

TOPOLOGICALLY TRANSITIVE MATRIX SEMIGROUPS

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Abstract. We determine conditions under which closed, topologically transitive, matrix semigroups must be transitive.

In this paper we shall consider (multiplicative) semigroups in $M_n(\mathbb{C})$ (or $M_n(\mathbb{R})$). Such a semigroup \mathcal{S} is *transitive* if for any two vectors \mathbf{x} and \mathbf{y} in \mathbb{C}^n (or \mathbb{R}^n) with $\mathbf{x} \neq \mathbf{0}$, there exists S in \mathcal{S} such that $S\mathbf{x} = \mathbf{y}$, and is *topologically transitive* if for any two vectors \mathbf{x} and \mathbf{y} in \mathbb{C}^n (or \mathbb{R}^n) with $\mathbf{x} \neq \mathbf{0}$, there exists a sequence S_n in \mathcal{S} such that $S_n\mathbf{x}$ converges to \mathbf{y} . We shall investigate the properties and structure of closed, topologically transitive matrix semigroups, and shall be particularly interested in the question: “What extra conditions must be imposed on such semigroups to guarantee transitivity?”

We say a vector \mathbf{x} in \mathbb{C}^n is *cyclic* for a set \mathcal{X} in $M_{m \times n}(\mathbb{C})$ if $\mathcal{X}\mathbf{x} = \{X\mathbf{x} : X \in \mathcal{X}\}$ is dense in \mathbb{C}^m , and *strictly cyclic* if $\mathcal{X}\mathbf{x} = \mathbb{C}^m$. So a set \mathcal{X} in $M_{m \times n}(\mathbb{C})$ is topologically transitive if every nonzero vector in \mathbb{C}^n is cyclic for \mathcal{X} , and is transitive if every nonzero vector in \mathbb{C}^n is strictly cyclic for \mathcal{X} . (In contexts where topological transitivity is being considered, transitivity is sometimes referred to as strict transitivity.)

We use standard notation and let $SL(n, \mathbb{C})$ (resp. $SL(n, \mathbb{R})$) denote the special linear group of determinant 1 matrices in $M_n(\mathbb{C})$ (resp. $M_n(\mathbb{R})$) and let $U(n)$ (resp. $O(n)$) denote the unitary (resp. orthogonal) group in $M_n(\mathbb{C})$ (resp. $M_n(\mathbb{R})$). It is well known that all these groups are closed and that the special linear group is transitive. (There is a slight technical problem with the definition here as invertibles cannot map a non-zero vector to zero, so we shall assume 0 is adjoined to all our groups and semigroups.) While the unitary group is not transitive, if we consider the semigroup of all multiples of unitaries, this is transitive.

We begin with some preliminary results connecting transitivity and topological transitivity.

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LEMMA 1. *If \mathcal{S} is a closed, topologically transitive matrix semigroup and there exists a rank-one matrix in \mathcal{S} then \mathcal{S} is transitive.*

Proof. If $S_0 = \mathbf{x}_0 \otimes \mathbf{y}_0$ is in \mathcal{S} with $S_0 \neq \mathbf{0}$, then $\mathbf{x}_0 \neq \mathbf{0}$ so for any \mathbf{z} there exists S_k in \mathcal{S} with $S_k \mathbf{x}_0 \rightarrow \mathbf{z}$. Then $S_k S_0 = (S_k \mathbf{x}_0) \otimes \mathbf{y}_0 \rightarrow \mathbf{z} \otimes \mathbf{y}_0$ so $\mathbf{z} \otimes \mathbf{y}_0$ is in \mathcal{S} for all \mathbf{z} . Given any $\mathbf{x} \neq \mathbf{0}$ and \mathbf{y} , choose S in \mathcal{S} such that $S\mathbf{x}$ is not orthogonal to \mathbf{y}_0 , and let $\mathbf{z} = \frac{\mathbf{y}}{\langle S\mathbf{x}, \mathbf{y}_0 \rangle}$. Then $(\mathbf{z} \otimes \mathbf{y}_0) S$ is in \mathcal{S} and

$$(\mathbf{z} \otimes \mathbf{y}_0) S\mathbf{x} = \langle S\mathbf{x}, \mathbf{y}_0 \rangle \mathbf{z} = \mathbf{y}$$

so \mathcal{S} is transitive. \square

In general, to deduce transitivity from topological transitivity, given any $\mathbf{x} \neq \mathbf{0}$ and \mathbf{y} , we will need enough information about the possible sequences S_k in \mathcal{S} with $S_k \mathbf{x} \rightarrow \mathbf{y}$, to be able to find a single S in \mathcal{S} with $S\mathbf{x} = \mathbf{y}$. In the above lemma, that is done by controlling the range. In the case where \mathcal{S} consists of multiples of unitaries, this can be done by using compactness.

LEMMA 2. *If \mathcal{S} is a closed, topologically transitive set of matrices, and every element of \mathcal{S} is a multiple of a unitary, then \mathcal{S} is transitive.*

Proof. Given $\mathbf{x} \neq \mathbf{0}$ and \mathbf{y} there exists $S_k = r_k U_k$ in \mathcal{S} with $S_k \mathbf{x} \rightarrow \mathbf{y}$ (where r_k is scalar and U_k is unitary). Then $\|S_k \mathbf{x}\| \rightarrow \|\mathbf{y}\|$ so $|r_k| \rightarrow \frac{\|\mathbf{y}\|}{\|\mathbf{x}\|}$. By compactness we can pass to a subsequence so that $r_k \rightarrow r$ and $U_k \rightarrow U$, and so $S_k \rightarrow S = rU$, so that S is in \mathcal{S} and $S\mathbf{x} = \mathbf{y}$. \square

The fact that \mathcal{S} is a semigroup will be key in cases where we can show that closed and topologically transitive imply transitive. If \mathcal{S} is just a set, very few if any results along these lines can be obtained as the following theorem illustrates.

THEOREM 3. *For, each $k = 1, 2, \dots$, let \mathcal{M}_k be a finite $\frac{1}{k}$ mesh for the unit sphere in \mathbb{C}^n , and $\mathcal{X}_k = \left\{ \sqrt{k} \mathbf{u} \otimes \mathbf{v} : \mathbf{u}, \mathbf{v} \in \mathcal{M}_k \right\}$. Then $\mathcal{X} = \bigcup_{k=1}^{\infty} \mathcal{X}_k$ is such that*

- (1) \mathcal{X} is closed (actually discrete)
- (2) \mathcal{X} is topologically transitive but not transitive.

Proof. Note that each \mathcal{X}_k is finite and each $\sqrt{k} \mathbf{u} \otimes \mathbf{v}$ in \mathcal{X}_k has norm \sqrt{k} so that \mathcal{X} is discrete and hence closed.

