mathscr{A}$ denote the subclass of $\mathscr{H}$ consisting of functions normalized by $f(0)=0$, $f'(0)=1$. The set of all functions $f\in\mathscr{A}$ that are starlike univalent in $\mathbb{D}$ will be denoted by $\mathscr{S}^*$. The set of all functions $f\in\mathscr{A}$ that are convex univalent in $\mathbb{D}$ by $\mathscr{K}$. Recall that a set $E\subset\mathbb{C}$ is said to be starlike with respect to a point $w_0\in E$ if and only if the linear segment joining w_0 to every other point $w\in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. Let the function f be analytic univalent in the unit disc $\mathbb{D}$ on the complex plane $\mathbb{C}$ with the normalization $f(0)=0$, $f'(0)=1$. Then f maps $\mathbb{D}$ onto a starlike domain with respect to $w_0=0$ if and only if [22]

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0, \quad (z\in\mathbb{D}), \tag{1}$$

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while f maps $\mathbb{D}$ onto a convex domain E if and only if [40]

$$\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0, \quad (z\in\mathbb{D}). \tag{2}$$

Such function f is said to be starlike in $\mathbb{D}$ with respect to $w_0=0$ (or briefly starlike) or, respectively, is said to be convex in $\mathbb{D}$ (or briefly convex). Robertson introduced in [32] the classes $\mathscr{S}^*(\alpha)$, $\mathscr{K}(\alpha)$ of starlike and convex functions of order $\alpha\leqslant 1$, which are defined by

$$\mathscr{S}^*(\alpha):=\left\{f\in\mathscr{A}:\ \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha, \quad (z\in\mathbb{D})\right\} \tag{3}$$

and

$$\mathscr{K}(\alpha):=\left\{f\in\mathscr{A}:\ \mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha, \quad (z\in\mathbb{D})\right\}. \tag{4}$$

If $\alpha\in[0;1)$, then a function in either of these sets is univalent. In particular we denote $\mathscr{S}^*(0)=\mathscr{S}^*$, $\mathscr{K}(0)=\mathscr{K}$. Let $\mathscr{S}$ denote the subset of $\mathscr{A}$ which is composed of univalent functions.

We say that the $f\in\mathscr{H}$ is subordinate to $g\in\mathscr{H}$ in the unit disc $\mathbb{D}$, written $f\prec g$ if and only if there exits an analytic function $w\in\mathscr{H}$ such that $w(0)=0, |w(z)|<1$ and $f(z)=g[w(z)]$ for $z\in\mathbb{D}$. The superordinate function g need not be univalent. Therefore $f\prec g$ in $\mathbb{D}$ implies $f(\mathbb{D})\subset g(\mathbb{D})$. In particular if g is univalent in $\mathbb{D}$, then (subordination principle)

$$f\prec g \quad \Leftrightarrow \quad [f(0)=g(0) \quad \text{and} \quad f(|z|<r)\subset g(|z|<r) \text{ for all } r, \text{ and } 0<r\leqslant 1]. \tag{5}$$

A typical direction in which this principle can be applied is to start with a given function F and discuss the properties of the class of all functions $f\prec F$. The theory of subordination, which is a straightforward generalization of the principle of maximum modulus, is of great importance in general theory of analytic functions. The idea of subordinations goes back to Lindelöf [16] and (4) usually is called the Lindelöf's principle. Subordination was more formally introduced and studied by Littelwood [17] and later by Rogosinski [33] and [34].

Applying the subordination, we may write that $f\in\mathscr{S}^*$ if $f\in\mathscr{A}$ if it satisfies the condition

