

WEYL MATRIX FUNCTIONS AND INVERSE PROBLEMS FOR DISCRETE DIRAC-TYPE SELF-ADJOINT SYSTEMS: EXPLICIT AND GENERAL SOLUTIONS

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Abstract. It is shown that the discrete Dirac-type self-adjoint system is equivalent to the block Szegő recurrence. A representation of the fundamental solution is obtained, inverse problems on the interval and semiaxis are solved. A Borg-Marchenko type result is obtained, too. Connections with block Toeplitz matrices are treated

1. Introduction

The continuous self-adjoint Dirac-type system

$$\frac{dY}{dx}(x, z) = i(zj + jV(x))Y(x, z), \quad j = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \quad (1.1)$$

is a classical object of analysis with various applications (in mathematical physics and nonlinear integrable equations, in particular). Here I_p is the $p \times p$ identity matrix and v is a $p \times p$ matrix function. In this paper, we treat a discrete self-adjoint Dirac-type system:

$$W_{k+1}(\lambda) - W_k(\lambda) = -\frac{i}{\lambda} j C_k W_k(\lambda) \quad (k \in \{0, 1, 2, \dots\}), \quad (1.2)$$

where C_k are $m \times m$ matrices, $m = 2p$, which are Hermitian and j -unitary:

$$C_k = C_k^*, \quad C_{kj}C_k = j, \quad (1.3)$$

To see that (1.2) is a discrete analog of system (1.1), notice that (1.1) is equivalent to a subclass of canonical systems $W_x = izjH(x)W$ (see [41, 45] and the references therein). One can follow also the arguments from [32], where the skew self-adjoint discrete Dirac-type system has been studied and explicit solutions of the isotropic

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Heisenberg magnet model have been obtained. As suggested in [32] we introduce matrix functions U and W by the relations

$$W(x, z) = U(x)Y(x, z), \quad \frac{dU}{dx}(x) = -iU(x)jV(x), \quad U(0) = I_m. \quad (1.4)$$

Since V is self-adjoint, we get from (1.4) that U is j -unitary, i.e., $UjU^* \equiv j$. Now (1.1) and the first relation in (1.4) yield

$$\frac{dW}{dx} = \frac{dU}{dx}U^{-1}W + iU(zj + jV)U^{-1}W = izjHW, \quad (1.5)$$

where $H = jUU^*j = H^*$, $HjH \equiv j$. Compare system (1.2), where the matrices C_k satisfy (1.3), and system (1.5) to see that (1.2) is an immediate discrete analog of (1.1).

When $p = 1$ and $C_k > 0$, system (1.2) is equivalent to the well-known self-adjoint Szegő recurrence, which plays an important role in the orthogonal polynomials theory and is also an auxiliary system for the Ablowitz-Ladik hierarchy (see, for instance, [26, 27, 47] and various references therein). The equivalence of system (1.2), where $C_k > 0$ and $C_kjC_k = j$, to the block (matrix-valued) Szegő recurrence studied in [15] - [18] is given in Proposition 2.1.

Discrete analogs for the Dirac systems are also of independent interest (see, for instance, [32] for applications to the isotropic Heisenberg magnet model). Weyl-Titchmarsh theory of the discrete systems is treated in a series of recent interesting papers by F. Gesztesy and coauthors. There are direct connections between Weyl functions of the discrete self-adjoint Dirac-type systems and extensions of Toeplitz matrices (see Section 6).

We consider a representation of the fundamental solution of system (1.2) and solve direct and inverse problems directly in terms of the Weyl functions. Both explicit and general solutions are obtained. First, we obtain explicit solutions of the direct and inverse problems for system (1.2) for the case of the so called *pseudo-exponential potentials* C_k (the case of the rational Weyl functions). Our case includes as a subcase the rapidly decaying *strictly pseudo-exponential potentials*, see also Remark 4.5. Recall that discrete and continuous systems with potentials, which belong to the subclass of the strictly pseudo-exponential potentials, have been actively studied in [1]-[6], [8]-[10]. In particular, direct and inverse problems for Szegő recurrence on the semiaxis $\{0, 1, 2, \dots\}$ with scalar ($p = 1$) strictly pseudo-exponential potentials have been treated in [5, 6]. Direct and inverse problems for pseudo-exponential potentials (continuous case) have been studied in a series of Gohberg-Kaashoek-Sakhnovich papers [29, 30] (see the references therein and see also [24] for the case of the generalized pseudo-exponential potentials). The case of the discrete skew-self-adjoint Dirac system has been studied in [32]. Notice that similar to [29, 39] (see also [5, 6, 8, 29, 32]) we start our explicit constructions with the explicit formula for the fundamental solution.

For a more general (non-rational) situation of the Weyl functions $\phi(z) = \sum_{k=0}^{\infty} \phi_k z^k$

such that $\sum_{k=0}^{\infty} \|\phi_k\| < \infty$ the direct problem for a block (matrix-valued) Szegő recurrence on the semiaxis (including non-self-adjoint case and under some additional

conditions) is treated in a recent important Alpay-Gohberg paper [7]. In Sections 5 and 6 we solve direct and inverse problems for the general type potentials $C_k > 0$ (and thus for the general type self-adjoint block Szegő recurrence) on the interval and semiaxis. A Borg-Marchenko type uniqueness result for system (1.2) is obtained, too. Connections with the well known Toeplitz matrices appear. For interesting discussions on the connections between Toeplitz matrices, Szegő recurrences and orthogonal polynomials see also [3, 17, 18] and the references therein. Interesting spectral theoretical results on the discrete canonical systems, where $C_k j C_k = 0$, one can find in [35, 44]. A complete Weyl theory for Jacobi matrices and various useful references are contained in [48].

2. Preliminaries

An important discrete analog for Dirac-type systems takes the form

$$X_{k+1}(z) = \mathcal{D}_k H_k \begin{bmatrix} z I_p & 0 \\ 0 & I_p \end{bmatrix} X_k(z), \quad (2.1)$$

where

$$H_k = \begin{bmatrix} I_p & -\rho_k \\ -\rho_k^* & I_p \end{bmatrix}, \quad \mathcal{D}_k = \text{diag} \left\{ (I_p - \rho_k \rho_k^*)^{-\frac{1}{2}}, (I_p - \rho_k^* \rho_k)^{-\frac{1}{2}} \right\}, \quad (2.2)$$

and the $p \times p$ matrices ρ_k are strictly contractive, that is $\|\rho_k\| < 1$.

Notice that the matrices $\mathcal{D}_k H_k$ are j -unitary, i.e.,

$$\mathcal{D}_k H_k j H_k \mathcal{D}_k = H_k \mathcal{D}_k j \mathcal{D}_k H_k = j. \quad (2.3)$$

System (2.1) can be rewritten in the form (1.2) by using the transformation

$$W_k(\lambda) = (i - \lambda^{-1})^k U_k \left(I_m - \frac{i}{\lambda} j \right) X_k(z), \quad C_k = j U_{k+1} j U_{k+1}^{-1} = j U_{k+1} U_{k+1}^* j, \quad (2.4)$$

where $U_0 := I_m$ ($m = 2p$),

$$U_k := (i H_0 \mathcal{D}_0 j) (i H_1 \mathcal{D}_1 j) \times \dots \times (i H_{k-1} \mathcal{D}_{k-1} j) \quad (k > 0), \quad z = \frac{1 + i\lambda}{1 - i\lambda}. \quad (2.5)$$

A particular scalar case ($p = 1$) of system (2.1) is the well known Szegő recurrence, where

$$\mathcal{D}_k = \frac{1}{\sqrt{1 - |\rho_k|^2}} I_2, \quad H_k = \begin{bmatrix} 1 & -\rho_k \\ -\rho_k & 1 \end{bmatrix} > 0, \quad |\rho_k| < 1. \quad (2.6)$$

When $p = 1$ one easily removes the factor $(\sqrt{1 - |\rho_k|^2})^{-1}$ in (2.1) to obtain systems as in [5, 6].

Coefficients ρ_k are called Schur (or sometimes Verblunsky) coefficients (see, for instance, [17, 47] and various references therein). Notice that the matrices C_k given by the second relation in (2.4) are positive definite. Vice versa, if the matrices C_k are positive definite, the Szegő recurrence is uniquely recovered from system (1.2). The same is true for the block Szegő recurrences (2.1).

PROPOSITION 2.1. *There is a one to one correspondence between the subclass of systems (1.2), where the matrices $C_k > 0$ satisfy (1.3), and block Szegő recurrences (2.1), where H_k and \mathcal{D}_k are defined via (2.2) and $\|\rho_k\| < 1$. This correspondence is given by (2.4), (2.5) to map block Szegő recurrences into Dirac-type systems. The inverse mapping of the Dirac-type systems into block Szegő recurrences is given by the first relation in (2.4) and equalities*

$$\rho_k = -\left(\tilde{\beta}_2(k)^{-1}\tilde{\beta}_1(k)\right)^*, \quad \tilde{\beta}(k) = [0, \quad I_p]R_k, \quad R_k = \left(jU_k^*C_kU_kj\right)^{\frac{1}{2}}. \quad (2.7)$$

where $\tilde{\beta}_1(k)$ and $\tilde{\beta}_2(k)$ are $p \times p$ blocks of $\tilde{\beta}(k)$. We recover H_l and \mathcal{D}_l (hence, U_{l+1}) from ρ_l successively, starting from $l = 0$.

Proof. By (2.5) the matrices U_k are j -unitary. Therefore, the matrices C_k defined by the second relation in (2.4) are positive definite and $C_kjC_k = j$. The first relation in (2.4) easily follows from (1.2), (2.1), the second relation in (2.4) and (2.5). Thus, the first part of the proposition is proved.

To prove the second part of the proposition notice that according to (2.7) we have $R_k > 0$. Next, assume that the second part of the proposition is true for all $l < k$. In particular, the j -unitary matrix U_k is defined via (2.5), and so $R_k > 0$ is well-defined. It follows that $R_k^2jR_k^2 = j$. Moreover, R_k^2 is unitary equivalent to a diagonal matrix $D_k > 0$:

$$R_k^2 = \widehat{U}_k^*D_k\widehat{U}_k, \quad \widehat{U}_k^*\widehat{U}_k = \widehat{U}_k\widehat{U}_k^* = I_m. \quad (2.8)$$

Then, formula (2.8) implies that $\widehat{U}_k^*D_k\widehat{U}_kj\widehat{U}_k^*D_k\widehat{U}_k = j$, i.e.,

$$D_k^{-1} = J_kD_kJ_k, \quad J_k := \widehat{U}_kj\widehat{U}_k^*, \quad J_k = J_k^* = J_k^{-1}. \quad (2.9)$$

Taking square roots in both parts of the first equality in (2.9) we get

$$D_k^{-\frac{1}{2}} = J_kD_k^{\frac{1}{2}}J_k, \text{ i.e.,}$$

$$D_k^{\frac{1}{2}}J_kD_k^{\frac{1}{2}} = J_k. \quad (2.10)$$

By (2.8) we have $R_k = \widehat{U}_k^*D_k^{\frac{1}{2}}\widehat{U}_k$, and, taking into account (2.10), we get

$$R_kjR_k = \widehat{U}_k^*D_k^{\frac{1}{2}}J_kD_k^{\frac{1}{2}}\widehat{U}_k = \widehat{U}_k^*J_k\widehat{U}_k = \widehat{U}_k^*\widehat{U}_kj\widehat{U}_k^*\widehat{U}_k = j. \quad (2.11)$$

In particular, formula (2.11) yields

$$\tilde{\beta}(k)j\tilde{\beta}(k)^* = -I_p, \quad (2.12)$$

and so $\tilde{\beta}_2(k)$ is invertible and $\rho_k^*\rho_k < I_p$. From (2.7) and (2.11) it follows that

$$jU_k^*C_kU_kj - j = R_k^2 - R_kjR_k = R_k(I_m - j)R_k = 2\tilde{\beta}(k)^*\tilde{\beta}(k). \quad (2.13)$$

According to (2.7) and (2.12) it is true that $(I_p - \rho_k^*\rho_k)^{-1} = \tilde{\beta}_2^*\tilde{\beta}_2$. Hence, in view of (2.2) and (2.7) we get

$$H_kD_k^2H_k - j = H_kD_k(I_m - j)D_kH_k = 2\tilde{\beta}(k)^*\tilde{\beta}(k). \quad (2.14)$$

Formulas (2.13) and (2.14) imply

$$H_k D_k^2 H_k = j U_k^* C_k U_{kj}. \quad (2.15)$$

Notice that the second relation in (2.4) is equivalent to

$$C_k = j U_k H_k D_k^2 H_k U_{kj}^*. \quad (2.16)$$

Therefore, (2.15) means that (2.7) defines a transformation inverse to the transformation defined by the second relation in (2.4). Moreover, (2.16) implies (2.14). By (2.14) we have $[-\rho_k^* \ I_p] = u \hat{\beta}(k)$. Thus there is a unique ρ_k with the property (2.16).

As $U_0 = I_m$, the same proof as above is valid for $l = 0$. In this way the second part of proposition is proved by induction. \square

REMARK 2.2. By the proof of Proposition 2.1 we obtain for any j -unitary matrix $\mathcal{C} > 0$ a representation $\mathcal{C} = H \mathcal{D}^2 H$, where H and \mathcal{D} have the form (2.2). This representation follows from the equality $C^{\frac{1}{2}} j C^{\frac{1}{2}} = j$, which is proved similar to $R_k j R_k = j$, and from the well-known (see, for instance, [17, 19]) Halmos extension representation.