If \mathbf{x} and \mathbf{y} are non-zero vectors, and $\varepsilon > 0$, we can choose k large enough that $\frac{\|\mathbf{y}\|}{\sqrt{k}\|\mathbf{x}\|} \leq 1$ and $\frac{2\|\mathbf{x}\|}{\sqrt{k}} < \varepsilon$. Then choose \mathbf{u} in \mathcal{M}_k such that

$$\left\| \mathbf{u} - \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\| < \frac{1}{k}$$

and choose \mathbf{v} in \mathcal{M}_k such that

$$\left| \left\langle \frac{\mathbf{x}}{\|\mathbf{x}\|}, \mathbf{v} \right\rangle - \frac{\|\mathbf{y}\|}{\sqrt{k}\|\mathbf{x}\|} \right| \leq \frac{1}{k}.$$

We can do this since, for a fixed unit vector \mathbf{z} in \mathbb{C}^n , the map from the unit sphere in \mathbb{C}^n to \mathbb{D} , the unit disk in \mathbb{C} , given by $\mathbf{x} \rightarrow \langle \mathbf{z}, \mathbf{x} \rangle$ is a contractive surjection and so $\{\langle \mathbf{z}, \mathbf{u} \rangle : \mathbf{u} \in \mathcal{M}_k\}$ is a $\frac{1}{k}$ mesh for \mathbb{D} . Then

$$\begin{aligned} \left\| \left(\sqrt{k} \mathbf{u} \otimes \mathbf{v} \right) \mathbf{x} - \mathbf{y} \right\| &= \left\| \sqrt{k} \langle \mathbf{x}, \mathbf{v} \rangle \mathbf{u} - \mathbf{y} \right\| \\ &= \left\| \sqrt{k} \langle \mathbf{x}, \mathbf{v} \rangle \mathbf{u} - \sqrt{k} \langle \mathbf{x}, \mathbf{v} \rangle \frac{\mathbf{y}}{\|\mathbf{y}\|} + \sqrt{k} \langle \mathbf{x}, \mathbf{v} \rangle \frac{\mathbf{y}}{\|\mathbf{y}\|} - \mathbf{y} \right\| \\ &\leq \left\| \sqrt{k} \langle \mathbf{x}, \mathbf{v} \rangle \mathbf{u} - \sqrt{k} \langle \mathbf{x}, \mathbf{v} \rangle \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\| + \left\| \sqrt{k} \langle \mathbf{x}, \mathbf{v} \rangle \frac{\mathbf{y}}{\|\mathbf{y}\|} - \mathbf{y} \right\| \\ &\leq \sqrt{k} |\langle \mathbf{x}, \mathbf{v} \rangle| \left\| \mathbf{u} - \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\| + \left| \left\langle \frac{\mathbf{x}}{\|\mathbf{x}\|}, \mathbf{v} \right\rangle - \frac{\|\mathbf{y}\|}{\sqrt{k}\|\mathbf{x}\|} \right| \sqrt{k} \|\mathbf{x}\| \\ &\leq \frac{\|\mathbf{x}\|}{\sqrt{k}} + \frac{\|\mathbf{x}\|}{\sqrt{k}} < \varepsilon \end{aligned}$$

hence \mathcal{X} is topologically transitive. However, \mathcal{X} is countable and so cannot be transitive. \square

The above theorem can be modified to give a similar example in $M_n(\mathbb{R})$, and if we arrange that $\langle \mathbf{u}, \mathbf{v} \rangle \neq \pm \frac{1}{\sqrt{k}}$ when \mathbf{u}, \mathbf{v} are in \mathcal{M}_k (which can be done by perturbations), then the above proof need only be slightly adapted to show that

$$\bigcup_{k=1}^{\infty} \left\{ I + \sqrt{k} \mathbf{u} \otimes \mathbf{v} : \mathbf{u}, \mathbf{v} \in \mathcal{M}_k \right\}$$

is a set of invertible matrices which is discrete and topologically transitive, but not transitive.

Recall that an *ideal* of a semigroup \mathcal{S} is a subsemigroup \mathcal{I} such that $\mathcal{S}\mathcal{I} \subseteq \mathcal{S}$ and $\mathcal{I}\mathcal{S} \subseteq \mathcal{S}$.

LEMMA 4. *If \mathcal{I} is a non-zero ideal of a transitive (resp. topologically transitive) semigroup \mathcal{S} in $M_n(\mathbb{C})$ or $M_n(\mathbb{R})$ then \mathcal{I} is transitive (resp. topologically transitive).*

Proof. Let $\mathbf{x} \neq 0$ and \mathbf{y} be vectors in \mathbb{C}^n . Fix a non-zero element X in \mathcal{I} . Since $\mathcal{S}\mathbf{x}$ is either \mathbb{C}^n (in the transitive case) or dense in \mathbb{C}^n (in the topologically transitive case), there must be an element T of \mathcal{S} such that $T\mathbf{x}$ is not in the kernel of X . Then in the transitive case there exists S in \mathcal{S} which maps $XT\mathbf{x}$ to \mathbf{y} so SXT , which is in \mathcal{I} , maps \mathbf{x} to \mathbf{y} . In the topologically transitive case there exists a sequence S_n in \mathcal{S} such that $S_n(XT\mathbf{x})$ converges to \mathbf{y} , so S_nXT is a sequence in \mathcal{I} which, when applied to \mathbf{x} , converges to \mathbf{y} . \square

In any closed semigroup \mathcal{S} , if r is the the minimal rank of non-zero elements of \mathcal{S} , the set $\mathcal{I}_r = \{S \in \mathcal{S} : \text{rank}(S) = r \text{ or } 0\}$ is a closed ideal which shares the transitivity (or topological transitivity) properties of \mathcal{S} . Because of this we will often be able to assume \mathcal{S} has constant rank. A natural question is what minimal ranks are possible in topologically transitive semigroups. The semigroup of all rank-ones shows that minimal rank one is realizable in a topologically transitive semigroup. The special linear group shows minimal rank n is realizable in a topologically transitive semigroup. The following shows that all other minimal ranks r between 2 and n are also realizable.