$$\frac{zf'(z)}{f(z)}\prec q(z), \quad (z\in\mathbb{D}), \tag{6}$$

where $q(z)=(1+z)/(1-z)$. Many subclasses of $\mathscr{S}^*$ have been defined by the condition (6) with a convex univalent function p, given arbitrary. If we restrict considerations to the absorbing geometric shape of $p(\mathbb{D})$, then it is proper to recall the papers [10, 11], where $p(\mathbb{D})$ is a disc. In [1, 39] the set $p(\mathbb{D})$ is an angle while in [8, 18, 35] $p(\mathbb{D})$ is a parabolic domain. In [13, 14, 15] the set $p(\mathbb{D})$ is an interior of hyperbola or of an elliptic domain. For the case when $p(\mathbb{D})$ is an interior of the right loop of the Lemniscate of Bernoulli see [36, 37] or when $p(\mathbb{D})$ is a leaf-like domain see [31]. An interesting case when the function p is convex but is not univalent was considered in [12]. A class related to a function p that is not univalent and is not convex and maps unit circle onto the trisectrix of Maclaurin was considered in [4], [6] and [5]. Littlewood [17] proved that if $f,g \in \mathscr{H}$ and $g \prec f$, $g(z) = f(w(z))$, then for $0 < t < \infty$

$$\frac{1}{2\pi}\int_0^{2\pi}\left|g(re^{i\theta})\right|^t \mathrm{d}\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^t \mathrm{d}\theta \quad 0 \leqslant r < 1,$$

and the strict inequality holds for $0 < r < 1$ unless f is constant or $w(z) = \alpha z$, $|\alpha| = 1$.

2. Materials and methods

In [28] we have established a sufficient condition for a function analytic in $|z| < 1$ to be p-valent starlike in $|z| < 1$. In this paper we shall determine the new sufficient conditions for the starlikeness of order α for p-valent functions and and some related results. We will apply a result well known as the Jack's lemma.

LEMMA 1. [9] *Let $w(z)$ be non-constant and analytic function in the unit disc $\mathbb{D}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the disc $|z| \leqslant r$ at the point z_0, $|z_0| = r$, then $z_0 w'(z_0) = kw(z_0)$ and $k \geqslant 1$.*

The Jack's lemma has found several of the applications and generalizations in the theory of differential subordinations, see for instance [19], [20], [21] and [29]. For a local version of Jack's lemma we refer to [27].

The following lemma is a simple generalization of Nunokawa's Lemma [23], which together with the lemma from [24] have a surprising number of important applications in the theory of univalent functions.

LEMMA 2. *Let $p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n$ be analytic function in $\mathbb{D}$. Suppose also that there exists a point $z_0 \in \mathbb{D}$, such that*

$$\mathfrak{Re}\{p(z)\} > 0 \ \textit{for} \ |z| < |z_0|$$

and

$$\mathfrak{Re}\{p(z_0)\} = 0 \ \textit{and} \ p(z_0) \neq 0.$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where k is real number and

$$k \geqslant \frac{m}{2}\left(a + \frac{1}{a}\right) \geqslant m \geqslant 1 \quad \textit{when} \ \operatorname{Arg}\{p(z_0)\} = \frac{\pi}{2}$$

and

$$k \leqslant -\frac{m}{2}\left(a + \frac{1}{a}\right) \leqslant -m \leqslant -1 \quad \textit{when} \ \operatorname{Arg}\{p(z_0)\} = -\frac{\pi}{2},$$

where

$$p(z_0) = \pm ia, \ \textit{and} \ a > 0.$$

3. Results

Let $\mathscr{A}_p \subset \mathscr{H}$ be the class of analytic functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (z \in \mathbb{D}). \tag{7}$$

So we have $\mathscr{A} = \mathscr{A}_1$. A function f which is analytic in a domain $D \subset \mathbb{C}$ is called p-valent in D if for every complex number w, the equation $f(z) = w$ have at most p roots in D and there will be a complex number w_0 such that the equation $f(z) = w_0$, has exactly p roots in D.

THEOREM 1. *Let $f \in \mathscr{A}_p$. If*

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{p(1+\alpha)^2 - (1-\alpha)}{2(1+\alpha)}, \quad (z \in \mathbb{D}), \tag{8}$$

then

$$\mathfrak{Re}\left\{\frac{f(z)}{z^p}\right\} > \frac{1}{1 + p(1-\alpha)}, \quad (z \in \mathbb{D}) \tag{9}$$

for all $\alpha \in [0,1)$.

