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CONSERVATIVE REALIZATIONS OF THE FUNCTIONS ASSOCIATED WITH SCHUR'S ALGORITHM FOR THE SCHUR CLASS OPERATOR–VALUED FUNCTION

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Dedicated to the memory of Peter Jonas,
remarkable human being and mathematician

(*communicated by D. Alpay*)

Abstract. Let $\mathfrak{M}$ and $\mathfrak{N}$ be separable Hilbert spaces and let $\Theta(\lambda)$ be a function from the Schur class $\mathbf{S}(\mathfrak{M},\mathfrak{N})$ of contractive functions holomorphic on the unit disk. The operator generalization of the classical Schur algorithm associates with Θ the sequence of contractions (the Schur parameters of Θ) $\Gamma_0=\Theta(0)\in\mathbf{L}(\mathfrak{M},\mathfrak{N})$, $\Gamma_n\in\mathbf{L}(\mathfrak{D}_{\Gamma_{n-1}},\mathfrak{D}_{\Gamma^*_{n-1}})$ and the sequence of functions $\Theta_0=\Theta$, $\Theta_n\in\mathbf{S}(\mathfrak{D}_{\Gamma_n},\mathfrak{D}_{\Gamma^*_n})$ $n=1,\ldots$ (the Schur iterates of Θ) connected by the relations

$$\Gamma_n=\Theta_n(0),\ \Theta_n(\lambda)=\Gamma_n+\lambda D_{\Gamma^*_n}\Theta_{n+1}(\lambda)(I+\lambda\Gamma^*_n\Theta_{n+1}(\lambda))^{-1}D_{\Gamma_n},\ |\lambda|<1.$$

It is well known that the function $\Theta(\lambda)\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$ can be realized as the transfer function

$$\Theta(\lambda)=D+\lambda C(I-\lambda A)^{-1}B$$

of a linear conservative and simple discrete time-invariant system $\tau=\left\{\begin{bmatrix}D & C\\ B & A\end{bmatrix};\mathfrak{M},\mathfrak{N},\mathfrak{H}\right\}$ with the state space $\mathfrak{H}$ and the input and output spaces $\mathfrak{M}$ and $\mathfrak{N}$, respectively.

In this paper we give a construction of conservative and simple realizations of the Schur iterates Θ_n by means of the conservative and simple realization of Θ.

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1. Introduction

The Schur class $\mathbf{S}$ of scalar analytic functions and bounded by one in the unit disc $\mathbb{D}=\{\lambda\in\mathbb{C}:|\lambda|<1\}$ plays a prominent role in complex analysis and operator theory as well in their applications in linear system theory and mathematical engineering. Given a Schur function $f(\lambda)$, which is not a finite Blaschke product, define inductively

$$f_0(\lambda)=f(\lambda),\ f_{n+1}(\lambda)=\frac{f_n(\lambda)-f_n(0)}{\lambda(1-\overline{f_n(0)}f_n(\lambda))},\ n\geqslant 0.$$

It is clear that $\{f_n(\lambda)\}$ is an infinite sequence of Schur functions and neither of its terms is a finite Blaschke product. The numbers $\gamma_n:=f_n(0)$ are called the *Schur parameters:*

$$\mathscr{S}f=\{\gamma_0,\gamma_1,\ldots\}.$$

Note that

$$f_n(\lambda)=\frac{\gamma_n+\lambda f_{n+1}(\lambda)}{1+\bar{\gamma}_n\lambda f_{n+1}}=\gamma_n+(1-|\gamma_n|^2)\frac{\lambda f_{n+1}(\lambda)}{1+\bar{\gamma}_n\lambda f_{n+1}(\lambda)},\ n\geqslant 0.$$

The method of labeling $f\in\mathbf{S}$ by its Schur parameters is known as the *Schur algorithm* and is due to I. Schur [40]. The transformation

$$\mathbf{S}\ni f(\lambda)\to\varphi(\lambda)=\frac{f(\lambda)-f(0)}{\lambda(1-\overline{f(0)}f(\lambda))}\in\mathbf{S}$$

is called now the *Schur transformation* [4]. In the case when

$$f(\lambda)=e^{i\varphi}\prod_{k=1}^{N}\frac{\lambda-\lambda_k}{1-\bar{\lambda}_k\lambda}$$

is a finite Blaschke product of order N, the Schur algorithm terminates at the N-th step. The sequence of Schur parameters $\{\gamma_n\}_{n=0}^{N}$ is finite, $|\gamma_n|<1$ for $n=0,1,\ldots,N-1$, and $|\gamma_N|=1$.