THEOREM 5. For a fixed $r \in \mathbb{N}$ with $2 \leq r < n$, let $\mathcal{L} \subseteq M_{r \times (n-r)}(\mathbb{R})$ and $\mathcal{M} \subseteq M_{(n-r) \times r}(\mathbb{R})$ be non-zero closed sets containing the zero matrices of the appropriate sizes, with \mathcal{L} having the property that for each non-zero vector \mathbf{x} in \mathbb{R}^{n-r} , the set $\mathcal{L}\mathbf{x}$ is not a singleton, and \mathcal{M} having the property that it has a dense set of cyclic vectors. Assume, in addition, that \mathcal{L} and \mathcal{M} are such that $\mathcal{L}\mathcal{M}$ consists of nilpotents (for example: if elements of \mathcal{L} have non-zero entries only in the last row and elements of \mathcal{M} have non-zero entries only in the first column). Define

$$S_r = \left\{ \begin{bmatrix} A & AL \\ MA & MAL \end{bmatrix} : A \in SL(r, \mathbb{R}), L \in \mathcal{L}, M \in \mathcal{M} \right\},$$

then

- (1) S_r is a closed semigroup
- (2) S_r has constant rank r
- (3) S_r is topologically transitive but not transitive.

Proof. Note that

$$\begin{bmatrix} A & AL \\ MA & MAL \end{bmatrix} = \begin{bmatrix} I & 0 \\ M & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & L \\ 0 & 0 \end{bmatrix}$$

which gives immediately that elements of S_r have constant rank r , and also shows that S_r is a semigroup since any product of two elements in S_r can be written as follows:

$$\begin{aligned} & \begin{bmatrix} A_1 & A_1L_1 \\ M_1A_1 & M_1A_1L_1 \end{bmatrix} \begin{bmatrix} A_2 & A_2L_2 \\ M_2A_2 & M_2A_2L_2 \end{bmatrix} \\ = & \begin{bmatrix} I & 0 \\ M_1 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & L_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ M_2 & 0 \end{bmatrix} \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & L_2 \\ 0 & 0 \end{bmatrix} \\ = & \begin{bmatrix} I & 0 \\ M_1 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I + L_1M_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & L_2 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Now the hypotheses of the theorem gives that the middle three terms have an element of $SL(r, \mathbb{R})$ in the $(1, 1)$ entry and so the product is also in S_r and so S_r is a semigroup.

To show that S_r is closed, suppose that S_n is a sequence in S_r such that S_n converges to some X . Then

$$S_n = \begin{bmatrix} A_n & A_nL_n \\ M_nA_n & M_nA_nL_n \end{bmatrix} \rightarrow X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and so $A_n \rightarrow A$. Hence A is in $SL(r, \mathbb{R})$ and so A_n^{-1} is a bounded sequence which converges to A^{-1} . Thus

$$\begin{aligned} \|L_n - A^{-1}B\| &= \|A_n^{-1}(A_nL_n - B) + (A_n^{-1} - A^{-1})B\| \\ &\leq \|A_n^{-1}\| \|A_nL_n - B\| + \|A_n^{-1} - A^{-1}\| \|B\| \end{aligned}$$

converges to zero, so $A^{-1}B$ is in \mathcal{L} and hence B is of the required form of the $(1, 2)$ entry of an element of S_r . Similarly, it can be shown the remaining entries are of the required form and so S_r is closed.

The above decomposition of an element of \mathcal{S}_r also shows that a vector in the range of an element of \mathcal{S}_r is of the form

$$\begin{bmatrix} \mathbf{x} \\ M\mathbf{x} \end{bmatrix}$$

for some element M of \mathcal{M} , and so no vector can be mapped to

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{y} \end{bmatrix}$$

for nonzero \mathbf{y} , so \mathcal{S}_r is not transitive.

On the other hand, given $\epsilon > 0$, a nonzero vector

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

and a vector

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix},$$

we can choose L in \mathcal{L} such that $\mathbf{x} + L\mathbf{y} \neq \mathbf{0}$, then choose A in $SL(r, \mathbb{R})$ such that $A(\mathbf{x} + L\mathbf{y}) = \mathbf{m}$ is a cyclic vector for \mathcal{M} within $\frac{\epsilon}{2}$ of \mathbf{u} , then choose M in \mathcal{M} such that $M\mathbf{m}$ is within $\frac{\epsilon}{2}$ of \mathbf{v} . Then

$$\left\| \begin{bmatrix} A & AL \\ MA & MAL \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} - \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \right\| < \epsilon$$

so \mathcal{S}_r is topologically transitive. \square

In the above theorem, in the case where \mathcal{L} consists of matrices (of the appropriate size) which have non-zero entries only in the last row and \mathcal{M} consists of matrices which have nonzero entries only in the first column, we can see from symmetry that \mathcal{S}_r^* is also topologically transitive. We say a set \mathcal{S} in $M_n(\mathbb{C})$ (or $M_n(\mathbb{R})$) is bitransitive if both \mathcal{S} and \mathcal{S}^* (or \mathcal{S}') are transitive. Topological bitransitivity is defined similarly. So the semigroups \mathcal{S}_r serves as examples of topologically bitransitive semigroups with arbitrary minimal rank.

Some other examples of bitransitive semigroups follow.

EXAMPLE 1. Over \mathbb{R} ,

$$\left\{ [z_{ij}]_{i,j=1}^n : z_{ij} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \text{ for some } a, b \in \mathbb{R} \right\}$$

is a bitransitive semigroup (actually algebra) of minimal rank 2.

EXAMPLE 2. Over \mathbb{C} ,

$$\left\{ [z_{ij}]_{i,j=1}^n : z_{ij} = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \text{ for some } \alpha, \beta \in \mathbb{C} \right\}$$

is a bitransitive semigroup (actually an \mathbb{R} algebra) of minimal rank 2.

DEFINITION 1. A set \mathcal{S} in $M_n(\mathbb{C})$ (or $M_n(\mathbb{R})$) is *homogeneous* if \mathcal{S} is closed under multiplication by \mathbb{R}^+ (the non-negative real numbers).

For homogeneous topologically transitive semigroups, we have the following result.

THEOREM 6. *If $\mathcal{S} \subseteq M_n(\mathbb{C})$ is a closed, topologically transitive homogeneous semigroup then \mathcal{S} is transitive.*

Proof. First note that for any $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, there is a nonzero S in \mathcal{S} with either $S\mathbf{x} = \mathbf{y}$ or $S\mathbf{x} = \mathbf{0}$. To see this, choose S_k in \mathcal{S} with $S_k\mathbf{x} \rightarrow \mathbf{y}$. If $\{S_k\}$ is bounded then by passing to a subsequence (using the Bolzano-Weierstrass Theorem) we may assume that $S_k \rightarrow S$ (in \mathcal{S} since \mathcal{S} is closed), but then $S\mathbf{x} = \lim S_k\mathbf{x} = \mathbf{y}$. If, on the other hand $\{S_k\}$ is unbounded then by passing to a subsequence we may assume that $\|S_{k_j}\| \rightarrow \infty$. Passing to a further subsequence (again using the Bolzano-Weierstrass Theorem) we may assume that $\frac{1}{\|S_{k_j}\|}S_{k_j}$ converges to some S in \mathcal{S} (since \mathcal{S} is closed and homogeneous) where S has norm one. Then $S\mathbf{x} = \lim \frac{1}{\|S_{k_j}\|}S_{k_j}\mathbf{x} = \lim \frac{1}{\|S_{k_j}\|}\mathbf{y} = \mathbf{0}$.