Proof. First, we prove that

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{p}{2}(1+\alpha), \quad (z \in \mathbb{D}). \tag{10}$$

For this, it suffices to show the subordination

$$\frac{zf'(z)}{pf(z)} \prec \frac{1 - \alpha z}{1 - z}.$$

or to find $w \in \mathscr{H}$ with $w(0) = 0$ such that

$$|w(z)| < 1, \quad (z \in \mathbb{D}), \tag{11}$$

and such that

$$\frac{zf'(z)}{pf(z)} = \frac{1 - \alpha w(z)}{1 - w(z)}. \tag{12}$$

If (11) doesn't hold, then $|w(z)|$ takes its local maximum $|w(z_0)| = 1$ at a point $z_0 \in \mathbb{D}$ such that

$$\mathfrak{Re}\left\{\frac{1 - \alpha w(z_0)}{1 - w(z_0)}\right\} = \frac{1+\alpha}{2}.$$

Then from Jack's Lemma [9], we can write

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \geqslant 1$$

and therefore, from (12) we have

$$1+\frac{z_0 f''(z_0)}{f'(z_0)} = p\frac{1-\alpha w(z_0)}{1-w(z_0)} - \frac{\alpha k w(z_0)}{1-\alpha w(z_0)} + \frac{k w(z_0)}{1-w(z_0)}.$$

From this, we have

$$\begin{aligned}
\mathfrak{Re}\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}\right\} &= \mathfrak{Re}\left\{p\frac{1-\alpha w(z_0)}{1-w(z_0)} - \frac{\alpha k w(z_0)}{1-\alpha w(z_0)} + \frac{k w(z_0)}{1-w(z_0)}\right\} \\
&\leqslant \frac{p}{2}(1+\alpha) + \frac{k\alpha}{1+\alpha} - \frac{k}{2} \\
&= \frac{p}{2}(1+\alpha) + k\frac{\alpha-1}{2(1+\alpha)} \\
&\leqslant \frac{p}{2}(1+\alpha) - \frac{1-\alpha}{2(1+\alpha)} \\
&= \frac{p(1+\alpha)^2-(1-\alpha)}{2(1+\alpha)}.
\end{aligned}$$

This contradicts (8), and shows that f satisfies the condition

$$\begin{aligned}
&\mathfrak{Re}\left\{1+\frac{z f''(z)}{f'(z)}\right\} > \frac{p(1+\alpha)^2-(1-\alpha)}{2(1+\alpha)} \\
\Rightarrow\quad &\mathfrak{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > \frac{p}{2}(1+\alpha), \quad (z\in\mathbb{D}).
\end{aligned} \tag{13}$$

Now, we prove that

$$\mathfrak{Re}\left\{\frac{f(z)}{z^p}\right\} > \frac{1}{1+p(1-\alpha)}, \quad (z\in\mathbb{D}).$$

For this, it suffices to show that

$$\frac{f(z)}{z^p} \prec \frac{1-\gamma z}{1-z}$$

or to find $\omega\in\mathscr{H}$ with $\omega(0)=0$ such that

$$|\omega(z)|<1, \quad (z\in\mathbb{D}),$$

and such that

$$\frac{f(z)}{z^p} = \frac{1-\gamma\omega(z)}{1-\omega(z)} \tag{14}$$

with γ such that

$$\frac{1+\gamma}{2} = \frac{1}{1+p(1-\alpha)}.$$

It is clear that for $|\omega(z)|<1$, we have

$$\mathfrak{Re}\left\{\frac{1-\gamma\omega(z)}{1-\omega(z)}\right\} > \frac{1+\gamma}{2} = \frac{1}{1+p(1-\alpha)}.$$

If $|\omega(z)|<1$ doesn't hold in $\mathbb{D}$, then $|\omega(z)|$ takes its local maximum $|\omega(z_1)|=1$ at a point $z_1\in\mathbb{D}$ such that

$$\mathfrak{Re}\left\{\frac{1-\gamma\omega(z_1)}{1-\omega(z_1)}\right\}=\frac{1+\gamma}{2}.$$

Then from Jack's Lemma [9], we can write

$$\frac{z_1\omega'(z_1)}{\omega(z_1)}=k\geqslant 1$$

and therefore, from (14) we have

$$\frac{z_1 f'(z_1)}{f(z_1)}=p-\frac{\gamma k\omega(z_1)}{1-\gamma\omega(z_1)}+\frac{k\omega(z_1)}{1-\omega(z_1)}.$$