The Schur algorithm for matrix valued Schur class functions has been considered in the paper of Delsarte, Genin, and Kamp [32] and in the book of Dubovoj, Fritzsche, and Kirstein [34]. An operator extension of the Schur algorithm was developed by T. Constantinescu in [29] and with numerous applications is presented in the monographs of Bakonyi and Constantinescu [20] and Constantinescu [30].

In what follows the class of all continuous linear operators defined on a complex Hilbert space $\mathfrak{H}_1$ and taking values in a complex Hilbert space $\mathfrak{H}_2$ is denoted by $\mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ and $\mathbf{L}(\mathfrak{H}):=\mathbf{L}(\mathfrak{H},\mathfrak{H})$. The domain, the range, and the null-space of a linear operator T are denoted by $\operatorname{dom}T$, $\operatorname{ran}T$, and $\ker T$, respectively. The set of all regular points of a closed operator T is denoted by $\rho(T)$. We denote by $I_{\mathscr{H}}$ the identity operator in a Hilbert space $\mathscr{H}$ and by $P_{\mathscr{L}}$ the orthogonal projection onto the subspace (the closed linear manifold) $\mathscr{L}$. The notation $T\upharpoonright\mathscr{L}$ means the restriction of a linear operator T on the set $\mathscr{L}$. The positive integers will be denoted by $\mathbb{N}$. An operator $T\in\mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ is said to be

(a) *contractive* if $\|T\| \leqslant 1$;

(b) *isometric* if $\|Tf\| = \|f\|$ for all $f \in \mathfrak{H}_1 \iff T^*T = I_{\mathfrak{H}_1}$;

(c) *co-isometric* if T^* is isometric $\iff TT^* = I_{\mathfrak{H}_2}$;

(d) *unitary* if it is both isometric and co-isometric.

Given a contraction $T \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$, the operators

$$D_T := (I - T^*T)^{1/2}, \qquad D_{T^*} := (I - TT^*)^{1/2}$$

are called the *defect operators* of T, and the subspaces $\mathfrak{D}_T = \overline{\operatorname{ran}} D_T$, $\mathfrak{D}_{T^*} = \overline{\operatorname{ran}} D_{T^*}$ the *defect subspaces* of T. The dimensions $\dim \mathfrak{D}_T$, $\dim \mathfrak{D}_{T^*}$ are known as the *defect numbers* of T. The defect operators satisfy the following intertwining relations

$$TD_T = D_{T^*}T, \qquad T^*D_{T^*} = D_T T^*. \tag{1.1}$$

It follows from (1.1) that $T\mathfrak{D}_T \subset \mathfrak{D}_{T^*}$, $T^*\mathfrak{D}_{T^*} \subset \mathfrak{D}_T$, and $T(\ker D_T) = \ker D_{T^*}$, $T^*(\ker D_{T^*}) = \ker D_T$. Moreover, the operators $T \restriction \ker D_T$ and $T^* \restriction \ker D_{T^*}$ are isometries and $T \restriction \mathfrak{D}_T$ and $T^* \restriction \mathfrak{D}_{T^*}$ are *pure* contractions, i.e., $||Tf|| < ||f||$ for $f \in \mathfrak{H} \setminus \{0\}$.

The *Schur class* $\mathbf{S}(\mathfrak{H}_1, \mathfrak{H}_2)$ is the set of all function $\Theta(\lambda)$ analytic on the unit disk $\mathbb{D}$ with values in $\mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ and such that $\|\Theta(\lambda)\| \leqslant 1$ for all $\lambda \in \mathbb{D}$. The next theorem goes back to Shmul'yan [41], [42] and T. Constantinescu [29] (see also [20], [10]) and plays a key role in the operator Schur algorithm.