Let r denote the minimal rank of a non-zero element of \mathcal{S} . By Lemma 3.1.6 of [3] there exists an idempotent E of rank r in \mathcal{S} and (after a similarity) $ESE|_{\text{Ran}(E)}$ consists of multiples of unitaries. So by Lemma 2 we may assume $E \neq I$. Given $\mathbf{x} \neq \mathbf{0}$ and \mathbf{y} , we must find S in \mathcal{S} with $S\mathbf{x} = \mathbf{y}$. First, by topological transitivity we can find S_1 in \mathcal{S} with $S_1\mathbf{x}$ not in the kernel of E . By considering $ES_1\mathbf{x}$ in place of \mathbf{x} we can assume \mathbf{x} is in $\text{Ran}(E)$. Now there exists S_k in \mathcal{S} with $S_k\mathbf{x} \rightarrow \mathbf{y}$ so $S_kE\mathbf{x} \rightarrow \mathbf{y}$. Consider 2×2 block matrix representations of linear maps on \mathbb{C}^n with respect to the decomposition $\text{Ran}(E) \oplus \ker(E)$. Then

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \text{ and } S_kE = \begin{bmatrix} r_k U_k & 0 \\ X_k & 0 \end{bmatrix}.$$

Writing $\mathbf{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ we see that $r_k U_k x_1 \rightarrow y_1$ and $X_k x_1 \rightarrow y_2$. Hence, as in Lemma 2 by passing to a subsequence we may assume that $r_k U_k \rightarrow rU$. If $y_2 = 0$, use ES_kE , and we are done as in Lemma 2, so with no loss of generality assume $y_2 \neq 0$. Consider the possibilities: the set $\{\|X_k\|\}$ is either bounded or unbounded. If the set is unbounded, by passing to a subsequence we may assume that $\|X_k\| \rightarrow \infty$. But then, by passing to a subsequence again we may assume that $\frac{X_k}{\|X_k\|} \rightarrow X \neq 0$ and so

$$\frac{1}{\|X_k\|}S_kE = \begin{bmatrix} \frac{r_k U_k}{\|X_k\|} & 0 \\ \frac{X_k}{\|X_k\|} & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix}.$$

But also, from above $\begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $Xx_1 = 0$ but this implies that the rank of X is less than r which is a contradiction. Hence $\{\|X_k\|\}$ is bounded, and by passing to a subsequence we can assume $X_k \rightarrow X$. But then $S_kE \rightarrow \begin{bmatrix} rU & 0 \\ X & 0 \end{bmatrix} = S \in \mathcal{S}$ and $S\mathbf{x} = \mathbf{y}$. \square

In contrast to Theorem 5, which shows that there are no restrictions on the minimal (non-zero) rank in a closed transitive semigroup, if the semigroup is also homogeneous, then the possible minimal ranks are restricted, as the following Corollary shows.

COROLLARY 7. *If S is a closed, topologically transitive, homogeneous semigroup then the minimal rank divides the spatial dimension.*

Proof. In [1] it is shown that the minimal rank of a transitive semigroup in $M_n(\mathbb{C})$ divides n . This result, along with Theorem 6 establishes the Corollary. \square

A natural example of a constant rank r closed transitive semigroup, where r divides the spatial dimension, is the set of all $\frac{n}{r} \times \frac{n}{r}$ block matrices with the property that any element in the set has only one block column with non-zero entries, and those entries are chosen independently from the set of all multiples of elements of a closed transitive unitary group. Other homogeneous examples can be constructed using block matrices where the entries are independently chosen from multiples of a closed, transitive, orthogonal or unitary group. However, as the following example shows, there are more unusual examples.

EXAMPLE 3. Fix a unitary V in $U(n)$ and consider

$$S_V = \left\{ r \begin{bmatrix} U & UX \\ YU & YUX \end{bmatrix}, r \begin{bmatrix} 0 & 0 \\ U & UX \end{bmatrix} : X = 0 \text{ or } V, Y = V^{-1}(sW - I) \right\}.$$

This semigroup is homogeneous, closed and transitive, but blocks cannot be chosen independently.

Homogeneity of a semigroup ensures that the semigroup contains many strict contractions. Even without homogeneity, the existence of even a single strict contraction in a topologically transitive semigroup in $M_n(\mathbb{C})$ (or $M_n(\mathbb{R})$) ensures the existence of some cyclic vectors, and, in conjunction with additional weak hypotheses, gives transitivity. The following simple lemma is key in proving the two theorems that follow.

LEMMA 8. *If $\{Z_k\}_{k \in \mathbb{N}}$ is a sequence of matrices in $M_n(\mathbb{C})$ (resp. $M_n(\mathbb{R})$) such that $\{\|Z_k\|\}_{k \in \mathbb{N}}$ is unbounded, and A is an invertible strict contraction in $M_n(\mathbb{C})$ (resp. $M_n(\mathbb{R})$), then there exist strictly increasing sequences $\{k_i\}_{i=1}^\infty$ and $\{m_i\}_{i=1}^\infty$ such that for $i = 1, 2, \dots$, $\|Z_{k_i}A^{m_i}\|$ lies in the interval $(1, \|A^{-1}\|]$.*

Proof. Choose k_1 such that $\|Z_{k_1}\| > \|A^{-1}\|$ and consider $\{\|Z_{k_1}A^m\|\}_{m=1}^\infty$. Since $\|Z_{k_1}A^m\| \leq \|Z_{k_1}\| \|A\|^m$, this sequence converges to 0, but $\frac{1}{\|A^{-1}\|} \|Z_{k_1}A^m\| \leq \|Z_{k_1}A^{m+1}\|$, so as we increase m , $\|Z_{k_1}A^m\|$ will eventually drop into the interval $(1, \|A^{-1}\|]$ and cannot jump over it. Let

$$m_1 = \min \{m \in \mathbb{N} : \|Z_{k_1}A^m\| \in (1, \|A^{-1}\|]\}.$$

Now choose k_2 such that, $k_2 > k_1$ and $\|Z_{k_2}A^{m_1}\| > \|A^{-1}\|$ and repeat the above argument to get $m_2 > m_1$ with $\|Z_{k_2}A^{m_2}\|$ in $(1, \|A^{-1}\|]$. We can continue in this fashion to construct the desired sequences. \square

Note, by considering adjoints, we can obtain the same result for sequences $\|B^{m_i}Z_{k_i}\|$. Also, the hypotheses of this lemma could be weakened in this case, by assuming only that B is invertible on its range, and that each $\{Z_k\}_{k \in \mathbb{N}}$ leaves the range of B invariant. In this case we obtain that $\|B_{m_i}Z^{k_i}\|$ lies in the interval $(1, \frac{1}{m_B}]$, where $m_B = \min \{\|B\mathbf{x}\| : \|\mathbf{x}\| = 1, \mathbf{x} \in \text{Ran}(B)\}$.