The spectral theory of discrete and continuous systems is strongly related to the construction of the fundamental solutions (see, for instance, [6]-[10], [29, 30, 32, 34], [39]-[45] and the references therein). The j -properties of the fundamental solutions play an important role [10, 17, 21, 22, 23, 31, 34, 44, 45].

For the case of the explicit construction the version of the Bäcklund-Darboux transformation (BDT) introduced in [37, 38, 39] proves to be very fruitful. Now we are going to present a corresponding principle of constructing of sequences $\{C_k\}_{k=0}^{\infty}$.

Choose $n > 0$, two $n \times n$ parameter matrices A ($\det A \neq 0$) and $S_0 = S_0^*$, and an $n \times m$ parameter matrix Π_0 such that

$$A S_0 - S_0 A^* = i \Pi_0 j \Pi_0^*. \quad (2.17)$$

Define recursively sequences $\{\Pi_k\}$ and $\{S_k\}$ ($k > 0$) by the relations

$$\Pi_{k+1} = \Pi_k + i A^{-1} \Pi_k j, \quad (2.18)$$

$$S_{k+1} = S_k + A^{-1} S_k (A^*)^{-1} + A^{-1} \Pi_k \Pi_k^* (A^*)^{-1}. \quad (2.19)$$

It follows that the matrix identity

$$A S_{k+1} - S_{k+1} A^* = i \Pi_{k+1} j \Pi_{k+1}^* \quad (k \geq 0) \quad (2.20)$$

is true. Following the lines of the discrete BDT version for the skew self-adjoint discrete Dirac-type system presented in [32], we get the following result.

THEOREM 2.3. *Suppose $\det S_r \neq 0$ ($0 \leq r \leq N$). Then the fundamental solution W_{k+1} of system (1.2), where*

$$C_k := I_m + \Pi_k^* S_k^{-1} \Pi_k - \Pi_{k+1}^* S_{k+1}^{-1} \Pi_{k+1}, \quad (2.21)$$

admits the representation

$$W_{k+1}(\lambda) = w_A(k+1, \lambda) \left(I_m - \frac{i}{\lambda} j \right)^{k+1} w_A(0, \lambda)^{-1} \quad (0 \leq k < N). \quad (2.22)$$

Here W_{k+1} is normalized by the condition $W_0(\lambda) = I_m$, and

$$w_A(k, \lambda) := I_m - ij\Pi_k^*S_k^{-1}(A - \lambda I_n)^{-1}\Pi_k, \tag{2.23}$$

The right hand side of (2.23) with fixed k is a so called transfer matrix function in Lev Sakhnovich form [43]-[45].

We say that a system (1.2), where the matrices C_k are given by (2.21), is *determined* by the parameter matrices A , S_0 and Π_0 .

Proof of Theorem 2.3. Formula (2.22) easily follows from the equality

$$w_A(k + 1, \lambda) \left(I_m - \frac{i}{\lambda}j \right) = \left(I_m - \frac{i}{\lambda}jC_k \right) w_A(k, \lambda), \tag{2.24}$$

which is basic for this proof. We shall derive now formula (2.24). Taking into account (2.23), one can see that (2.24) is equivalent to the equality

$$\begin{aligned} -\frac{i}{\lambda}j(I_m - C_k) &= -\left(I_m - \frac{i}{\lambda}jC_k \right) ij\Pi_k^*S_k^{-1}(A - \lambda I_n)^{-1}\Pi_k \\ &\quad + ij\Pi_{k+1}^*S_{k+1}^{-1}(A - \lambda I_n)^{-1}\Pi_{k+1} \left(I_m - \frac{i}{\lambda}j \right), \end{aligned} \tag{2.25}$$

i.e., the Taylor coefficients at infinity of the matrix functions at both sides of (2.25) coincide. Hence, by the series expansion

$$(A - \lambda I_n)^{-1} = -\lambda^{-1} \sum_{r=0}^{\infty} (\lambda^{-1}A)^r$$

and formula (2.18), formula (2.25) is equivalent to a family of identities:

$$I_m - C_k = -\Pi_k^*S_k^{-1}\Pi_k + \Pi_{k+1}^*S_{k+1}^{-1}\Pi_{k+1} \tag{2.26}$$

and

$$K_k A^{r-2} \Pi_k = 0 \quad (r > 0), \tag{2.27}$$

where

$$K_k := \Pi_{k+1}^*S_{k+1}^{-1}(A^2 + I_n) - \Pi_k^*S_k^{-1}A^2 + iC_kj\Pi_k^*S_k^{-1}A. \tag{2.28}$$

Notice that (2.26) is immediate from (2.21). If we prove also $K_k = 0$, then (2.27) will follow, and so we will get (2.25) or equivalently (2.24), which implies (2.22). It remains to show that $K_k = 0$. For this purpose we shall rewrite (2.28) using (2.18) and (2.21):

$$\begin{aligned} K_k &= \Pi_{k+1}^*S_{k+1}^{-1}(A^2 + I_n) - \Pi_k^*S_k^{-1}A^2 + ij\Pi_k^*S_k^{-1}A + i\Pi_k^*S_k^{-1}\Pi_kj\Pi_k^*S_k^{-1}A \\ &\quad - i\Pi_{k+1}^*S_{k+1}^{-1}(\Pi_k + iA^{-1}\Pi_kj)j\Pi_k^*S_k^{-1}A. \end{aligned} \tag{2.29}$$

According to (2.20) we have $i\Pi_kj\Pi_k^*S_k^{-1} = A - S_kA^*S_k^{-1}$. Therefore, from (2.29) we derive

$$K_k = \Pi_{k+1}^*S_{k+1}^{-1}(I_n + S_kA^*S_k^{-1}A + A^{-1}\Pi_k\Pi_k^*S_k^{-1}A) - \Pi_k^*A^*S_k^{-1}A + ij\Pi_k^*S_k^{-1}A.$$

In view of (2.19) we simplify our last formula:

$$K_k = \Pi_{k+1}^*A^*S_k^{-1}A - \Pi_k^*A^*S_k^{-1}A + ij\Pi_k^*S_k^{-1}A. \tag{2.30}$$

Finally, by (2.18) and (2.30) we have $K_k = 0$.

PROPOSITION 2.4. *Suppose $\det S_r \neq 0$ ($0 \leq r \leq N$). Then the matrices C_k ($0 \leq k < N$) given by (2.21) satisfy conditions (1.3).*

Proof. The first equality in (1.3) is immediate. To prove the second equality notice that by the standard calculations in S -node theory [43]-[45] (see also, for instance, formula (2.10) in [24]) it follows from (2.17) and (2.19) that

$$w_A(r, \lambda)^* j w_A(r, \lambda) = j + i(\bar{\lambda} - \lambda) \Pi_r^* (A^* - \bar{\lambda} I_n)^{-1} S_r^{-1} (A - \lambda I_n)^{-1} \Pi_r. \quad (2.31)$$

In particular, we have

$$w_A(r, \bar{\lambda})^* j w_A(r, \lambda) = j. \quad r \geq 0. \quad (2.32)$$

It is easily checked also that

$$\left(I_m + \frac{i}{\lambda} j \right) j \left(I_m - \frac{i}{\lambda} j \right) = \left(1 + \frac{1}{\lambda^2} \right) j. \quad (2.33)$$

According to (2.24) formulas (2.32) and (2.33) yield the equality

$$\left(I_m + \frac{i}{\lambda} C_k j \right) j \left(I_m - \frac{i}{\lambda} j C_k \right) = \left(1 + \frac{1}{\lambda^2} \right) j. \quad (2.34)$$

Therefore the second equality in (1.3) holds. \square

3. Auxiliary propositions

Recall that the invertibility of the Hermitian matrices S_k is essential for our constructions. On the other hand the important subcase of Szegő recursion corresponds to system (1.2), where $C_k > 0$. A natural condition, when all $S_k > 0$ and $C_k > 0$ is given in our next proposition.

PROPOSITION 3.1. *Let the parameter matrix S_0 be positive definite, i.e., $S_0 > 0$. Then we have*

$$S_k > 0 \quad (k \geq 0), \quad C_k > 0 \quad (k \geq 0). \quad (3.1)$$

Proof. The inequalities for S_k in (3.1) follow from (2.19) by induction. To derive the relations $C_k > 0$, introduce first two block matrices:

$$G = \begin{bmatrix} S_k & \Pi_k \\ \Pi_k^* & c I_m \end{bmatrix}, \quad F = \begin{bmatrix} A^{-1} & a A^{-1} \Pi_k \\ 0 & -i b j \end{bmatrix}, \quad (3.2)$$

where

$$a(2 + ac) = 1, \quad b(1 + ac) = 1, \quad (3.3)$$

and, moreover, c is sufficiently large so that $G > 0$. We shall discuss the choice of a and b satisfying (3.3) later on. According to (2.18), (2.19), (3.2) and (3.3), direct calculations show that

$$G + F G F^* = \begin{bmatrix} S_{k+1} & \Pi_{k+1} \\ \Pi_{k+1}^* & c(1 + b^2) I_m \end{bmatrix}. \quad (3.4)$$

As $G + FGF^* > G > 0$, we have $G^{-1} > (G + FGF^*)^{-1}$, and therefore, the inequality holds also for the $m \times m$ right lower blocks of these matrices: $(G^{-1})_{22} > ((G + FGF^*)^{-1})_{22}$. Finally, we obtain

$$\left((G^{-1})_{22} \right)^{-1} < \left(((G + FGF^*)^{-1})_{22} \right)^{-1}. \tag{3.5}$$

Taking into account (3.2), we can rewrite (3.5) in the form

$$cI_m - \Pi_k^* S_k^{-1} \Pi_k < c(1 + b^2)I_m - \Pi_{k+1}^* S_{k+1}^{-1} \Pi_{k+1}. \tag{3.6}$$

Let us fix c and choose a root a ($0 < a < 1/2$), of the equation

$$a^2 + \frac{2}{c}a - \frac{1}{c} = 0,$$

which is always possible. Putting also $b = a(1 - a)^{-1}$, we see that relations (3.3) hold. Moreover, the first relation in (3.3) means that $a^2c = 1 - 2a$. Hence,

$$cb^2 = ca^2(1 - a)^2 = (1 - 2a)(1 - a)^2 < 1. \tag{3.7}$$

From (3.6) and (3.7) it follows that

$$I_m + \Pi_k^* S_k^{-1} \Pi_k - \Pi_{k+1}^* S_{k+1}^{-1} \Pi_{k+1} > 0 \tag{3.8}$$

Recall the definition (2.21) of C_k to see that inequality (3.8) implies $C_k > 0$. □

In this section we shall need as well another property of C_k .

PROPOSITION 3.2. *Let relations (1.3) hold, and assume that $C_k > 0$. Then we have $C_k \pm j \geq 0$.*

Proof. It follows from (1.3) that

$$(C_k + \varepsilon j)j(C_k + \varepsilon j) = 2\varepsilon \left(C_k + \frac{1 + \varepsilon^2}{2\varepsilon}j \right). \tag{3.9}$$

If $(C_k + \varepsilon j)f = 0$, then by (3.9) we have also $\left(C_k + \frac{1 + \varepsilon^2}{2\varepsilon}j \right)f = 0$, and so $(1 - \varepsilon^2)jf = 0$. Therefore, we have $\det(C_k + \varepsilon j) \neq 0$, when $|\varepsilon| < 1$. Thus, the inequality $C_k > 0$ yields $(C_k + \varepsilon j) \geq 0$ for $|\varepsilon| \leq 1$. □

REMARK 3.3. Under the conditions of Proposition 3.1, formulas (2.18), (2.19), and (2.21) explicitly define system (1.2), where the matrices C_k are positive definite and j -unitary. The Schur coefficients of the corresponding Szegő recurrence are then explicitly defined via (2.7). The matrices U_k in the third relation in (2.7) are iteratively defined using (2.5).

4. Weyl functions, direct and inverse problem: the case of the pseudoexponential potentials

Guided by the definitions of the Weyl functions for Sturm-Liouville, Dirac-type and canonical systems on the semiaxis (see, for instance, [33, 45] and the references therein), we can define also corresponding modifications of Weyl functions for system (1.2). Namely, let the matrices $C_k > 0$ satisfy (1.3). Then, a $p \times p$ matrix function φ holomorphic in the lower halfplane \mathbb{C}_- is said to be a Weyl function for system (1.2) on the semiaxis $k \in \{0, 1, 2, \dots\}$, if the inequality

$$\sum_{k=0}^{\infty} [\varphi(\lambda)^* \quad I_p] q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) \begin{bmatrix} \varphi(\lambda) \\ I_p \end{bmatrix} < \infty \quad (4.1)$$

holds, where $q(\lambda) = |\lambda^2|(|\lambda^2| + 1)^{-1}$.

REMARK 4.1. Similar to the continuous case we have a summation formula:

$$\sum_{k=0}^r q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) = \frac{|\lambda^2| + 1}{i(\lambda - \bar{\lambda})} (q(\lambda)^{r+1} W_{r+1}(\lambda)^* j W_{r+1}(\lambda) - j). \quad (4.2)$$

Indeed, according to (1.2) and (1.3) we have

$$\begin{aligned} W_{k+1}(\lambda)^* j W_{k+1}(\lambda) &= W_k(\lambda)^* \left(I_m + \frac{i}{\lambda} C_k j \right) j \left(I_m - \frac{i}{\bar{\lambda}} j C_k \right) W_k(\lambda) \\ &= q(\lambda)^{-1} W_k(\lambda)^* j W_k(\lambda) + \frac{i(\lambda - \bar{\lambda})}{|\lambda^2|} W_k(\lambda)^* C_k W_k(\lambda), \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{|\lambda^2| + 1}{i(\lambda - \bar{\lambda})} \left(q(\lambda)^{k+1} W_{k+1}(\lambda)^* j W_{k+1}(\lambda) - q(\lambda)^k W_k(\lambda)^* j W_k(\lambda) \right) \\ = q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda). \end{aligned} \quad (4.3)$$

Formula (4.3) yields (4.2).