From this, we have

$$\begin{aligned}\mathfrak{Re}\left\{\frac{z_1 f'(z_1)}{f(z_1)}\right\}&=\mathfrak{Re}\left\{p-\frac{\gamma k\omega(z_1)}{1-\gamma\omega(z_1)}+\frac{k\omega(z_1)}{1-\omega(z_1)}\right\}\\&\leqslant p+\frac{k\gamma}{1+\gamma}-\frac{k}{2}\\&=p+\frac{k(\gamma-1)}{2(1+\gamma)}\\&\leqslant p-\frac{1-\gamma}{2(1+\gamma)}\\&=\frac{p}{2}(1+\alpha).\end{aligned}$$

This contradicts (10), and shows that f satisfies condition (9). $\square$

THEOREM 2. *Let $f(z)=z^p+\sum_{n=m}^{\infty}a_nz^n$ be analytic in the unit disc $\mathbb{D}$, and $zf'(z)/f(z)\neq -1$ in $\mathbb{D}$. Suppose also that $m>2+p$ and*

$$\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\leqslant\frac{m-p}{2},\quad (z\in\mathbb{D}). \tag{15}$$

Then we have

$$\left|\frac{zf'(z)}{pf(z)}-1\right|<1,\quad (z\in\mathbb{D}). \tag{16}$$

Proof. The function $zf'(z)/(pf(z))$ is analytic in $\mathbb{D}$, thus we can define the function s by the following condition

$$\frac{zf'(z)}{pf(z)}-1=\frac{1-s(z)}{1+s(z)},\quad (z\in\mathbb{D}), \tag{17}$$

or

$$\frac{zf'(z)}{f(z)}=\frac{2p}{1+s(z)},\quad (z\in\mathbb{D}),$$

where $s(0)=1$, and $s(z)=1+s_{m-p}z^{m-p}+\cdots$, $z\in\mathbb{D}$. Then it follows that

$$1+\frac{zf''(z)}{f'(z)}=\frac{2p}{1+s(z)}-\frac{s(z)}{1+s(z)}\frac{zs'(z)}{s(z)}.$$

If there exists a point $z_0\in\mathbb{D}$, such that

$$\left|\frac{zf'(z)}{pf(z)}-1\right|<1 \ \text{ for } \ |z|<|z_0|$$

and

$$\left|\frac{z_0 f'(z_0)}{pf(z_0)}-1\right|=1,$$

then by (17)

$$\mathfrak{Re}\{s(z)\}>0 \ \text{ for } \ |z|<|z_0|$$

and

$$\mathfrak{Re}\{s(z_0)\}=0$$

and $s(z_0)\neq 0$. Then applying Lemma 2, we have

$$\frac{z_0 s'(z_0)}{s(z_0)}=ik,$$

where

$$k\geqslant\frac{(m-p)(a^2+1)}{2a} \quad \text{when } \ \mathrm{Arg}\{s(z_0)\}=\frac{\pi}{2} \tag{18}$$

and

$$k\leqslant-\frac{(m-p)(a^2+1)}{2a} \quad \text{when } \ \mathrm{Arg}\{s(z_0)\}=-\frac{\pi}{2},$$

and where $s(z_0)=\pm ia$ and $0<a$. For the case $\mathrm{Arg}\{s(z_0)\}=\pi/2$, $s(z_0)=ia$ and $0<a$ it follows that

$$\begin{aligned}\mathfrak{Re}\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}\right\}&=\mathfrak{Re}\frac{2p}{1+ia}-\mathfrak{Re}\frac{ia}{1+ia}ik\\&=\frac{2p}{1+a^2}+\frac{ak}{1+a^2}\\&=\frac{2p+ak}{1+a^2}.\end{aligned}$$

Therefore, we have from (18)

$$\begin{aligned}\mathfrak{Re}\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}\right\}&\geqslant\frac{4p+(m-p)(1+a^2)}{2(1+a^2)}\\&=\frac{2p}{a^2+1}+\frac{m-p}{2}\\&>\frac{m-p}{2} \quad \text{for } \ a>0.\end{aligned}$$

This contradicts the hypothesis (15), and therefore, we have

$$\mathfrak{Re}\{s(z)\} > 0, \quad (z \in \mathbb{D}). \tag{19}$$

Furthermore,

$$\left|\frac{zf'(z)}{pf(z)} - 1\right| = \left|\frac{1-s(z)}{1+s(z)}\right| < 1, \quad (z \in \mathbb{D}). \tag{20}$$