THEOREM 9. *If S is a closed, topologically transitive, semigroup in $M_n(\mathbb{C})$ (resp. $M_n(\mathbb{R})$) and there exists a nonzero element A of S such that*

- (1) *the rank of A is minimal among the nonzero elements of S*
- (2) *A is similar to a strict contraction*
- (3) *$A\mathbf{e} = \alpha\mathbf{e}$ where $\alpha \neq 0$ and \mathbf{e} is a non-zero vector in $M_n(\mathbb{C})$ (resp. $M_n(\mathbb{R})$); then $S\mathbf{e} = \mathbb{C}^n$ (resp. \mathbb{R}^n).*

Proof. Let r denote the rank of A . If $r = 1$, then by passing to the rank-one ideal the result follows from Lemma 1, so with no loss of generality $r > 1$. After a similarity, we may assume that there exists ρ such that $\|A\| < \rho < 1$, and then, via the Jordan Canonical Form, we can assume that

$$A = \begin{bmatrix} \alpha & a & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with respect to a decomposition $\mathbb{C}^n = (\mathbb{C}\mathbf{e}) \oplus M \oplus N$, where $0 < |\alpha| < \rho$, $\|a\| < \rho$, $\|A_0\| < \rho$ and A_0 is invertible. Then, by topological transitivity, for any \mathbf{y} in \mathbb{C}^n there exists R_k in S such that $R_k\mathbf{e} \rightarrow \frac{1}{\alpha}\mathbf{y}$ and we may write R_k with respect to the above decomposition, as

$$R_k = \begin{bmatrix} \mathbf{u}_k & V_k & W_k \end{bmatrix}$$

where $\mathbf{u}_k \rightarrow \frac{1}{\alpha}\mathbf{y}$ as $k \rightarrow \infty$. Then $S_k = R_kA$ has the property that $S_k\mathbf{e} \rightarrow \mathbf{y}$ and S_k has the form

$$S_k = \begin{bmatrix} \mathbf{y}_k & Z_k & 0 \end{bmatrix}$$

where $\mathbf{y}_k \rightarrow \mathbf{y}$. If the sequence $\|Z_k\|$ for $k = 1, 2, \dots$ is bounded, we may choose a convergent subsequence of the S_k converging to some S in S such that $S\mathbf{e} = \mathbf{y}$.

We claim that $\|Z_k\|$ cannot be unbounded. To prove this by contradiction, suppose it is bounded. Note that

$$S_kA^m = \begin{bmatrix} \mathbf{y}_k & Z_k & 0 \end{bmatrix} \begin{bmatrix} \alpha^m & a_m & 0 \\ 0 & A_0^m & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha^m\mathbf{y}_k & \mathbf{y}_ka_m + Z_kA_0^m & 0 \end{bmatrix}$$

where $\|a_m\| < \rho^m$ as it is a submatrix of a strict contraction of norm less than ρ^m .

Applying Lemma 8 to $\{Z_k\}$ and A_0 , there exists increasing sequences $\{k_i\}$ and $\{m_i\}$ such that $\|Z_{k_i}A_0^{m_i}\|$ is in $(1, \|A_0^{-1}\|]$ for all $i = 1, 2, \dots$. Let $X_i = S_{k_i}A^{m_i}$, then by construction X_i is bounded, so by passing to a subsequence we may assume that $X_i \rightarrow X \neq 0$. However, clearly $\alpha^{m_i}\mathbf{y}_{k_i}$ and $\mathbf{y}_{k_i}a_{m_i}$ converge to 0 but then, the existence of X contradicts the minimality of non-zero rank r and so our assumption that $\|Z_k\|$ is unbounded was incorrect and the theorem is proven. \square

COROLLARY 10. *If \mathcal{G} is a group in $M_n(\mathbb{C})$ (resp. $M_n(\mathbb{R})$), $\mathcal{G} \cup \{0\}$ is closed and topologically transitive and there exists an element A of \mathcal{G} which is similar to a strict contraction and has $A\mathbf{e} = \alpha\mathbf{e}$ where \mathbf{e} is non-zero and $\alpha \in \mathbb{C}$ (resp. \mathbb{R}), then $\mathcal{G} \cup \{0\}$ is transitive.*

Proof. This follows from Theorem 9 since given vectors $\mathbf{x} \neq 0$ and $\mathbf{y} \neq 0$ there exists $G \in \mathcal{G}$ with $G\mathbf{e} = \mathbf{x}$ and $H \in \mathcal{G}$ with $H\mathbf{e} = \mathbf{y}$, but then $HG^{-1}\mathbf{x} = \mathbf{y}$. \square

THEOREM 11. *If S is a closed, topologically transitive semigroup in $M_n(\mathbb{C})$ which contains no non-zero nilpotents, and contains a nonzero matrix A which is similar to a strict contraction then S is transitive.*

Proof. Since there are no nilpotents in S , by consideration of the Jordan Canonical form, there exists m in \mathbb{N} such that $\text{Ran}(A^j) = \text{Ran}(A^m) \neq \{0\}$ and $\ker A^j = \ker A^m \neq \mathbb{C}^n$ for all $j \geq m$.

If \mathbf{x} is a non-zero vector in \mathbb{C}^n , by the topological transitivity of the minimal-rank ideal, there exists X in S of minimal rank such that $X\mathbf{x}$ is not in the kernel of A^m . Then for large enough j , $B = A^jX$ is a strict contraction of minimal rank such that $B\mathbf{x} = A^jX\mathbf{x} \neq 0$.