To construct the Weyl function, partition first our parameter matrix Π_0 and the matrix-function $w_A(0, \lambda)$, defined via the parameter matrices A , S_0 , and Π_0 satisfying (2.17), into blocks:

$$\Pi_0 = [\Phi \quad \Psi], \quad w_A(0, \lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix}. \quad (4.4)$$

Similar to the considerations in [29] it follows from (2.23) that

$$b(\lambda)d(\lambda)^{-1} = -i\Phi^* S_0^{-1} (A^\times - \lambda I_n)^{-1} \Psi, \quad A^\times = A + i\Psi\Psi^* S_0^{-1}. \quad (4.5)$$

To calculate $d(\lambda)^{-1}$ here, we use the following fact from system theory:

$$\left(I_p + C(\lambda I_n - A)^{-1} B \right)^{-1} = I_p - C(\lambda I_n - (A - BC))^{-1} B. \quad (4.6)$$

THEOREM 4.2. *Let parameter matrices be fixed, assume $S_0 > 0$, and define C_k by (2.21). Then system (1.2) is well-defined on the semiaxis and its unique Weyl function, which satisfies (4.1), takes the form*

$$\varphi(\lambda) = -i\Phi^* S_0^{-1} (A^\times - \lambda I_n)^{-1} \Psi, \quad A^\times = A + i\Psi\Psi^* S_0^{-1}. \quad (4.7)$$

Proof. By Proposition 3.1 system (1.2) is well-defined. Now, relations (4.5) imply $\varphi = bd^{-1}$ for the matrix function φ given by (4.7). According to (2.31) we have

$$w_A(0, \lambda)^* j w_A(0, \lambda) \leq j \quad (\lambda \in \mathbb{C}_-),$$

and it follows, in particular, that $d(\lambda)^* d(\lambda) \geq I_p + b(\lambda)^* b(\lambda)$. Therefore, we get

$$\varphi(\lambda)^* \varphi(\lambda) < I_p \quad (\lambda \in \mathbb{C}_-), \quad (4.8)$$

and so φ is holomorphic in \mathbb{C}_- . Notice that the equality $\varphi = bd^{-1}$ is equivalent to the formula

$$\begin{bmatrix} \varphi(\lambda) \\ I_p \end{bmatrix} = w_A(0, \lambda) \begin{bmatrix} 0 \\ I_p \end{bmatrix} d(\lambda)^{-1}. \quad (4.9)$$

Taking into account (4.9) and $w_A(r+1, \lambda)^* j w_A(r+1, \lambda) \leq j$, we derive from representation (2.22) of $W_{r+1}(\lambda)$ that

$$\begin{aligned} [\varphi(\lambda)^* \quad I_p] W_{r+1}(\lambda)^* j W_{r+1}(\lambda) \begin{bmatrix} \varphi(\lambda) \\ I_p \end{bmatrix} &= |\lambda + i|^{2r+2} |\lambda|^{-2r-2} (d(\lambda)^*)^{-1} \\ &\times [0 \quad I_p] w_A(r+1, \lambda)^* j w_A(r+1, \lambda) \begin{bmatrix} 0 \\ I_p \end{bmatrix} d(\lambda)^{-1} < 0. \end{aligned} \quad (4.10)$$

By (4.2) and (4.10) the inequality

$$[\varphi(\lambda)^* \quad I_p] \sum_{k=0}^r q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) \begin{bmatrix} \varphi(\lambda) \\ I_p \end{bmatrix} < \frac{|\lambda^2| + 1}{i(\lambda - \bar{\lambda})} I_p \quad (4.11)$$

is true. From (4.11) inequality (4.1) is immediate, i.e., φ defined by (4.7) is a Weyl function.

Let us show that there is no other Weyl function. First notice that by Proposition 3.2 we have inequality $W_s^* C_s W_s \geq W_s^* j W_s$. Now, use relation (4.3) to derive inequality $q^s W_s^* j W_s \geq q^{s-1} W_{s-1}^* j W_{s-1}$. From the inequalities above we get

$$q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) \geq j. \quad (4.12)$$

Therefore, the following equality is immediate for any $f \in \mathbb{C}^p$:

$$\sum_{k=0}^{\infty} f^* [I_p \quad 0] q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) \begin{bmatrix} I_p \\ 0 \end{bmatrix} f = \infty. \quad (4.13)$$

According to (4.1) and (4.13), the dimension of the subspace L of \mathbb{C}^m , such that for all $h \in L$ we have

$$\sum_{k=0}^{\infty} h^* q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) h < \infty, \quad (4.14)$$

equals p . Now, suppose that there would be a Weyl function $\tilde{\varphi} \neq \varphi$, where φ is given by (4.7). Then the columns of $\begin{bmatrix} \varphi(\lambda) \\ I_p \end{bmatrix}$ and the columns of $\begin{bmatrix} \tilde{\varphi}(\lambda) \\ I_p \end{bmatrix}$ would belong to L . Therefore, $\dim L > p$ for those λ , where $\tilde{\varphi}(\lambda) \neq \varphi(\lambda)$, which would be a contradiction. \square

REMARK 4.3. If $S_0 > 0$, then by (2.18)-(2.21) we can substitute parameter matrices A , S_0 and Π_0 by the parameter matrices $S_0^{-\frac{1}{2}}AS_0^{\frac{1}{2}}$, I_n and $S_0^{-\frac{1}{2}}\Pi_0$, which determine the same system. For $S_0 = I_p$ formula (4.7) takes the form

$$\varphi(\lambda) = -i\Phi^*(A^\times - \lambda I_n)^{-1}\Psi, \quad A^\times = A + i\Psi\Psi^*, \quad (4.15)$$

and we have $A^\times - (A^\times)^* = i(\Phi\Phi^* + \Psi\Psi^*)$, $\det(A^\times - i\Psi\Psi^*) \neq 0$.

Recall also that by Proposition 3.1 the condition $S_0 > 0$ implies (3.1).

EXAMPLE 4.4. Consider the simplest example: $p = 1$, $n = 1$, $A = a \in \mathbb{R}$ ($a \neq 0$), $S_0 = 1$. From (2.17) and (2.18) it follows that $|\Phi| = |\Psi|$ and

$$\Pi_k = \begin{bmatrix} \left(\frac{a+i}{a}\right)^k \Phi & \left(\frac{a-i}{a}\right)^k \Psi \end{bmatrix}, \quad \Pi_k \Pi_k^* = 2|\Phi|^2 \left(\frac{a^2+1}{a^2}\right)^k. \quad (4.16)$$

Now, in view of $S_0 = 1$, (2.19) and the second relation in (4.16) one can check that

$$S_k = (k\zeta + 1) \left(\frac{a^2+1}{a^2}\right)^k, \quad \zeta = \frac{2|\Phi|^2}{a^2+1}. \quad (4.17)$$

Finally, using (2.21), (4.16) and (4.17) we get the entries $(C_k)_{ij}$ of C_k :

$$(C_k)_{11} = (C_k)_{22} = 1 + \zeta|\Phi|^2(k\zeta + 1)^{-1}((k+1)\zeta + 1)^{-1}, \quad (4.18)$$

$$(C_k)_{21} = \overline{(C_k)_{12}} = \Phi\bar{\Psi} \left((k\zeta + 1)^{-1} \left(\frac{a+i}{a-i}\right)^k - ((k+1)\zeta + 1)^{-1} \left(\frac{a+i}{a-i}\right)^{k+1} \right). \quad (4.19)$$

The Weyl function of system (1.2), where the matrices C_k are given by (4.18) and (4.19), is easily calculated using (4.15):

$$\varphi(\lambda) = i\bar{\Phi}\Psi(\lambda - a - i|\Psi|^2)^{-1}. \quad (4.20)$$

REMARK 4.5. Notice that our matrices C_k are determined by the parameter matrices A_0 , S_0 , and Π_0 via formulas (2.18), (2.19), and (2.21). Similar to the continuous case [29], the sets of matrices $\{C_k\}_{k \geq 0}$ are called pseudo-exponential potentials. In view of (2.19), we require that $0 \notin \sigma(A)$ (σ - spectrum). The class of the pseudo-exponential potentials contains an important subclass of the strictly pseudo-exponential potentials, which is characterized by the additional requirement $\sigma(A) \subset \mathbb{C}_-$. Such a subclass has been treated for Szegő recurrence ($p = 1$) in [5, 6]. In particular, for the strictly pseudo-exponential subcase the inequality $|\varphi(\lambda)| < 1$ for $\lambda \in \overline{\mathbb{C}_-}$ is true. On the other hand, in the simple example above we have $\sigma(A) = a \in \mathbb{R}$ and $|\varphi| = 1$ for $\lambda = a$.

According to (4.7) and (4.8) the Weyl function φ is a rational, strictly proper matrix function, which is contractive in \mathbb{C}_- . The rational, strictly proper matrix functions admit non-unique representations $\varphi(\lambda) = \mathcal{C}(\lambda I_n - \mathcal{A})^{-1} \mathcal{B}$, which are called realizations. Here n is some natural number and \mathcal{C} , \mathcal{A} , and \mathcal{B} are $p \times n$, $n \times n$, and $n \times p$ matrices, respectively. (See, for instance, [11], [14].) As the matrix function φ is also contractive in \mathbb{C}_- , so by the proof of Theorem 9.4 [30], there are such realizations

$$\varphi(\lambda) = -i\tilde{\Phi}^*(\theta - \lambda I_n)^{-1}\tilde{\Psi} \tag{4.21}$$

of φ that $\tilde{\Phi}$, θ , and $\tilde{\Psi}$ from these realizations satisfy the identity

$$\theta - \theta^* = i(\tilde{\Phi}\tilde{\Phi}^* + \tilde{\Psi}\tilde{\Psi}^*). \tag{4.22}$$

A direct calculation shows also that formulas (4.21) and (4.22) yield $I_p - \varphi^* \varphi \geq 0$ for $\lambda \in \mathbb{C}_-$. So, realization (4.21), (4.22) is equivalent to the fact that the function is rational, strictly proper, and contractive in \mathbb{C}_- .

THEOREM 4.6. *A matrix function φ is the Weyl function of some system (1.2) determined by the parameter matrices A , Π_0 , and $S_0 > 0$ if and only if it admits representation (4.21), (4.22) such that $\det(\theta - i\tilde{\Psi}\tilde{\Psi}^*) \neq 0$. In this case φ is the Weyl function of some system (1.2), where $C_k > 0$. To recover such system put*

$$S_0 = I_n, \quad A = \theta - i\tilde{\Psi}\tilde{\Psi}^*, \quad \Phi = \tilde{\Phi}, \quad \Psi = \tilde{\Psi}, \quad \Pi_0 = [\Phi \quad \Psi], \tag{4.23}$$

and define the matrices C_k by formula (2.21), where the matrices Π_k and S_k ($k > 0$) are given by formulas (2.18) and (2.19).

Proof. The necessity of the theorem’s conditions follows from Remark 4.3. Now, suppose that these conditions are fulfilled. Then, from (4.22) and (4.23) it follows that the identity (2.17) holds for the parameter matrices. Therefore, system (1.2) is defined. So, by Theorem 4.2 φ is the Weyl function of this system. □

REMARK 4.7. The Weyl functions in the upper halfplane can be treated in a quite similar way. That is, we define Weyl functions in \mathbb{C}_+ by the inequality

$$\sum_{k=0}^{\infty} [I_p \quad \varphi(\lambda)^*] q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) \begin{bmatrix} I_p \\ \varphi(\lambda) \end{bmatrix} < \infty. \tag{4.24}$$

Then the Weyl function of system (1.2), where matrices C_k are given by (2.21) and $S_0 > 0$, takes the form

$$\varphi(\lambda) = c(\lambda)a(\lambda)^{-1} = i\Psi^*S_0^{-1}(A^\times - \lambda I_n)^{-1}\Phi, \quad A^\times = A - i\Phi\Phi^*S_0^{-1}. \tag{4.25}$$

A definition of a Weyl function in \mathbb{C}_- can be also given in a more general form.

DEFINITION 4.8. Let the matrices $C_k > 0$ satisfy (1.3). Then, a $p \times p$ matrix function φ holomorphic in \mathbb{C}_- is said to be a Weyl function for system (1.2) on the semiaxis $k \in \{0, 1, 2, \dots\}$, if the following inequality holds:

$$\sum_{k=0}^{\infty} [i\varphi(\lambda)^* \quad I_p] q(\lambda)^k K W_k(\lambda)^* C_k W_k(\lambda) K^* \begin{bmatrix} -i\varphi(\lambda) \\ I_p \end{bmatrix} < \infty. \quad (4.26)$$

Here $K^* = K^{-1}$ and $q(\lambda) = |\lambda^2|(|\lambda^2| + 1)^{-1}$.