For the case $\operatorname{Arg}\{s(z_0)\} = -\pi/2$, $s(z_0) = -ia$ and $0 < a$, applying the same method as above, we also have (19). Therefore, we get (20), which completes the proof of Theorem 2. $\square$

4. Discussion

In a general case of parameters p and α, the bound in implication (13) and the bound in (9) seems to be not sharp. But for $p = 1$, $\alpha = 0$ implication (13) becomes the known sharp result that each convex function is starlike of order $1/2$:

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad \Rightarrow \quad \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2}, \quad (z \in \mathbb{D}).$$

On the other hand, if we put

$$\beta = \frac{p\alpha^2 + (2p+1)\alpha + p - 1}{2(1+\alpha)}, \quad \alpha \in [0,1),$$

then $\beta \in [(p-1)/2, p)$ and

$$\frac{p}{2}(1+\alpha) = \frac{1}{4}\left(2\beta - 1 + \sqrt{4\beta^2 - 4\beta + 8p + 1}\right) := \delta(\beta, p).$$

This gives the following corollary.

COROLLARY 1. *Let $f \in \mathscr{A}_p$. Then we have*

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta \quad \Rightarrow \quad \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta(\beta, p), \quad (z \in \mathbb{D}) \tag{21}$$

for all $\beta \in [(p-1)/2, p)$, where

$$\delta(\beta, p) = \frac{1}{4}\left(2\beta - 1 + \sqrt{4\beta^2 - 4\beta + 8p + 1}\right). \tag{22}$$

This gives for $\beta = 1/2$ and $p = 2$ the following corollary.

COROLLARY 2. *If $f \in \mathscr{A}_2$, then we have*

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2} \quad \Rightarrow \mathfrak{Re}\left\{\frac{zf'(z)}{2f(z)}\right\} > \frac{1}{2}, \quad (z \in \mathbb{D}). \tag{23}$$

Recall here the sharp result [41], for the case $p=1$, namely if $f\in\mathscr{K}_1(\alpha)$, then $f\in\mathscr{S}_1^*(\delta(\alpha))$, where

$$\delta(\alpha)=\begin{cases}\frac{1-2\alpha}{2^{2-2\alpha}-2} & \text{for } \alpha\neq 1/2,\\ \frac{1}{2\log 2} & \text{for } \alpha=1/2.\end{cases} \tag{24}$$

For $p=1$ and $\beta=0$ (24) and (22) give the same result $\delta(0)=1/2=\delta(0,1)$. For $p=1$ and $\beta=1/2$ (24) becomes just over (22). Namely (24) gives $\delta(1/2)=1/\log 4=0.72\ldots$, while (22) gives $\delta(1/2,1)=\sqrt{2}/2=0.707\ldots$. If $\beta=3/4$, then also (24) becomes just over (22):

$$\delta(3/4)=\frac{2+\sqrt{2}}{4}=0.85\ldots,\quad \delta(3/4,1)=\frac{1+\sqrt{33}}{8}=0.84\ldots.$$

For $p=1$, $\alpha=0$ Theorem 1 becomes the known result for convex functions:

$$\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \quad\Rightarrow\quad \mathfrak{Re}\left\{\frac{f(z)}{z}\right\}>\frac{1}{2},\quad (z\in\mathbb{D}),$$

which may be written as $\mathscr{K}\subset\overline{co}\mathscr{K}$. By $co\mathscr{K}$ we denote the convex hull of the class of convex functions $\mathscr{K}$, that is the set of all convex combinations of functions belonging to $\mathscr{K}$. Let us recall from [2] that the closure of the set $co\mathscr{K}$ is

$$\overline{co}\mathscr{K}=\left\{f\in\mathscr{H}: f(0)=0,\ f'(0)=1,\ \mathfrak{Re}\left\{\frac{f(z)}{z}\right\}>\frac{1}{2},\quad (z\in\mathbb{D})\right\}. \tag{25}$$

Note that in [26] it was considered a version of Theorem 1 with *Arg* in the place of $\mathfrak{Re}$, see also [38], [25] and [30].

Availability of data and materials. Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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