If \mathbf{y} is any vector in the range of B (so $B\mathbf{z} = \mathbf{y}$ for some \mathbf{z}), then by topological transitivity there exists X_n in S such that $X_nB\mathbf{x} \rightarrow \mathbf{z}$, so $BX_nB\mathbf{x} \rightarrow \mathbf{y}$. If BX_nB is bounded we may find a convergent subsequence and so an element S in S with $S\mathbf{x} = \mathbf{y}$. If BX_nB is unbounded, then, applying Lemma 8 (and the comments following the lemma), can find increasing sequences $\{k_i\}$ and $\{m_i\}$ such that $B^{m_i}BX_{k_i}B$ all have norm in $\left[1, \frac{1}{m_B}\right]$. But by passing to subsequences we may assume $B^{k_i}BX_{k_i}B \rightarrow T \neq 0$ and thus $T\mathbf{x} = \lim B^{k_i}BX_{k_i}B\mathbf{x} = \lim B^{k_i}\mathbf{y} = 0$. But we have now constructed a non-zero matrix T in S with rank below the minimal rank of B which is a contradiction. Thus we have shown that any non-zero vector in \mathbb{C}^n can be mapped onto a vector in the range of a strict contraction in S by an element of S . Thus we can map any vector \mathbf{x} to an eigenvector (corresponding to a nonzero eigenvalue) of a strict contraction, and then by Theorem 9 onto any vector, establishing transitivity. \square

Even in two spatial dimensions there are non-trivial problems. We begin by constructing a topologically transitive semigroup of invertibles which is closed in $M_2(\mathbb{R})$ but is not transitive.

For a matrix Q with rational entries, define the degree of Q to be

$$\text{deg}(Q) = \min \{n \in \mathbb{N} : nQ \in M_2(\mathbb{Z})\}$$

and note that $\text{deg}(PQ) \leq \text{deg}(P) \cdot \text{deg}(Q)$ for all P and Q in $M_2(\mathbb{Q})$.

THEOREM 12. *Let*

$$\mathcal{Q} = \{Q \in M_2(\mathbb{Q}) : \text{deg}(Q) \leq |\det(Q)|\}$$

then S is a closed (discrete) semigroup in $M_2(\mathbb{R})$ which is topologically transitive but not transitive.

Proof. If P and Q are in \mathcal{Q} then

$$\deg(PQ) \leq \deg(P)\deg(Q) \leq |\det(P)| |\det(Q)| = |\det(PQ)|$$

so \mathcal{Q} is a semigroup.

For each k in \mathbb{N} , let $\mathcal{Q}_k = \{Q \in \mathcal{Q} : |\deg(Q)| = k\}$. Then each \mathcal{Q}_k is discrete and $\mathcal{Q} = \bigcup_{k=1}^{\infty} \mathcal{Q}_k$, so if $\{Q_n\}_{n=1}^{\infty}$ is a sequence in \mathcal{S} with $Q_n \rightarrow Q$ and Q is not in \mathcal{Q} then $\deg(Q_n) \rightarrow \infty$. But then $|\det(Q_n)| \rightarrow \infty$ so $\{Q_n\}_{n=1}^{\infty}$ does not converge and hence \mathcal{Q} is closed.

The semigroup \mathcal{Q} is clearly not transitive since it is countable, but $\mathcal{Q} \cup \{0\}$ does act transitively on \mathbb{Q}^2 . The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ can be mapped to any non-zero vector $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ in \mathbb{Q}^2 by $\begin{bmatrix} q_1 & -nq_2 \\ q_2 & nq_1 \end{bmatrix}$, which will be in \mathcal{Q} if n in \mathbb{N} is chosen large enough, and any non-zero vector $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ can be mapped to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by $\frac{1}{p_1+p_2} \begin{bmatrix} p_1 & p_2 \\ -np_2 & np_1 \end{bmatrix}$ where again n can be chosen large enough to ensure this is in \mathcal{Q} . Composing these we can map any non-zero $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ to any non-zero $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$.

The proof will be complete once we show topological transitivity. To this end, let $x = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ be a non-zero vector in \mathbb{R}^2 (by symmetry, with no loss of generality we will assume $\alpha \neq 0$, and it is enough to show $\mathcal{Q} \begin{bmatrix} \alpha \\ 1 \end{bmatrix}$ is dense in \mathbb{R}^2 when α is irrational, since $\mathcal{Q} \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \alpha \mathcal{Q} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and for nonzero β , $\mathcal{Q} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \beta \mathcal{Q} \begin{bmatrix} \alpha/\beta \\ 1 \end{bmatrix}$).

Let $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ be a nonzero vector in \mathbb{Q}^2 . With no loss of generality assume $r_1 \neq 0$ (if not just reverse the role of $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$ in what follows). If, for each $\epsilon > 0$ we exhibit an element Q of \mathcal{Q} with $\left\| Q \begin{bmatrix} \alpha \\ 1 \end{bmatrix} - \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \right\| < \epsilon$ then topological transitivity will be established. Let $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$ be consecutive convergents of α (see [4] for information on convergents and continued fractions) and let

$$Q_n = \begin{bmatrix} q_n & -p_n + r_1 \\ q_{n+1} & -p_{n+1} + r_2 \end{bmatrix}.$$

Then

$$\left\| Q_n \begin{bmatrix} \alpha \\ 1 \end{bmatrix} - \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} q_n\alpha - p_n \\ q_{n+1}\alpha - p_{n+1} \end{bmatrix} \right\| < \frac{\sqrt{2}}{q_n}$$

due to the approximation properties of convergents (see page 163 of [4]) and

$$|\det(Q_n)| = |p_nq_{n+1} - p_{n+1}q_n + q_nr_2 - q_{n+1}r_1|.$$

Consecutive convergents satisfy $p_n q_{n+1} - p_{n+1} q_n = (-1)^n$ so

$$|\det(Q_n)| \geq |(-1)^n + q_n r_2 - q_{n+1} r_1| \geq q_{n+1} \left| \frac{q_n}{q_{n+1}} r_2 - r_1 \right| - 1.$$

Since $q_{n+1} \rightarrow \infty$ and $\frac{q_n}{q_{n+1}} \rightarrow 0$ we can choose n large enough to guarantee that $\frac{\sqrt{2}}{q_n} < \epsilon$ and

$$q_{n+1} \left| \frac{q_n}{q_{n+1}} r_2 - r_1 \right| - 1 > \deg \begin{bmatrix} 0 & r_1 \\ 0 & r_2 \end{bmatrix}$$

so Q_n is in \mathcal{Q} . \square

Our final example is of a closed group in $M_n(\mathbb{R})$ which is countable and hence not transitive and yet, computational evidence suggests is topologically transitive. Let