If $K = I_n$ then inequality (4.26) coincides with inequality (4.1). In general, the choice of the matrix K is related to the choice of the domain of the operator corresponding to the Dirac system, and usually K is chosen so that the Weyl functions are Herglotz functions. Further we assume that

$$K = \frac{1}{\sqrt{2}} \begin{bmatrix} I_p & -I_p \\ I_p & I_p \end{bmatrix}, \quad (4.27)$$

Simple transformations show that the contractive Weyl function φ_I defined via (4.1) and the Weyl function φ_K defined via (4.26) and (4.27) are connected by the relation

$$\varphi_K = -i(I_p - \varphi_I)(I_p + \varphi_I)^{-1}. \quad (4.28)$$

From (4.28) it follows that

$$\varphi_K(\lambda) - \varphi_K(\lambda)^* = -2i(I_p + \varphi_I(\lambda)^*)^{-1}(I_p - \varphi_I(\lambda)^* \varphi_I(\lambda))(I_p + \varphi_I(\lambda))^{-1}, \quad (4.29)$$

$\lambda \in \mathbb{C}_-$. Thus, according to (4.8) and (4.29) φ_K is a Herglotz function with a non-positive imaginary part in \mathbb{C}_- .

5. Weyl functions, direct and inverse problem on the interval: general case

In this section we shall consider the self-adjoint matricial discrete Dirac-type system (1.2) on the interval $k \in \{0, 1, 2, \dots\}$. We assume that (1.3) holds and $C_k > 0$. It was shown in Proposition 2.1 that these properties yield $C_k = jU_k R_k^2 U_k^* j$, where $R_k = R_k^*$ and $R_k j R_k = j$. Hence, we get

$$C_k + j = jU_k(R_k^2 + R_k j R_k)U_k^* j = jU_k R_k (I_m + j) R_k U_k^* j = 2\widehat{\beta}(k)^* \widehat{\beta}(k), \quad 0 \leq k \leq N,$$

where

$$\widehat{\beta}(k) = [I_p \quad 0] R_k U_k^* j, \quad \widehat{\beta}(k) j \widehat{\beta}(k)^* = I_p. \quad (5.1)$$

Further we shall use these relations:

$$C_k = 2\widehat{\beta}(k)^* \widehat{\beta}(k) - j, \quad \widehat{\beta}(k) j \widehat{\beta}(k)^* = I_p, \quad 0 \leq k \leq N. \quad (5.2)$$

REMARK 5.1. Relations (5.2) are equivalent to the relations $C_k > 0$ and (1.3) for $0 \leq k \leq N$. Indeed, we have just derived (5.2) from $C_k > 0$ and $C_k j C_k = j$ ($0 \leq k \leq N$), and vice versa: direct calculation shows that (5.2) yields (1.3). To derive from (5.2) also the inequality $C_k > 0$, choose a matrix $\check{\beta}(k)$ such that

$$\check{\beta}(k) j \widehat{\beta}(k)^* = 0, \quad \check{\beta}(k) j \check{\beta}(k)^* = -I_p. \tag{5.3}$$

Notice, that in view of the second relation in (5.2), the maximal subspace, which is j -orthogonal to the rows of $\widehat{\beta}_k$, proves to be p -dimensional and j -negative, i.e., $\check{\beta}(k)$ always exists. According to (5.2) and (5.3) we have

$$\begin{bmatrix} \widehat{\beta}(k) \\ \check{\beta}(k) \end{bmatrix} j \begin{bmatrix} \widehat{\beta}(k) \\ \check{\beta}(k) \end{bmatrix}^* = j = \begin{bmatrix} \widehat{\beta}(k) \\ \check{\beta}(k) \end{bmatrix}^* j \begin{bmatrix} \widehat{\beta}(k) \\ \check{\beta}(k) \end{bmatrix}. \tag{5.4}$$

Finally, by the first relation in (5.2) and by (5.4) we obtain

$$C_k = \widehat{\beta}(k)^* \widehat{\beta}(k) + \check{\beta}(k)^* \check{\beta}(k) = \begin{bmatrix} \widehat{\beta}(k) \\ \check{\beta}(k) \end{bmatrix}^* \begin{bmatrix} \widehat{\beta}(k) \\ \check{\beta}(k) \end{bmatrix} > 0. \tag{5.5}$$

From the second relation in (5.2), for $k \geq 0$, it follows also that

$$\det(\widehat{\beta}(k) j \widehat{\beta}(k+1)^*) \neq 0, \quad 0 \leq k \leq N-1. \tag{5.6}$$

Indeed, if (5.6) does not hold, we have $\widehat{\beta}(k) j \widehat{\beta}(k+1)^* f = 0$ for some $f \neq 0$. Then, in view of the second relations in (5.2) for $k \geq 0$, we see that the linear span of the rows of $\widehat{\beta}_k$ and of $f^* \widehat{\beta}_{k+1}$ forms a $p+1$ -dimensional j -positive subspace of \mathbb{C}_m , which is impossible.

Similar to the continuous case [45], the Weyl functions of the discrete system on the interval will be defined via Möbius (linear-fractional) transformation

$$\varphi(\lambda) = i(\mathscr{W}_{21}(\lambda)R(\lambda) + \mathscr{W}_{22}(\lambda)Q(\lambda))(\mathscr{W}_{11}(\lambda)R(\lambda) + \mathscr{W}_{12}(\lambda)Q(\lambda))^{-1}, \tag{5.7}$$

where R and Q are $p \times p$ analytic functions in the neighbourhood of $\lambda = -i$, and

$$\mathscr{W}(\lambda) = \{\mathscr{W}_{ij}(\lambda)\}_{i,j=1}^2 = KW_{N+1}(\overline{\lambda})^*. \tag{5.8}$$

Here, the coefficients \mathscr{W}_{ij} of the Möbius transformation are the $p \times p$ blocks of \mathscr{W} , the matrix K is given by (4.27) and

$$K^* = K^{-1}, \quad K j K^* = J, \quad J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}. \tag{5.9}$$

It is convenient to put $\beta(k) := \widehat{\beta}(k)K^*$ and rewrite (5.2) as

$$C_k = 2K^* \beta(k)^* \beta(k) K - j, \quad \beta(k) j \beta(k)^* = I_p, \quad 0 \leq k \leq N. \tag{5.10}$$

Similar to relations (5.2), by Remark 5.1 relations (5.10) are equivalent to $C_k > 0$ and $C_k j C_k = j$ for $0 \leq k \leq N$. We shall need the following analog (for the self-adjoint case) of Theorem 3.4 [42].

THEOREM 5.2. Suppose W ($W_0(\lambda) = I_m$) is the fundamental solution of system (1.2), which satisfies conditions (5.10). Suppose also that a $p \times p$ matrix function φ is given by formulas (5.7) and (5.8), where

$$\det(\mathscr{W}_{11}(-i)R(-i) + \mathscr{W}_{12}(-i)Q(-i)) \neq 0. \quad (5.11)$$

Then system (1.2) satisfies (1.3), $C_k > 0$ ($0 \leq k \leq N$), and the inequalities

$$\det(\beta(k)J\beta(k+1)^*) \neq 0, \quad 0 \leq k \leq N-1 \quad (5.12)$$

hold. Moreover, system (1.2) on the interval $0 \leq k \leq N$ is uniquely recovered from the first $N+1$ Taylor coefficients $\{\alpha_k\}_{k=0}^N$ of $i\varphi\left(i\left(\frac{z+1}{z-1}\right)\right)$ at $z=0$ by the following procedure.

First, introduce $(N+1)p \times p$ matrices Φ_1, Φ_2 :

$$\Phi_1 = \begin{bmatrix} I_p \\ I_p \\ \dots \\ I_p \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} \alpha_0 \\ \alpha_0 + \alpha_1 \\ \dots \\ \alpha_0 + \alpha_1 + \dots + \alpha_N \end{bmatrix}. \quad (5.13)$$

Then, introduce an $(N+1)p \times 2p$ matrix Π and an $(N+1)p \times (N+1)p$ block lower triangular matrix A by the blocks: $\Pi = [\Phi_1 \quad \Phi_2]$,

$$A := A(N) = \left\{ a_{j-k} \right\}_{k,j=0}^N, \quad a_r = \begin{cases} 0 & \text{for } r > 0 \\ \frac{i}{2} I_p & \text{for } r = 0 \\ i I_p & \text{for } r < 0 \end{cases}. \quad (5.14)$$

Next, we recover the $(N+1)p \times (N+1)p$ matrix S as a unique solution of the matrix identity

$$AS - SA^* = i\Pi J \Pi^*. \quad (5.15)$$

This solution is invertible and positive, i.e., $S > 0$.

Finally, the matrices $\beta(k)^* \beta(k)$ ($0 < k \leq N$) are recovered using the formula

$$\beta(k)^* \beta(k) = \Pi^* P_k^* (P_k S P_k^*)^{-1} P_k \Pi - \Pi^* P_{k-1}^* (P_{k-1} S P_{k-1}^*)^{-1} P_{k-1} \Pi, \quad (5.16)$$

where P_k is a $(k+1)p \times (N+1)p$ matrix:

$$P_k = \begin{bmatrix} I_{(k+1)p} & 0 \end{bmatrix}. \quad (5.17)$$

The matrix $\beta(0)^* \beta(0)$ is given by the relation

$$\beta(0)^* \beta(0) = \Pi^* P_0^* (P_0 S P_0^*)^{-1} P_0 \Pi. \quad (5.18)$$

Now, the matrices C_k and system (1.2) are defined via the first equality in (5.10).

Proof. Step 1. According to Remark 5.1 the relations $C_k > 0$ and (1.3) follow from (5.10). The relations (5.12) follow from (5.6). Now, we will show that

$$\Pi^* S^{-1} \Pi = B^* B, \quad B := B(N) = \begin{bmatrix} \beta(0) \\ \beta(1) \\ \dots \\ \beta(N) \end{bmatrix}. \quad (5.19)$$

Let

$$K(r) = \begin{bmatrix} K_0(r) \\ K_1(r) \\ \dots \\ K_r(r) \end{bmatrix} \quad (5.20)$$

be the square matrix, where $K_l(r)$ are $p \times (r+1)p$ matrices of the form

$$K_l(r) = i\beta(l)J[\beta(0)^* \dots \beta(l-1)^* \quad \beta(l)^*/2 \quad 0 \dots 0]. \quad (5.21)$$

From (5.19)-(5.21) it follows that

$$K(r) - K(r)^* = iB(r)JB(r)^*. \quad (5.22)$$

By induction we shall show in the next step that K is similar to A :

$$K(r) = V_-(r)A(r)V_-(r)^{-1} \quad (0 \leq r \leq N), \quad (5.23)$$

where $V_-(r)^{\pm 1}$ are block lower triangular matrices. Taking into account (5.23) and multiplying both sides of (5.22) by $V_-(r)^{-1}$ from the left and by $(V_-(r)^*)^{-1}$ from the right, we get

$$A(r)S(r) - S(r)A(r)^* = i\Pi(r)J\Pi(r)^*, \quad (5.24)$$

$$S(r) := V_-(r)^{-1}(V_-(r)^*)^{-1}, \quad \Pi(r) := V_-(r)^{-1}B(r). \quad (5.25)$$

Moreover, Step 3 will show that matrix $V_-(N)$ can be chosen so that the equality

$$\Pi = [\Phi_1 \quad \Phi_2] = V_-(N)^{-1}B(N) \quad (5.26)$$

holds, i.e., $\Pi = \Pi(N)$. (Here Φ_1 and Φ_2 are given by (5.13).)

Identities (5.24) have unique solutions $S(r)$ as the spectra of $A(r)$ and $A(r)^*$ do not intersect. (The statement follows from rewriting of (5.24) in the form

$$S(r)(A(r)^* - \lambda I)^{-1} - (A(r) - \lambda I)^{-1}S(r) = i(A(r) - \lambda I)^{-1}\Pi(r)J\Pi(r)^*(A(r)^* - \lambda I)^{-1},$$

and from the following integration of both sides of the obtained identity along a contour, such that the spectra of A is inside and the spectra of A^* outside it.) In particular, by (5.15) and (5.24), using $\Pi = \Pi(N)$, one can see that $S = S(N)$. Hence, we derive from (5.25) that $S > 0$. As $S = S(N)$ and $\Pi = \Pi(N)$, by (5.25) equality (5.19) holds, too.

Notice that $P_l A = A(l)P_l$ for $l \leq N$. So, formula (5.15) yields the operator identities

$$A(l)(P_l S P_l^*) - (P_l S P_l^*)A(l)^* = iP_l \Pi J \Pi^* P_l^*. \quad (5.27)$$

Thus, substituting l instead of N into the proof of (5.19) we immediately prove

$$\Pi^* P_l^* (P_l S P_l^*)^{-1} P_l \Pi = B^* P_l^* P_l B = \sum_{r=0}^l \beta(r)^* \beta(r) \quad (l \leq N), \quad (5.28)$$

and formulas (5.16) and (5.18) follow.

It remains only to prove (5.23) and (5.26).