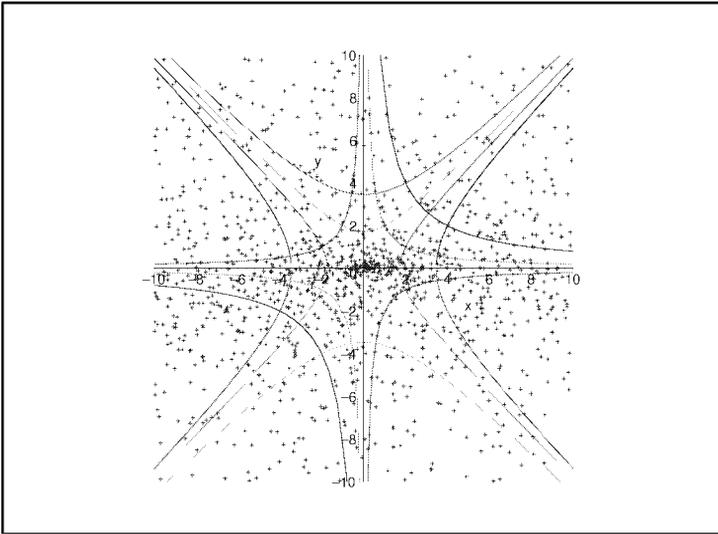
$$A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

and let

$$B = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

THEOREM 13. *The group \mathcal{G} generated by A and B is free and is a closed subgroup of $SL(2, \mathbb{R})$ which is discrete in the usual topology on $SL(2, \mathbb{R})$.*

The idea behind the proof of this theorem is quite straightforward. The image of the unit ball under a word $W = w(A, B)$ is an ellipse whose orientation is determined mainly by the initial letter in the word W and whose major radius is determined mainly by the length of the word W . For words beginning with A , the major axis of this ellipse is very close to the x -axis. For words beginning with A^{-1} , the major axis is very close to the y -axis. For words beginning with B , the major axis is very close to the line $y = x$, while for words beginning with B^{-1} , the major axis is very close to the line $y = -x$. In addition, the length of the major axis increases exponentially in relation to the word length (and hence so does the norm). The details of these claims are in the Lemma which follows. With these facts in hand, consider a sequence W_n in the group \mathcal{G} which converges to some W in $M_2(\mathbb{R})$. The length of the words W_n must be bounded and eventually all terms must map the unit ball to ellipses with major axis falling into one of the four cases above. But this implies that eventually all W_n have the same initial letter, which we may assume, with no loss of generality, is A . Then consider $A^{-1}W_n$ and repeat the above argument. Since word length is bounded, we must have that the sequence W_n is eventually constant, which proves discreteness.



Using Maple, we can plot the orbit of randomly chosen points in the plane under the above group. When A (or A^{-1}) acts on a vector \mathbf{x} it moves that vector along a hyperbola of the form $xy = c$, and when B (or B^{-1}) acts on a vector it moves it along a hyperbola of the form $x^2 - y^2 = c$. All evidence indicates that the group is topologically transitive. The plot below shows the orbit of the point $(1, 2)$ under all words in the group of length eight or less.

The image of the unit circle $\{\mathbf{x} : \|\mathbf{x}\| = 1\}$ under an invertible matrix A^{-1} is the ellipse $\{\mathbf{x} : \|\mathbf{A}\mathbf{x}\| = 1\} = \{\mathbf{x} : \langle \mathbf{A}^* \mathbf{A} \mathbf{x}, \mathbf{x} \rangle = 1\}$ and the principal axes of the ellipse are the eigenvectors of $\mathbf{A}^* \mathbf{A}$, lengths of the principal axis of the ellipse are the eigenvalues of $\mathbf{A}^* \mathbf{A}$. These facts and symmetry considerations, along with the following technical lemma, contains the details for the above claim that the image of the unit ball under a word $W = w(A, B)$ is an ellipse whose orientation is determined mainly by the initial letter in the word W and whose major radius is determined mainly by the length of the word W .

Let

$$D_\beta = \begin{bmatrix} \beta & 0 \\ 0 & \frac{1}{\beta} \end{bmatrix} \quad \text{and} \quad U_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

LEMMA 14. *If $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$, $\alpha > 2$, $\beta > 4$ and $B = U_\theta D_\beta U_{-\theta}$ then $D_\alpha B D_\alpha = U_{\theta_1} D_\gamma U_{-\theta_1}$ where $|\theta_1| < \frac{\pi}{6}$ and $\gamma \geq \frac{\sqrt{7}}{2} \beta$.*

Proof. Since $D_\alpha B D_\alpha$ is a positive matrix of determinant 1, it can be written as $U_{\theta_1} D_\gamma U_{-\theta_1}$ for some choice of γ and θ_1 . Computing both sides of $D_\alpha B D_\alpha =$

$U_{\theta_1} D_\gamma U_{-\theta_1}$ and equating entries we obtain that

$$\text{from the (1, 2) entry : } \cos \theta \sin \theta \left(\beta - \frac{1}{\beta} \right) = \cos \theta_1 \sin \theta_1 \left(\gamma - \frac{1}{\gamma} \right)$$

$$\text{from the (1, 1) entry : } \alpha^2 \left(\beta \cos^2 \theta + \frac{1}{\beta} \sin^2 \theta \right) = \gamma \cos^2 \theta_1 + \frac{1}{\gamma} \sin^2 \theta_1$$

$$\text{from the (2, 2) entry : } \frac{1}{\alpha^2} \left(\frac{1}{\beta} \cos^2 \theta + \beta \sin^2 \theta \right) = \frac{1}{\gamma} \cos^2 \theta_1 + \gamma \sin^2 \theta_1$$

Considering the (1, 1) entry minus the (2, 2) entry we see that

$$\begin{aligned} \cos(2\theta_1) \left(\gamma - \frac{1}{\gamma} \right) &= (\cos^2 \theta_1 - \sin^2 \theta_1) \left(\gamma - \frac{1}{\gamma} \right) \\ &= \beta \left(\alpha^2 \cos^2 \theta - \frac{1}{\alpha^2} \sin^2 \theta \right) + \frac{1}{\beta} \left(\alpha^2 \sin^2 \theta - \frac{1}{\alpha^2} \cos^2 \theta \right) \\ &\geq \beta \left(\alpha^2 \cos^2 \theta - \frac{1}{\alpha^2} \sin^2 \theta \right) \geq \beta \alpha^2 \cos^2 \theta \\ &\geq \beta \alpha^2 \cos^2 \left(\frac{\pi}{3} \right) = \frac{\beta \alpha^2}{4} \end{aligned}$$

while from the (1, 2) entry we see that

$$\begin{aligned} \sin(2\theta_1) \left(\gamma - \frac{1}{\gamma} \right) &= 2 \cos \theta_1 \sin \theta_1 \left(\gamma - \frac{1}{\gamma} \right) \\ &= 2 \cos \theta \sin \theta \left(\beta - \frac{1}{\beta} \right) = \sin(2\theta) \left(\beta - \frac{1}{\beta} \right) \\ &\geq \sin \left(\frac{\pi}{3} \right) \beta = \frac{\sqrt{3}\beta}{2}. \end{aligned}$$