Step 2. Now, we shall consider the block lower triangular matrices $V_-(k)$ ($k \in \{0, 1, 2, \dots\}$):

$$V_-(0) = v_-(0) = \beta_1(0), \quad V_-(k) = \begin{bmatrix} V_-(k-1) & 0 \\ X(k) & v_-(k) \end{bmatrix} \quad (k > 0), \quad (5.29)$$

where $v_-(k)$ are $p \times p$ matrices, where $\beta_1(k)$ and $\beta_2(k)$ are $p \times p$ blocks of $\beta(k) = [\beta_1(k) \ \beta_2(k)]$, and where $X(k) = [X_0(k) \ \tilde{X}(k)]$ are $p \times kp$ matrices. Here $X_0(k)$ are arbitrary $p \times p$ blocks, and the matrices $\tilde{X}(k)$, $v_-(k)$ are given by the formulas

$$\begin{aligned} \tilde{X}(k) &= i \left(\beta(k) J [\beta(0)^* \dots \beta(k-1)^*] V_-(k-1) \begin{bmatrix} I^{(k-1)p} \\ 0 \end{bmatrix} - v_-(k) [I_p \dots I_p] \right) \\ &\quad \times \left(A(k-2) + \frac{i}{2} I_{(k-1)p} \right)^{-1}, \end{aligned} \quad (5.30)$$

$$v_-(k) = \beta(k) J \beta(k-1)^* v_-(k-1).$$

According to (5.14) we have $A(0) = (i/2)I_p$. From the second relation in (5.10) and definitions (5.20) and (5.21) it is immediate that $K(0) = (i/2)I_p$, and so (5.23) is valid for $r = 0$. Assume that (5.23) is true for $r = k-1$, and let us show that (5.23) is true for $r = k$, too. It is easy to see that

$$V_-(k)^{-1} = \begin{bmatrix} V_-(k-1)^{-1} & 0 \\ -v_-(k)^{-1} X(k) V_-(k-1)^{-1} & v_-(k)^{-1} \end{bmatrix}. \quad (5.31)$$

Then, in view of definitions (5.14) and (5.29), our assumption implies

$$V_-(k) A(k) V_-(k)^{-1} = \begin{bmatrix} K(k-1) & 0 \\ Y(k) & \frac{i}{2} I_p \end{bmatrix}, \quad (5.32)$$

where

$$Y(k) = \left[(X(k) A(k-1) + i v_-(k) [I_p \dots I_p]) \quad \frac{i}{2} v_-(k) \right] \begin{bmatrix} V_-(k-1)^{-1} \\ -v_-(k)^{-1} X(k) V_-(k-1)^{-1} \end{bmatrix}.$$

Rewrite the product on the right-hand side of the last formula as

$$Y(k) = \left(X(k) \left(A(k-1) - \frac{i}{2} I_{kp} \right) + i v_-(k) [I_p \dots I_p] \right) V_-(k-1)^{-1}. \quad (5.33)$$

From (5.14) and (5.33) it follows that

$$Y(k) = \left[\left(\tilde{X}(k) \left(A(k-2) + \frac{i}{2} I_{(k-1)p} \right) + i v_-(k) [I_p \dots I_p] \right) \quad i v_-(k) \right] V_-(k-1)^{-1}. \quad (5.34)$$

Notice that the row $[I_p \dots I_p]$ of identity matrices in (5.34) is one block smaller than in (5.33). By (5.30) and (5.34) we have

$$Y(k) = i\beta(k)J \left[[\beta(0)^* \dots \beta(k-1)^*] V_-(k-1) \begin{bmatrix} I^{(k-1)p} \\ 0 \end{bmatrix} \beta(k-1)^* v_-(k-1) \right] \times V_-(k-1)^{-1}. \tag{5.35}$$

Finally, formulas (5.29) and (5.35) imply

$$Y(k) = i\beta(k)J[\beta(0)^* \dots \beta(k-1)^*] \quad (k > 0). \tag{5.36}$$

According to the second relation in (5.10) and formulas (5.21) and (5.36) we get

$$\begin{bmatrix} Y(k) & i \\ & 2I_p \end{bmatrix} = K_k(k). \tag{5.37}$$

Now, using (5.20) and (5.37), one can see that the right-hand side of (5.32) equals $K(k)$. Thus, (5.23) is true for $r = k$ and, therefore, it is true for all $0 \leq r \leq n$.

Step 3. To derive (5.26) we shall first prove that the matrices $V_-(r)$ given by (5.29) and (5.30) can be chosen so that

$$V_-(r)^{-1}B_1(r) = \begin{bmatrix} I_p \\ \dots \\ I_p \end{bmatrix}, \quad B_1(r) := B(r) \begin{bmatrix} I_p \\ 0 \end{bmatrix} = \begin{bmatrix} \beta_1(0) \\ \dots \\ \beta_1(r) \end{bmatrix}. \tag{5.38}$$

In other words, the blocks $X_0(r)$, arbitrary till now, can be chosen so. Indeed, by the definition in (5.19) and the first equality in (5.29) formula (5.38) is true for $r = 0$. Assume that (5.38) is true for $r = k - 1$. Then, from (5.31) it follows that (5.38) is true for $r = k$, if only

$$-v_-(k)^{-1}X(k) \begin{bmatrix} I_p \\ \dots \\ I_p \end{bmatrix} + v_-(k)^{-1}\beta_1(k) = I_p. \tag{5.39}$$

It implies that we get equality (5.38) for $r = k$ by letting

$$X_0(k) = \beta_1(k) - v_-(k) - \tilde{X}(k) \begin{bmatrix} I_p \\ \dots \\ I_p \end{bmatrix}. \tag{5.40}$$

Hence, by a proper choice of the matrices $X_0(r)$ we obtain (5.38) for all $r \leq N$.

It remains to prove that

$$V_-(N)^{-1}B_2(N) = \Phi_2, \quad B_2(N) := \begin{bmatrix} \beta_2(0) \\ \dots \\ \beta_2(N) \end{bmatrix}. \tag{5.41}$$

For that purpose we shall consider the matrix function $W_{N+1}(\lambda)$, which is used in (5.8) to define the coefficients of the Möbius transformation (5.7). Namely, we shall prove the transfer matrix function representation of $W_{N+1}(\lambda)$:

$$W_{N+1}(\lambda) = \left(\frac{\lambda + i}{\lambda} \right)^{N+1} K^* w_A \left(N, -\frac{\lambda}{2} \right) K, \tag{5.42}$$

where

$$w_A(r, \lambda) = I_{2p} - iJ\Pi(r)^*S(r)^{-1}(A(r) - \lambda I_{(r+1)p})^{-1}\Pi(r). \quad (5.43)$$

Identity (5.24) implies an equality similar to (2.31), namely

$$\begin{aligned} w_A(r, \mu)^*Jw_A(r, \lambda) &= J + i(\bar{\mu} - \lambda)\Pi(r)^*(A(r)^* - \bar{\mu}I_{(r+1)p})^{-1} \\ &\quad \times S(r)^{-1}(A(r) - \lambda I_{(r+1)p})^{-1}\Pi(r). \end{aligned} \quad (5.44)$$

Moreover, according to factorization Theorem 4 from [43] (see also [45], p. 188) we have

$$\begin{aligned} w_A(r, \lambda) &= \left(I_{2p} - iJ\Pi(r)^*S(r)^{-1}P^*(PA(r)P^* - \lambda I_p)^{-1}(PS(r)^{-1}P^*)^{-1} \right. \\ &\quad \left. \times PS(r)^{-1}\Pi(r) \right) w_A(r-1, \lambda), \quad P = \begin{bmatrix} 0 & \dots & 0 & I_p \end{bmatrix}. \end{aligned} \quad (5.45)$$

Taking into account (5.14), (5.25), and (5.29) we obtain

$$(PA(r)P^* - \lambda I_p)^{-1} = \left(\frac{i}{2} - \lambda \right)^{-1} I_p, \quad PS(r)^{-1}P^* = v_-(r)^*v_-(r), \quad (5.46)$$

$$PS(r)^{-1}\Pi(r) = v_-(r)^*PB(r) = v_-(r)^*\beta(r). \quad (5.47)$$

Substitute (5.46) and (5.47) into (5.45) to get

$$w_A\left(r, \frac{\lambda}{2}\right) = \left(I_{2p} - \frac{2i}{i-\lambda}J\beta(r)^*\beta(r) \right) w_A\left(r-1, \frac{\lambda}{2}\right). \quad (5.48)$$

From the definitions (5.14), (5.25), and (5.43) we also easily derive

$$w_A\left(0, \frac{\lambda}{2}\right) = I_{2p} - \frac{2i}{i-\lambda}JB(0)^*B(0) = I_{2p} - \frac{2i}{i-\lambda}J\beta(0)^*\beta(0). \quad (5.49)$$

On the other hand system (1.2) with additional conditions (5.10) can be rewritten as

$$W_{r+1}(\lambda) = \frac{\lambda + i}{\lambda} \left(I_{2p} - \frac{2i}{i + \lambda}jK^*\beta(r)^*\beta(r)K \right) W_r(\lambda). \quad (5.50)$$

In view of the normalization $W(0) = I_{2p}$, formulas (5.48)-(5.50) imply (5.42).

From (5.42) and (5.44) it follows that

$$W_{N+1}(\lambda)jW_{N+1}(\bar{\lambda})^* = \left(\frac{\lambda + i}{\lambda} \right)^{N+1} \left(\frac{\lambda - i}{\lambda} \right)^{N+1} j. \quad (5.51)$$

Let us include the functions φ into consideration. Introduce

$$\mathcal{A}(\lambda) := \left| \frac{\lambda}{\lambda + i} \right|^{2N+2} [i\varphi(\lambda)^* \quad I_p]K W_{N+1}(\lambda)^*jW_{N+1}(\lambda)K^* \begin{bmatrix} -i\varphi(\lambda) \\ I_p \end{bmatrix}. \quad (5.52)$$

According to (5.7), (5.8), and (5.51) we have

$$\begin{aligned} \mathcal{A}(\lambda) &= \left| \frac{\lambda - i}{\lambda} \right|^{2N+2} ((\mathcal{W}_{11}(\lambda)R(\lambda) + \mathcal{W}_{12}(\lambda)Q(\lambda))^*)^{-1} \\ &\quad \times (R(\lambda)^*R(\lambda) - Q(\lambda)^*Q(\lambda)) (\mathcal{W}_{11}(\lambda)R(\lambda) + \mathcal{W}_{12}(\lambda)Q(\lambda))^{-1}. \end{aligned} \quad (5.53)$$

By (5.11) and (5.53) \mathcal{A} is bounded in the neighbourhood of $\lambda = -i$:

$$\|\mathcal{A}(\lambda)\| = O(1) \quad \text{for } \lambda \rightarrow -i. \tag{5.54}$$

Now, substitute (5.42) and (5.44) into (5.52) to obtain

$$\begin{aligned} \mathcal{A}(\lambda) = & [i\varphi(\lambda)^* \quad I_p] \left(J + \frac{i}{2}(\lambda - \bar{\lambda})\Pi(N)^* \left(A(N)^* + \frac{\bar{\lambda}}{2}I_{(N+1)p} \right)^{-1} S(N)^{-1} \right. \\ & \left. \times \left(A(N) + \frac{\lambda}{2}I_{(N+1)p} \right)^{-1} \Pi(N) \right) \begin{bmatrix} -i\varphi(\lambda) \\ I_p \end{bmatrix}. \end{aligned} \tag{5.55}$$

Notice that $S(N) > 0$. Hence, formulas (5.54) and (5.55) imply that

$$\left\| \left(A(N) + \frac{\lambda}{2}I_{(N+1)p} \right)^{-1} \Pi(N) \begin{bmatrix} -i\varphi(\lambda) \\ I_p \end{bmatrix} \right\| = O(1) \quad \text{for } \lambda \rightarrow -i. \tag{5.56}$$

Recall that $\Pi(N) = V_-(N)^{-1}B(N)$ and that $A(N)$ is denoted by A . Now, represent $\Pi(N)$ in the block form

$$\Pi(N) = [\Phi_1(N) \quad \Phi_2(N)], \quad \Phi_k(N) = V_-(N)^{-1}B_k(N) \quad (k = 1, 2). \tag{5.57}$$

According to (5.13) and (5.38) we have $\Phi_1(N) = \Phi_1$. Hence, multiplying the matrix function on the left-hand side of (5.56) by $i \left(\Phi_1^* \left(A + \frac{\lambda}{2}I_{(N+1)p} \right)^{-1} \Phi_1 \right)^{-1} \Phi_1^*$ we derive

$$\begin{aligned} & \left\| \varphi(\lambda) + i \left(\Phi_1^* \left(A + \frac{\lambda}{2}I_{(N+1)p} \right)^{-1} \Phi_1 \right)^{-1} \Phi_1^* \left(A + \frac{\lambda}{2}I_{(N+1)p} \right)^{-1} \Phi_2(N) \right\| \\ & = O \left(\left\| \left(\Phi_1^* \left(A + \frac{\lambda}{2}I_{(N+1)p} \right)^{-1} \Phi_1 \right)^{-1} \right\| \right) \quad \text{for } \lambda \rightarrow -i. \end{aligned} \tag{5.58}$$

The matrix $A + \frac{\lambda}{2}I_{(N+1)p}$ is easily inverted explicitly (see, for instance, formula (1.10) in [40]). As a result one obtains

$$\Phi_1^* \left(A + \frac{\lambda}{2}I_{(N+1)p} \right)^{-1} = \frac{2}{i + \lambda} [\hat{q}^N \quad \hat{q}^{N-1} \quad \dots \quad \hat{q} \quad I_p], \quad \hat{q} := \frac{\lambda - i}{\lambda + i} I_p. \tag{5.59}$$

Moreover, we get

$$\Phi_1^* \left(A + \frac{\lambda}{2}I_{(N+1)p} \right)^{-1} \Phi_1 = \frac{2}{i + \lambda} (\hat{q}^{N+1} - I_p) (\hat{q} - I_p)^{-1}. \tag{5.60}$$

Let $\lambda = i \left(\frac{z + 1}{z - 1} \right)$, i.e., $z = \left(\frac{\lambda + i}{\lambda - i} \right)$. Then, we derive from (5.60) that

$$\left(\Phi_1^* \left(A + \frac{\lambda}{2}I_{(N+1)p} \right)^{-1} \Phi_1 \right)^{-1} = (-iz^{N+1} + O(z^{2N+2})) I_p \quad (z \rightarrow 0). \tag{5.61}$$

Taking into account (5.59) and (5.61), we rewrite (5.58) as

$$\left\| \varphi \left(i \left(\frac{z+1}{z-1} \right) \right) + i(1-z) [I_p \quad zI_p \quad z^2I_p \quad \dots] \Phi_2(N) \right\| = O(z^{N+1}) \quad (5.62)$$

for $z \rightarrow 0$. From (5.13) and (5.62) it follows that $\Phi_2(N) = \Phi_2$, i.e., (5.41) is true. As $\Phi_1(N) = \Phi_1$ and $\Phi_2(N) = \Phi_2$, so $\Pi(N) = \Pi$ and formula (5.26) is finally proved. \square

DEFINITION 5.3. Let the matrices C_k satisfy (5.10). Then, a $p \times p$ matrix function φ is said to be a Weyl function for system (1.2) on the interval $0 \leq k \leq N$, if φ is holomorphic in \mathbb{C}_- and admits representation (5.7), where the pair $[R, Q]$ is meromorphic in \mathbb{C}_- , well-defined at $\lambda = -i$, and nonsingular with j -property, i.e.,

$$R(\lambda)^*R(\lambda) + Q(\lambda)^*Q(\lambda) > 0, \quad R(\lambda)^*R(\lambda) \leq Q(\lambda)^*Q(\lambda). \quad (5.63)$$

The set of Weyl functions is denoted by $\mathcal{N}(N)$.

Using notation (5.8), we deduce from (4.2) the inequality

$$q(\lambda)^{N+1} \mathcal{W}(\lambda) j \mathcal{W}(\lambda)^* \leq J, \quad \lambda \in \mathbb{C}_-. \quad (5.64)$$

According to [34] we can change the order of factors in (5.64):

$$q(\lambda)^{N+1} \mathcal{W}(\lambda)^* J \mathcal{W}(\lambda) \leq j, \quad \lambda \in \mathbb{C}_-. \quad (5.65)$$

Moreover, after excluding $\lambda = -i$, the inequality is strict

$$q(\lambda)^{N+1} \mathcal{W}(\lambda)^* J \mathcal{W}(\lambda) < j, \quad \lambda \in \mathbb{C}_- \setminus -i. \quad (5.66)$$

In view of (1.2), (5.8), (5.4), and (5.5), at $\lambda = -i$ we get

$$\mathcal{W}(-i) = KW_{N+1}(i)^* = (-2)^{N+1} K \prod_{k=0}^N \left(\check{\beta}_k^* \check{\beta}_{kj} \right). \quad (5.67)$$

From the second relation in (5.3) and from (5.63) we, analogously to the proof of (5.6), derive:

$$\det [I_p \quad I_p] j \check{\beta}_0^* \neq 0, \quad \det \check{\beta}_{kj} \check{\beta}_{k+1}^* \neq 0, \quad \det \check{\beta}_{Nj} \begin{bmatrix} R(-i) \\ Q(-i) \end{bmatrix} \neq 0. \quad (5.68)$$

By (5.67) and (5.68) the next proposition is valid.

PROPOSITION 5.4. *Let the pair $[R, Q]$ satisfy (5.63) Then inequality (5.11) is fulfilled.*

By Proposition 5.4 and the proof of Theorem 5.2 we get a corollary.

COROLLARY 5.5. *Weyl functions of system (1.2), which satisfies conditions (5.10), are Herglotz functions and admit the Taylor representation*

$$\varphi \left(i \left(\frac{z+1}{z-1} \right) \right) = -i (\psi_0 + (\psi_1 - \psi_0)z + \dots + (\psi_N - \psi_{N-1})z^N) + O(z^{N+1}), \tag{5.69}$$

where $z \rightarrow 0$ and the $p \times p$ matrices ψ_k are the blocks of

$$\Phi_2 = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \dots \\ \psi_N \end{bmatrix} = V_-(N)^{-1}B(N). \tag{5.70}$$

Proof. From (5.7), (5.63) and (5.65) it follows that

$$[I_p \quad i\varphi^*]J \begin{bmatrix} I_p \\ -i\varphi \end{bmatrix} \leq 0,$$

i.e., $\Im\varphi(\lambda) \leq 0$ for $\lambda \in \mathbb{C}_-$, and so φ is a Herglotz function.

By Proposition 5.4 the Weyl functions satisfy conditions of Theorem 5.2. Then, by the second relation in (5.13) we have representation (5.69) of φ via the blocks of Φ_2 . By the proof of Theorem 5.2 we get also $\Phi_2 = \Phi_2(N)$, i.e., (5.70) holds. Here $V_-(N)$ and $B(N)$ are recovered from the matrices $\beta(k)$ and do not depend on φ . \square

REMARK 5.6. As Weyl functions φ satisfy conditions of Theorem 5.2, so the procedure given in Theorem 5.2 provides a recovery of system (1.2) from a Weyl function (i.e., provides a solution of the inverse problem on a finite interval).

The following proposition is also true:

PROPOSITION 5.7. *The set $\mathcal{N}(N)$ ($N > M$) is imbedded in $\mathcal{N}(M)$, i.e., $\mathcal{N}(N) \subset \mathcal{N}(M)$.*

Proof. By (4.2) we have

$$q(\lambda)^{N+1}W_{N+1}(\lambda)^*jW_{N+1}(\lambda) \leq q(\lambda)^{M+1}W_{M+1}(\lambda)^*jW_{M+1}(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{5.71}$$

Insert the length N of the interval into the notation \mathscr{W} :

$$\mathscr{W}(N, \lambda) = \mathscr{W}(\lambda) = KW_{N+1}(\bar{\lambda})^*. \tag{5.72}$$

From (5.71) and (5.72) it follows that

$$q(\lambda)^{N-M} \left(\mathscr{W}(M, \lambda)^{-1} \mathscr{W}(N, \lambda) \right)^* j \mathscr{W}(M, \lambda)^{-1} \mathscr{W}(N, \lambda) \leq j. \tag{5.73}$$

Moreover, in view of (1.2) and (5.72) we have

$$\mathscr{W}(M, \lambda)^{-1} \mathscr{W}(N, \lambda) = \prod_{k=M+1}^N \left(I_m + \frac{i}{\lambda} C_{kj} \right), \tag{5.74}$$

and the expression on the left-hand side of (5.74) is analytic at $\lambda = -i$. Suppose now that $\varphi \in \mathcal{N}(N)$ is a Weyl function generated by some pair $[R, Q]$, which satisfies (5.63). Then, according to (5.63), (5.73) and (5.74) the pair

$$\begin{bmatrix} \tilde{R}(\lambda) \\ \tilde{Q}(\lambda) \end{bmatrix} = \mathcal{W}(M, \lambda)^{-1} \mathcal{W}(N, \lambda) \begin{bmatrix} R(\lambda) \\ Q(\lambda) \end{bmatrix} \quad (5.75)$$

satisfies conditions of Definition 5.3 too. Moreover, it is easy to see that

$$\begin{aligned} & i \left(\mathcal{W}_{21}(M, \lambda) \tilde{R}(\lambda) + \mathcal{W}_{22}(M, \lambda) \tilde{Q}(\lambda) \right) \left(\mathcal{W}_{11}(M, \lambda) \tilde{R}(\lambda) + \mathcal{W}_{12}(M, \lambda) \tilde{Q}(\lambda) \right)^{-1} \\ &= i \left(\mathcal{W}_{21}(N, \lambda) R(\lambda) + \mathcal{W}_{22}(N, \lambda) Q(\lambda) \right) \left(\mathcal{W}_{11}(N, \lambda) R(\lambda) + \mathcal{W}_{12}(N, \lambda) Q(\lambda) \right)^{-1} \\ &= \varphi(\lambda), \end{aligned} \quad (5.76)$$

which completes the proof. \square

Theorem 5.2 and Proposition 5.7 imply a Borg-Marchenko type result:

THEOREM 5.8. *Let $\tilde{\varphi}$ and $\hat{\varphi}$ be Weyl functions of the two discrete Dirac-type systems (1.2), which satisfy conditions (5.10). Denote by \tilde{C}_k ($0 \leq k \leq \tilde{N}$) the potentials C_k of the first system and by \hat{C}_k ($0 \leq k \leq \hat{N}$) the potentials of the second system. Denote the Taylor coefficients of $i\tilde{\varphi}\left(i\left(\frac{z+1}{z-1}\right)\right)$ and $i\hat{\varphi}\left(i\left(\frac{z+1}{z-1}\right)\right)$ at $z = 0$ by $\{\tilde{\alpha}_k\}$ and $\{\hat{\alpha}_k\}$, respectively, and assume that $\tilde{\alpha}_k = \hat{\alpha}_k$ for all $k \leq N \leq \min\{\tilde{N}, \hat{N}\}$. Then we have $\tilde{C}_k = \hat{C}_k$ for $k \leq N$.*

Proof. According to Proposition 5.7, $\tilde{\varphi}$ and $\hat{\varphi}$ are Weyl functions of the first and second systems, respectively, on the interval $0 \leq k \leq N$. By Theorem 5.2 these systems on the interval $0 \leq k \leq N$ are uniquely recovered by the first $N + 1$ Taylor coefficients of the Weyl functions. \square

The present interest in the Borg-Marchenko type uniqueness results was initiated by a series of papers by F. Gesztesy, B. Simon and coauthors (see, for instance, [28, 46]). Further results and various references, including the most recent ones, one can find in [12, 13]. Both local and global versions of the Borg-Marchenko type uniqueness theorems for CMV operators were obtained in [13]. A procedure for the unique recovery of the reflectionless matrix-valued Jacobi operators from their spectrum was given in [12]. (The characterisation of the non-unique recovery of the related reflectionless supersymmetric Dirac operators was given in [12] too.) One can also find in [12] the results on the asymptotic expansions of the half and full-line Weyl-Titchmarsh functions for Jacobi operators and supersymmetric Dirac operators. In the next Section we shall treat the expansions of the Weyl functions for our systems (1.2) and their connections with the Toeplitz matrices.

6. Toeplitz matrices and Dirac systems on the semiaxis

By [36], p. 116 it is easy to recover a block Toeplitz matrix S which satisfies (5.15), where the blocks Φ_1 and Φ_2 of Π are given by (5.13). Namely, we have

$$S = \{s_{j-k}\}_{k,j=0}^N, \quad s_{-k} = \alpha_k = s_k^* \quad (k > 0), \quad s_0 = s_0^* = \alpha_0 + \alpha_0^*. \quad (6.1)$$

Moreover, this S is a unique solution of (5.15). A description of all extensions of S preserving the number of negative eigenvalues, which uses the transfer matrix function w_A , is given in [36] (see also Theorem 4.1 in [40]). It is given in terms of the linear fractional transformation

$$\widehat{\varphi}(\lambda) = \left(\widehat{R}(\lambda)w_{11}(\lambda) + \widehat{Q}(\lambda)w_{21}(\lambda) \right)^{-1} \left(\widehat{R}(\lambda)w_{12}(\lambda) + \widehat{Q}(\lambda)w_{22}(\lambda) \right), \quad (6.2)$$

where $\{w_{kj}(\lambda)\}_{k,j=1}^2 = w_A(N, \lambda)$, the transfer matrix function w_A is defined by formula (2.23), and the meromorphic pairs $[\widehat{R}, \widehat{Q}]$ have J -property, i.e.,

$$\widehat{R}(\lambda)\widehat{R}(\lambda)^* + \widehat{Q}(\lambda)\widehat{Q}(\lambda)^* > 0, \quad \widehat{R}(\lambda)\widehat{Q}(\lambda)^* + \widehat{Q}(\lambda)\widehat{R}(\lambda)^* \geq 0, \quad \lambda \in \mathbb{C}_+. \quad (6.3)$$

In particular, for the case $S > 0$, which is treated here, the matrix functions $\widehat{\varphi} \left(-\frac{i(z+1)}{2(z-1)} \right)$ are always analytic at $z = 0$ and admit the Taylor representation

$$\widehat{\varphi} \left(-\frac{i(z+1)}{2(z-1)} \right) = \widehat{s}_0 + \widehat{s}_{-1}z + \widehat{s}_{-2}z^2 + \dots \quad (6.4)$$

Our next statement is a reformulation of Theorem 4.1 [40] for the subcase $S > 0$.

THEOREM 6.1. *Assume that $S = \{s_{j-k}\}_{k,j=0}^N > 0$, and fix α_0 such that $\alpha_0 + \alpha_0^* = s_0$. Using (5.13), (5.43), and (6.1) introduce $\Pi = [\Phi_1 \quad \Phi_2]$ and $\{w_{kj}(\lambda)\}_{k,j=1}^2 = w_A(N, \lambda)$. Now, let matrix functions $\widehat{\varphi}$ be given by (6.2), where the pairs $[\widehat{R}, \widehat{Q}]$ satisfy (6.3) and are well defined at $\lambda = \frac{i}{2}$. Then the Taylor coefficients \widehat{s}_{-k} at $z = 0$ of the matrix functions $\widehat{\varphi} \left(-\frac{i(z+1)}{2(z-1)} \right)$ satisfy relations*

$$\widehat{s}_{-k} = s_{-k} \quad (0 < k \leq N), \quad \widehat{s}_0 = \alpha_0. \quad (6.5)$$

Moreover, putting $s_{-k} = s_{-k}^* = \widehat{s}_{-k}$ for $k > N$, we have $\{s_{j-k}\}_{k,j=0}^M \geq 0$ for all $M > N$. In other words, the Taylor coefficients of the matrix functions $\widehat{\varphi} \left(-\frac{i(z+1)}{2(z-1)} \right)$ generate nonnegative extensions of S . All the nonnegative extensions of S are generated in this way.