Hence

$$\begin{aligned} \left(\gamma - \frac{1}{\gamma} \right)^2 &= \cos^2(2\theta_1) \left(\gamma - \frac{1}{\gamma} \right)^2 + \sin^2(2\theta_1) \left(\gamma - \frac{1}{\gamma} \right)^2 \\ &\geq \beta^2 \left(\frac{\alpha^4}{16} + \frac{3}{4} \right) = \beta^2 \left(\frac{\alpha^4 + 12}{16} \right) \end{aligned}$$

Since $\gamma > 1$,

$$\gamma > \gamma - \frac{1}{\gamma} \geq \beta \left(\frac{\sqrt{\alpha^4 + 12}}{4} \right)$$

Isolating $\sin^2 \theta_1$ in the (2, 2) entry we see that

$$\begin{aligned} \sin^2 \theta_1 &= \frac{1}{\gamma} \left(\frac{1}{\alpha^2} \left(\frac{1}{\beta} \cos^2 \theta + \beta \sin^2 \theta \right) - \frac{1}{\gamma} \cos^2 \theta_1 \right) \\ &\leq \frac{1}{\gamma} \frac{1}{\alpha^2} \left(\frac{1}{\beta} \cos^2 \theta + \beta \sin^2 \theta \right) \leq \frac{1}{\gamma} \frac{1}{\alpha^2} (2\beta) \\ &\leq \frac{1}{\alpha^2} \frac{4}{\sqrt{\alpha^4 + 12}} \end{aligned}$$

Since $\alpha \geq 2$, we obtain that $\gamma \geq \frac{\sqrt{7}}{2}\beta$ and that $\sin \theta_1 \leq \frac{1}{\sqrt[3]{28}} < \frac{1}{2}$, so $\theta_1 < \frac{\pi}{6}$, and the Lemma is proven. \square

REMARK 1. The above example can be generalized as follows. Let D_β and U_θ be as above. For $n \in \mathbb{N}$, with $\theta = \frac{\pi}{2n}$ and β large enough, one can show similarly to above that the group generated by

$$\{D_\beta, U_\theta D_\beta U_{-\theta}, U_\theta^2 D_\beta U_{-\theta}^2, \dots, U_\theta^{n-1} D_\beta U_{-\theta}^{n-1}\}$$

is a closed subgroup of $SL(2, \mathbb{R})$ which gives a new representation of the free group on n letters in $SL(2, \mathbb{R})$. By permuting the angle θ slightly so that $\cos^2 \theta$ and $\cos \theta \sin \theta$ are rational and choosing β rational as well, we can arrange for this group to be in $SL(2, \mathbb{Q})$. Also, this representation has the property that all the generators (and their inverses) are unitarily equivalent.

With transitivity there is a concept of sharp transitivity. If S is transitive and the function mapping $S \rightarrow Sx$ is one-to-one for each non-zero x then we say S is sharply transitive. For topological transitivity we can use the same definition: a matrix semigroup S is *sharply topologically transitive* if S is topologically transitive and the function mapping $S \rightarrow Sx$ is one-to-one for each non-zero x .

Closed, sharply transitive groups have been classified in [2] and in [1] sharply transitive matrix semigroups are classified. Transitive groups (or semigroups) with the property of sharpness seem to be more tractable. Our final result shows that, at least in some cases, semigroups which are topologically transitive but not transitive have the property of sharpness.

LEMMA 15. *If \mathcal{G} is a closed topologically transitive group in $SL(2, \mathbb{R})$ which is not sharply topologically transitive then \mathcal{G} is transitive.*

Proof. Assume \mathcal{G} is not sharply topologically transitive. Then we must have some non-zero \mathbf{x} in \mathbb{R}^2 such that $S\mathbf{x} = T\mathbf{x}$ for distinct S and T in \mathcal{G} . It follows that $G\mathbf{x} = \mathbf{x}$ for some G in \mathcal{G} besides the identity. We will show that $\mathcal{G}\mathbf{x}$ is the whole space for this particular \mathbf{x} , therefore for every vector because \mathcal{G} is a group; this will show \mathcal{G} is transitive. Suppose not. It is not difficult to see that there is a \mathbf{y} which is not a multiple of \mathbf{x} and doesn't belong to $\mathcal{G}\mathbf{x}$. (For, if $\mathcal{G}\mathbf{x}$ contains every vector \mathbf{v} that is not a multiple of \mathbf{x} , and if t is nonzero, first choose a member S of \mathcal{G} with $S\mathbf{x}$ not a multiple of \mathbf{x} and then choose T in \mathcal{G} such that $T\mathbf{x}$ is $tS\mathbf{x}$, giving $(S^{-1}T)\mathbf{x} = t\mathbf{x}$. Thus every vector of the form $t\mathbf{x}$ is in $\mathcal{G}\mathbf{x}$. If all vectors which are not multiples of $t\mathbf{x}$ are also in $\mathcal{G}\mathbf{x}$ then \mathcal{G} is transitive and we are done.) Next assume with no loss of generality that \mathbf{y} is perpendicular to \mathbf{x} . Write matrices with respect to the basis $\{\mathbf{x}, \mathbf{y}\}$. Now G has matrix

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

with t not zero, so \mathcal{G} contains all matrices

$$G^k = \begin{bmatrix} 1 & kt \\ 0 & 1 \end{bmatrix}$$

for all integers k .

Given any $\mathbf{z} \in \mathbb{R}^2$, by topological transitivity there exists a sequence of matrices S_n in \mathcal{G} such that $S_n \mathbf{x} \rightarrow \mathbf{z}$. So the first column of S_n converges to \mathbf{z} . However we can replace this sequence with $T_n = S_n G^{k_n}$ so that the first column of T_n is the same as the first column of S_n and the k_n are chosen large enough (with plus or minus sign) so that the inner products $(T_n \mathbf{y}, \mathbf{y})$, (i.e, the the southeast entry of the matrix of each T_n) can be made smaller than $\epsilon \|\mathbf{z}\|$ in absolute value. Now $\det(T_n) = 1$ implies that the northeast entry is also bounded. Thus $\{T_n\}$ is bounded and we can extract a convergent subsequence converging to $T \in \mathcal{G}$ such that $T \mathbf{x} = \mathbf{z}$. \square

There are a number of open questions regarding closed topologically transitive semigroups. We close with just a few:

- (1) Is the existence of a single strict contraction in the semigroup enough to yield transitivity? Our Theorem 11 required the additional condition that there be no non-zero nilpotents in the semigroup, but this condition may not be necessary.
- (2) Is the example given in Theorem 13 topologically transitive?
- (3) Is there a deeper general connection between sharpness and lack of transitivity?

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