For results related to Theorem 6.1 we refer to the series of papers [20].

Taking into account that $w_A(N, \lambda)Jw_A(N, \bar{\lambda})^* = J$, we derive the equality $\widehat{\varphi} = \widetilde{\varphi}$ for the matrix function

$$\widetilde{\varphi}(\lambda) = -\left(w_{12}(\bar{\lambda})^*\widetilde{R}(\lambda) + w_{22}(\bar{\lambda})^*\widetilde{Q}(\lambda) \right) \left(w_{11}(\bar{\lambda})^*\widetilde{R}(\lambda) + w_{21}(\bar{\lambda})^*\widetilde{Q}(\lambda) \right)^{-1}, \quad (6.6)$$

where

$$\widetilde{R}(\lambda)^*\widetilde{R}(\lambda) + \widetilde{Q}(\lambda)^*\widetilde{Q}(\lambda) > 0, \quad \widetilde{R}(\lambda)\widetilde{Q}(\lambda) + \widetilde{Q}(\lambda)\widetilde{R}(\lambda) = 0, \quad \lambda \in \mathbb{C}_+. \quad (6.7)$$

Notice also that relations (6.3) and (6.7) yield

$$\widetilde{R}(\lambda)^*\widetilde{Q}(\lambda) + \widetilde{Q}(\lambda)^*\widetilde{R}(\lambda) \leq 0, \quad (6.8)$$

and vice versa relations (6.7) and (6.8) yield the second relation in (6.3). Hence, Theorem 6.1 can be reformulated in terms of the linear fractional transformations (6.6), where $[\tilde{R}, \tilde{Q}]$ have J -property (6.8). Finally, use (5.8), (5.9) and (5.42) to rewrite (5.7) in the form

$$i\varphi(\lambda) = - \left(w_{12}(-\bar{\lambda}/2)^* \tilde{R}(-\lambda/2) + w_{22}(-\bar{\lambda}/2)^* \tilde{Q}(-\lambda/2) \right) \times \left(w_{11}(-\bar{\lambda}/2)^* \tilde{R}(-\lambda/2) + w_{21}(-\bar{\lambda}/2)^* \tilde{Q}(-\lambda/2) \right)^{-1}, \quad (6.9)$$

where we put

$$\begin{bmatrix} \tilde{R}(-\lambda/2) \\ \tilde{Q}(-\lambda/2) \end{bmatrix} = K \begin{bmatrix} R(\lambda) \\ Q(\lambda) \end{bmatrix}. \quad (6.10)$$

Here, formula (6.10) is a one to one mapping of the pairs satisfying (5.63) into pairs satisfying the first relation in (6.7) and relation (6.8). By (6.6) and (6.9) we have $\hat{\varphi} \left(-\frac{i(z+1)}{2(z-1)} \right) = \tilde{\varphi} \left(-\frac{i(z+1)}{2(z-1)} \right) = i\varphi \left(i\frac{(z+1)}{(z-1)} \right)$ Therefore Theorem 6.1 can be rewritten.

THEOREM 6.2. Assume that $S = \{s_{j-k}\}_{k,j=0}^N > 0$, fix α_0 such that $\alpha_0 + \alpha_0^* = s_0$, and introduce \mathscr{W} via (5.8) and (5.42). Let matrix functions φ be given by (5.7), where the pairs $[R, Q]$ satisfy (5.63) and are well defined at $\lambda = -i$. Then $i\varphi(-i) = \alpha_0$, and the next following Taylor coefficients α_k at $z = 0$ of the matrix functions $i\varphi \left(i\frac{(z+1)}{(z-1)} \right)$ satisfy the relations

$$\alpha_k = s_{-k} \quad (0 < k \leq N). \quad (6.11)$$

Moreover, putting $s_{-k} = s_k^* = \alpha_k$ for $k > N$, we have $\{s_{j-k}\}_{k,j=0}^M \geq 0$ for all $M > N$. In other words, the Taylor coefficients of the matrix functions $i\varphi \left(i\frac{(z+1)}{(z-1)} \right)$ generate nonnegative extensions of S . All the nonnegative extensions of S are generated in this way.

REMARK 6.3. By Definition 5.3 and Theorem 6.2 the Weyl functions from the Weyl disk $\mathcal{N}(N)$ generate all the nonnegative extensions of S . It provides, in particular, an alternative proof of Proposition 5.7.

Consider system (1.2), which satisfies (5.10) on the semiaxis $k \in \{0, 1, 2, \dots\}$. Recall that the equalities in (5.10) are equivalent to $C_k > 0$, $C_k j C_k = j$, and that $\beta(k)$ in (5.10) is given by the formulas $\beta(k) = \hat{\beta}(k)K^*$ and (5.1).

THEOREM 6.4. Let system (1.2) be given on the semiaxis $k \in \{0, 1, 2, \dots\}$ and let matrices C_k satisfy (5.10). Then, there is a unique function φ_∞ , which belongs to all the Weyl discs $\mathcal{N}(N)$:

$$\bigcap_{N=0}^{\infty} \mathcal{N}(N) = \varphi_\infty. \quad (6.12)$$

Proof. According to Corollary 5.5 the matrices $\{C_k\}_{k=0}^N$ ($N < \infty$) or equivalently the matrices $\{\beta(k)\}_{k=0}^N$ uniquely define the blocks $\{s_{-k}\}_{k=0}^N$, where $s_{-k} = \alpha_k =$

$\psi_k - \psi_{k-1}$ for $k > 0$ and $s_0 = \alpha_0 + \alpha_0^*$ ($\alpha_0 = \psi_0$). Moreover, by Proposition 5.7 these s_{-k} do not depend on $N \geq k$, and so system (1.2) on the semiaxis determines an infinite sequence $\{s_{-k}\}_{k=0}^\infty$. By Theorem 5.2 we have $\{s_{j-k}\}_{k,j=0}^N > 0$ for all $N \geq 0$. Apply now Theorem 6.2 to see that

$$i\varphi \left(i \frac{(z+1)}{(z-1)} \right) = \alpha_0 + \sum_{k=1}^\infty s_{-k} z^k = i\varphi_\infty \left(i \frac{(z+1)}{(z-1)} \right), \tag{6.13}$$

i.e., this φ belongs to $\bigcap_{N=0}^\infty \mathcal{N}(N)$. Moreover, as the sequence $\{s_{-k}\}_{k=0}^\infty$ is unique, so by Theorem 6.2 the function $\varphi \in \bigcap_{N=0}^\infty \mathcal{N}(N)$ is unique. \square

Recall that a Weyl function on the semiaxis is defined by Definition 4.8, where K is given by formula (4.27). Theorem 6.4 yields our next result.

THEOREM 6.5. *Let system (1.2) be given on the semiaxis $k \geq 0$ and let matrices C_k satisfy (5.10). Then, the matrix function φ_∞ given by (6.12) is the unique Weyl function of system (1.2) on the semiaxis.*

Proof. By (5.7) and (6.12), we have

$$\begin{bmatrix} -i\varphi_\infty(\lambda) \\ I_p \end{bmatrix} = J\mathcal{W}(r+1, \lambda) \begin{bmatrix} R(\lambda) \\ Q(\lambda) \end{bmatrix} \tag{6.14}$$

for all $r \geq 0$ and for some depending on r pairs $[R, Q]$, which satisfy (5.63). In view of (5.8), (5.9), (5.51), and (6.14) we obtain

$$\begin{aligned} & [i\varphi_\infty(\lambda)^* \quad I_p] (q(\lambda)^{r+1} K W_{r+1}(\lambda)^* j W_{r+1}(\lambda) K^* - J) \begin{bmatrix} -i\varphi_\infty(\lambda) \\ I_p \end{bmatrix} \\ &= i(\varphi_\infty(\lambda) - \varphi_\infty(\lambda)^*) + \left(\frac{|\lambda|^2 + 1}{|\lambda|^2(|\lambda|^2 + 1)} \right)^{r+1} [R(\lambda)^* \quad Q(\lambda)^*] j \begin{bmatrix} R(\lambda) \\ Q(\lambda) \end{bmatrix}. \end{aligned} \tag{6.15}$$

Now, formulas (5.63) and (6.15) imply

$$\begin{aligned} & [i\varphi_\infty(\lambda)^* \quad I_p] (q(\lambda)^{r+1} K W_{r+1}(\lambda)^* j W_{r+1}(\lambda) K^* - J) \begin{bmatrix} -i\varphi_\infty(\lambda) \\ I_p \end{bmatrix} \\ & \leq i(\varphi_\infty(\lambda) - \varphi_\infty(\lambda)^*). \end{aligned} \tag{6.16}$$

It follows from (4.2) and (6.16) that

$$\begin{aligned} & \sum_{k=0}^r [i\varphi_\infty(\lambda)^* \quad I_p] q(\lambda)^k K W_k(\lambda)^* C_k W_k(\lambda) K^* \begin{bmatrix} -i\varphi_\infty(\lambda) \\ I_p \end{bmatrix} \\ & \leq \frac{|\lambda|^2 + 1}{\lambda - \bar{\lambda}} (\varphi_\infty(\lambda) - \varphi_\infty(\lambda)^*). \end{aligned} \tag{6.17}$$

Finally, by (6.17) the inequality (4.26) is immediate, and φ_∞ given by (6.12) is a Weyl function.

To prove the uniqueness of the Weyl function notice that by Proposition 3.2 and by relation (4.3) we have

$$q^k W_k^* C_k W_k \geq q^k W_k^* j W_k \geq q^{k-1} W_{k-1}^* j W_{k-1} \geq \dots \geq W_0^* j W_0 = j \quad (\lambda \in \mathbb{C}_-). \quad (6.18)$$

Hence, in view of (6.18) we obtain

$$\sum_{k=0}^r [I_p \quad I_p] q(\lambda)^k K W_k(\lambda)^* C_k W_k(\lambda) K^* \begin{bmatrix} I_p \\ I_p \end{bmatrix} \geq 2(r+1)I_p,$$

and it follows that

$$\sum_{k=0}^{\infty} [I_p \quad I_p] q(\lambda)^k K W_k(\lambda)^* C_k W_k(\lambda) K^* \begin{bmatrix} I_p \\ I_p \end{bmatrix} = \infty. \quad (6.19)$$

Taking into account Definition 4.8 and inequality (6.19), we can show the uniqueness of the Weyl function similar to the proof of the uniqueness in Theorem 4.2. \square

Now, we formulate a solution of the inverse problem.

THEOREM 6.6. *The set of the Weyl functions $\varphi(\lambda)$ of systems (1.2), given on the semiaxis $k \geq 0$ and such that the matrices C_k satisfy (5.10), coincides with the set of functions φ such that*

$$i\varphi\left(i\frac{(z+1)}{(z-1)}\right) = \alpha_0 + \sum_{k=1}^{\infty} s_{-k} z^k \quad (6.20)$$

are Caratheodory matrix functions in the unit disk and $\{s_{j-k}\}_{k,j=0}^N > 0$ for all $0 \leq N < \infty$ ($s_0 := \alpha_0 + \alpha_0^*$). These systems (1.2) are uniquely recovered from their Weyl functions via the procedure given in Theorem 5.2.

Proof. According to Theorem 6.5 the Weyl function on the semiaxis is also a Weyl function on the intervals. Hence, the procedure to construct a solution of the inverse problem follows from Theorem 5.2. It follows from Theorem 5.2 also, that the matrices $\{s_{j-k}\}_{k,j=0}^N$ generated by the Weyl functions are positive definite.

Hence, it remains to show that all the functions such that (6.20) holds and $\{s_{j-k}\}_{k,j=0}^r > 0$ ($r \geq 0$) are Weyl functions. Indeed, fixing such a matrix function φ , we get a sequence of matrices $S(r) = \{s_{j-k}\}_{k,j=0}^r > 0$. Therefore we get a sequence of the transfer matrix functions $w_A(r, \lambda)$ of the form (5.43), where

$$\Pi(r) = \begin{bmatrix} I_p & \alpha_0 \\ I_p & \alpha_0 + s_{-1} \\ \dots & \dots \\ I_p & \alpha_0 + s_{-1} + \dots + s_{-r} \end{bmatrix}, \quad (6.21)$$

and (5.24) holds. Compare formulas (5.43) ($r = 0$) and (5.49), and compare also formulas (5.45) and (5.48) to get

$$\beta(r)^* \beta(r) = \Pi(r)^* S(r)^{-1} P^* (P S(r)^{-1} P^*)^{-1} P S(r)^{-1} \Pi(r) \quad (r \geq 0). \quad (6.22)$$

In view of the matrix identity (5.24) we have

$$\begin{aligned}
 PS(r)^{-1}\Pi(r)J\Pi(r)^*S(r)^{-1}P^* &= -iP\left(S(r)^{-1}A(r) - A(r)^*S(r)^{-1}\right)P^* \\
 &= PS(r)^{-1}P^*.
 \end{aligned}
 \tag{6.23}$$

Thus, we can put

$$\beta(r) := (PS(r)^{-1}P^*)^{-\frac{1}{2}}PS(r)^{-1}\Pi(r) \quad (r \geq 0),
 \tag{6.24}$$

so that the matrices $\beta(r)$ will satisfy (6.22) and the second relation in (5.10). Therefore formulas $C_r = 2K\beta(r)^* \beta(r)K - j$ define a system of our class on the semiaxis. Similar to the proof of Theorem 5.2 we derive from (6.22) the equality (5.42). Compare now Definition 5.3 and Theorem 6.2 to see that $\varphi \in \mathcal{N}(N)$ for any N . According to Theorems 6.4 and 6.5 it means that φ is the Weyl function. \square

Formula (6.24) provides a somewhat different way in comparison with formulas (5.16) and (5.18) way to recover system (1.2).

Finally, consider the upper halfplane and define holomorphic Weyl functions in \mathbb{C}_+ via relations (4.26) and (4.27), too. Put

$$\overline{\mathcal{N}}(N) := \{\varphi(\overline{\lambda})^* : \varphi \in \mathcal{N}(N)\}.
 \tag{6.25}$$

REMARK 6.7. Similar to the proof that $\widehat{\varphi} = \widetilde{\varphi}$, where $\widehat{\varphi}$ and $\widetilde{\varphi}$ are given by (6.2) and (6.6), respectively, one can show that the set $\overline{\mathcal{N}}(N)$ consists of linear fractional transformations (5.7), where the pairs $[R, Q]$ are meromorphic in \mathbb{C}_+ , are well defined at $\lambda = i$, and have the property

$$R(\lambda)^*R(\lambda) + Q(\lambda)^*Q(\lambda) > 0, \quad R(\lambda)^*R(\lambda) \geq Q(\lambda)^*Q(\lambda), \quad \lambda \in \mathbb{C}_+.
 \tag{6.26}$$

In view of Remark 6.7, we obtain in \mathbb{C}_+ the analog of Theorem 6.5, and the proof is similar.

THEOREM 6.8. *Let system (1.2) be given on the semiaxis $k \geq 0$ and let matrices C_k satisfy (5.10). Then, the matrix function $\varphi_\infty(\overline{\lambda})^* = \bigcap_{N=0}^\infty \overline{\mathcal{N}}(N)$ is the unique Weyl function in \mathbb{C}_+ of system (1.2) on the semiaxis.*

Proof. Substitute $\varphi_\infty(\overline{\lambda})^*$ instead of $\varphi_\infty(\lambda)$ into (6.15) and take into account (6.26) to derive

$$\begin{aligned}
 [i\varphi_\infty(\overline{\lambda}) \quad I_p] \left(J - q(\lambda)^{r+1}KW_{r+1}(\lambda)^*jW_{r+1}(\lambda)K^* \right) \begin{bmatrix} -i\varphi_\infty(\overline{\lambda})^* \\ I_p \end{bmatrix} \\
 \leq i\left(\varphi_\infty(\overline{\lambda}) - \varphi_\infty(\overline{\lambda})^*\right).
 \end{aligned}
 \tag{6.27}$$

Now, inequalities (6.17) and (4.26) are straightforward, i.e., $\varphi_\infty(\overline{\lambda})^*$ is a Weyl function.

Instead of (6.18) we use the inequality

$$q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) \geq -q(\lambda)^k W_k(\lambda)^* j W_k(\lambda) \geq -j \quad (\lambda \in \mathbb{C}_+),
 \tag{6.28}$$

which yields inequality

$$\sum_{k=0}^{\infty} [I_p \quad -I_p] q(\lambda)^k K W_k(\lambda)^* C_k W_k(\lambda) K^* \begin{bmatrix} I_p \\ -I_p \end{bmatrix} = \infty. \quad (6.29)$$

The uniqueness of the Weyl function follows from (6.29) \square

Theorem 6.8 for the scalar case $p = 1$ has been proved earlier in [25, 31] (see also Theorem 3.2.11 [47]).

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REFERENCES

- [1] D. ALPAY, I. GOHBERG, *Inverse spectral problems for difference operators with rational scattering matrix function*, Integral Equations Operator Theory **20** (1994), 125–170.
- [2] D. ALPAY, I. GOHBERG, *Inverse spectral problem for differential operators with rational scattering matrix functions*, J. Differ. Equations **118** (1995), 1–19.
- [3] D. ALPAY, I. GOHBERG, *Connections between the Carathéodory-Toeplitz and the Nehari extension problems: the discrete scalar case*, Integral Equations Operator Theory **37** (2000), 125–142.
- [4] D. ALPAY, I. GOHBERG, *Inverse problems associated to a canonical differential system*, in: Oper. Theory, Adv. Appl. **127** (2001), Birkhäuser, 1–27.
- [5] D. ALPAY, I. GOHBERG, *Discrete analogs of canonical systems with pseudo-exponential potential. Definitions and formulas for the spectral matrix functions*, in: Oper. Theory, Adv. Appl. **161** (2005), Birkhäuser, 1–47.
- [6] D. ALPAY, I. GOHBERG, *Discrete analogs of canonical systems with pseudo-exponential potential. Inverse Problem*, in: Oper. Theory, Adv. Appl. **165** (2005), Birkhäuser, 31–65.
- [7] D. ALPAY, I. GOHBERG, *Discrete systems and their characteristic spectral functions*, Mediterr. J. Math., **4** (2007), 1–32.
- [8] D. ALPAY, I. GOHBERG, M.A. KAASHOEK, A.L. SAKHNOVICH, *Direct and inverse scattering problem for canonical systems with a strictly pseudo-exponential potential*, Math. Nachrichten **215** (2000), 5–31.
- [9] T. AKTOSUN, M. KLAUS, C. VAN DER MEE, *Direct and inverse scattering for selfadjoint Hamiltonian systems on the line*, Integral Equations Operator Theory **38** (2000) 129–171.
- [10] D.Z. AROV, H. DYM, *J -inner matrix functions, interpolation and inverse problems for canonical systems V*, Integral Equations Operator Theory **43** (2002) 68–129.
- [11] H. BART, I. GOHBERG, M.A. KAASHOEK, *Minimal factorization of matrix and operator functions*, Operator Theory: Adv. and Appl. **1**, Birkhäuser Verlag, Basel, 1979.
- [12] S. CLARK, F. GESZTESY, W. RENGGER, *Trace formulas and Borg-type theorems for matrix-valued Jacobi and Dirac finite difference operators*, J. Differ. Equations **219** (2005), 144–182.
- [13] S. CLARK, F. GESZTESY, M. ZINCHENKO, *Weyl-Titchmarsh theory and Borg-Marchenko-type uniqueness results for CMV operators with matrix-valued Verblunsky coefficients*, Operators and Matrices **1** (2007), 535–592.
- [14] M.J. CORLESS, A.E. FRAZHO, *Linear Systems and Control - An Operator Perspective*, Marcel Dekker, New York, 2003.
- [15] PH. DELSARTE, Y. GENIN, Y. KAMP, *Orthogonal polynomial matrices on the unit circles*, IEEE Trans. Circuits and Systems, **CAS-25** (1978), 149–160.
- [16] PH. DELSARTE, Y. GENIN, Y. KAMP, *Schur parametrization of positive definite block-Toeplitz systems*, SIAM J. Appl. Math. **36:1** (1979), February, 34–46.
- [17] V.K. DUBOVOI, B. FRITZSCHE, B. KIRSTEIN, *Matricial version of the classical Schur problem*, in: Teubner-Texte zur Mathematik [Teubner Texts in Mathematics] **129**, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1992.

- [18] H. DYM, *Hermitian block Toeplitz matrices, orthogonal polynomials, reproducing kernel Pontryagin spaces, interpolation and extension*, in: Oper. Theory, Adv. Appl. **34** (1988), Birkhäuser, 79–135.
- [19] H. DYM, *J-contractive matrix functions, reproducing kernel Hilbert spaces and interpolation*, CBMS Regional Conference Series in Mathematics 71, American Mathematical Society, Providence, RI, 1989.
- [20] B. FRITZSCHE, B. KIRSTEIN, *An extension problem for non-negative Hermitian block Toeplitz matrices*, Math. Nachr., Part I: **130** (1987), 121–135; Part II: **131** (1987), 287–297; Part III: **135** (1988), 319–341; Part IV: **143** (1989), 329–354; Part V: **144** (1989), 283–308.
- [21] B. FRITZSCHE, B. KIRSTEIN, *On the Weyl matrix balls associated with nondegenerate matrix-valued Carathéodory functions*, Z. Anal. Anwendungen **12** (1993), 239–261.
- [22] B. FRITZSCHE, B. KIRSTEIN, M. MOSCH, *On block completion problems for Arov-normalized j_{qq} - J_q -elementary factors*, Linear Algebra Appl. **346** (2002), 273–291.
- [23] B. FRITZSCHE, B. KIRSTEIN, K. MÜLLER, *An analysis of the block structure of certain subclasses of j_{qq} -inner functions*, Z. Anal. Anwendungen **17** (1998), 459–478.
- [24] B. FRITZSCHE, B. KIRSTEIN, A.L. SAKHNOVICH, *Completion problems and scattering problems for Dirac type differential equations with singularities*, J. Math. Anal. Appl. **317** (2006), 510–525.
- [25] J.S. GERONIMO, *Polynomials orthogonal on the unit circle with random recurrence coefficients*, in: Lecture Notes in Math. **1550** (1993), Springer, Berlin, 43–61.
- [26] J.S. GERONIMO, F. GESZTESY, H. HOLDEN, *Algebro-geometric solutions of the Baxter-Szegő difference equation*, Comm. Math. Phys. **258** (2005), 149–177.
- [27] F. GESZTESY, H. HOLDEN, J. MICHOR, G. TESCHL, *Algebro-Geometric Finite-Gap Solutions of the Ablowitz-Ladik Hierarchy*, arxiv: nlin.SI/0611055 (Int. Math. Res. Not. to appear).
- [28] F. GESZTESY, B. SIMON, *On local Borg-Marchenko uniqueness results*, Commun. Math. Phys. **211** (2000), 273–287.
- [29] I. GOHBERG, M.A. KAASHOEK, A.L. SAKHNOVICH, *Canonical systems with rational spectral densities: explicit formulas and applications*, Math. Nachrichten **194** (1998), 93–125.
- [30] I. GOHBERG, M.A. KAASHOEK, A.L. SAKHNOVICH, *Scattering problems for a canonical system with a pseudo-exponential potential*, Asymptotic Analysis **29:1** (2002), 1–38.
- [31] L. GOLINSKII, P. NEVAI, *Szegő difference equations, transfer matrices and orthogonal polynomials on the unit circle*, Comm. Math. Phys. **223** (2001), 223–259.
- [32] M.A. KAASHOEK, A.L. SAKHNOVICH, *Discrete skew self-adjoint canonical system and the isotropic Heisenberg magnet model*, J. Functional Anal. **228** (2005), 207–233.
- [33] B.M. LEVITAN, I.S. SARGSIAN, *Introduction to spectral theory: selfadjoint ordinary differential operators*, Translations of Mathematical Monographs, Vol. 39, American Mathematical Society, Providence, R.I., 1975.
- [34] V.P. POTAPOV, *The multiplicative structure of J-contractive matrix functions* Am. Math. Soc. Transl. II Ser. **15** (1960), 131–243.
- [35] J. ROVNYAK, L.A. SAKHNOVICH, *Some indefinite cases of spectral problems for canonical systems of difference equations*, Linear Algebra Appl. **343/344** (2002), 267–289.
- [36] A.L. SAKHNOVICH, *On the continuation of the block Toeplitz matrices*, Functional Analysis (Ulanovsk) **14** (1980), 116–127.
- [37] A.L. SAKHNOVICH, *Exact solutions of nonlinear equations and the method of operator identities*, Linear Algebra Appl. **182** (1993), 109–126.
- [38] A.L. SAKHNOVICH, *Dressing procedure for solutions of nonlinear equations and the method of operator identities*, Inverse Problems **10** (1994), 699–710.
- [39] A.L. SAKHNOVICH, *Iterated Bäcklund-Darboux transform for canonical systems*, J. Functional Anal. **144** (1997), 359–370.
- [40] A.L. SAKHNOVICH, *Toeplitz matrices with an exponential growth of entries and the first Szegő limit theorem*, J. Functional Anal. **171** (2000), 449–482.
- [41] A.L. SAKHNOVICH, *Dirac type and canonical systems: spectral and Weyl-Titchmarsh functions, direct and inverse problems*, Inverse Problems **18** (2002), 331–348.
- [42] A.L. SAKHNOVICH, *Skew-self-adjoint discrete and continuous Dirac-type systems: inverse problems and Borg-Marchenko theorems*, Inverse Problems **22** (2006), 2083–2101.
- [43] L.A. SAKHNOVICH, *On the factorization of the transfer matrix function*, Dokl. Akad. Nauk SSSR **226** (1976), 781 – 784. English transl. in Sov. Math. Dokl. **17** (1976), 203–207.
- [44] L.A. SAKHNOVICH, *Interpolation theory and its applications*, Mathematics and its Applications **428**, Kluwer Academic Publishers, Dordrecht, 1997.
- [45] L.A. SAKHNOVICH, *Spectral theory of canonical differential systems, method of operator identities*, Oper. Theory, Adv. Appl. **107**, Birkhäuser Verlag, Basel-Boston, 1999.

- [46] B. SIMON, *A new approach to inverse spectral theory I. Fundamental formalism*, Ann. of Math. **150** (1999), 1029–1057.
- [47] B. SIMON, *Orthogonal polynomials on the unit circle*, Parts 1,2, Colloquium Publications, American Mathematical Society **51, 54**, Providence, RI, 2005.
- [48] G. TESCHL, *Jacobi operators and completely integrable nonlinear lattices*, Mathematical Surveys and Monographs **72**, Providence, RI: AMS, 2000.

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