THEOREM 1.1. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be separable Hilbert spaces and let the function $\Theta(\lambda)$ be from the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$. Then there exists a function $Z(\lambda)$ from the Schur class $\mathbf{S}(\mathfrak{D}_{\Theta(0)}, \mathfrak{D}_{\Theta^*(0)})$ such that*

$$\Theta(\lambda) = \Theta(0) + D_{\Theta^*(0)} Z(\lambda)(I + \Theta^*(0)Z(\lambda))^{-1} D_{\Theta(0)}, \ \lambda \in \mathbb{D}. \tag{1.2}$$

In what follows we will call the representation (1.2) of a function $\Theta(\lambda)$ from the Schur class *the Möbius representation* of $\Theta(\lambda)$ and the function $Z(\lambda)$ we will call *the Möbius parameter* of $\Theta(\lambda)$. Clearly, $Z(0) = 0$ and by Schwartz's lemma we obtain that

$$||Z(\lambda)|| \leqslant |\lambda|, \ \lambda \in \mathbb{D}.$$

The operator Schur's algorithm [20]. Fix $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$, put $\Theta_0(\lambda) = \Theta(\lambda)$ and let $Z_0(\lambda)$ be the Möbius parameter of Θ. Define

$$\Gamma_0 = \Theta(0), \ \Theta_1(\lambda) = \frac{Z_0(\lambda)}{\lambda} \in \mathbf{S}(\mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma_0^*}), \ \Gamma_1 = \Theta_1(0) = Z_0'(0).$$

If $\Theta_0(\lambda), \ldots, \Theta_n(\lambda)$ and $\Gamma_0, \ldots, \Gamma_n$ have been chosen, then let $Z_{n+1}(\lambda) \in \mathbf{S}(\mathfrak{D}_{\Gamma_n}, \mathfrak{D}_{\Gamma_n^*})$ be the Möbius parameter of Θ_n. Put

$$\Theta_{n+1}(\lambda) = \frac{Z_{n+1}(\lambda)}{\lambda}, \ \Gamma_{n+1} = \Theta_{n+1}(0).$$

The contractions $\Gamma_0 \in \mathbf{L}(\mathfrak{M},\mathfrak{N})$, $\Gamma_n \in \mathbf{L}(\mathfrak{D}_{\Gamma_{n-1}},\mathfrak{D}_{\Gamma^*_{n-1}})$, $n = 1,2,\ldots$ are called the *Schur parameters* of $\Theta(\lambda)$ and the function $\Theta_n(\lambda) \in \mathbf{S}(\mathfrak{D}_{\Gamma_{n-1}},\mathfrak{D}_{\Gamma^*_{n-1}})$ we will call the $n-th$ *Schur iterate* of $\Theta(\lambda)$.

Formally for the operator analog of the Schur transformation we have

$$\Theta_{n+1}(\lambda)\upharpoonright \operatorname{ran} D_{\Gamma_n} = \frac{1}{\lambda} D_{\Gamma^*_n}(I_{\mathfrak{D}_{\Gamma^*_n}} - \Theta_n(\lambda)\Gamma^*_n)^{-1}(\Theta_n(\lambda) - \Gamma_n)D^{-1}_{\Gamma_n}\upharpoonright \operatorname{ran} D_{\Gamma_n}.$$

Clearly, the sequence of Schur parameters $\{\Gamma_n\}$ is infinite if and only if the operators Γ_n are non-unitary. The sequence of Schur parameters consists of finite number operators $\Gamma_0,\ \Gamma_1,\ldots,\Gamma_N$ if and only if $\Gamma_N \in \mathbf{L}(\mathfrak{D}_{\Gamma_{N-1}},\mathfrak{D}_{\Gamma^*_{N-1}})$ is unitary. If Γ_N is isometric (co-isometric) then $\Gamma_n = 0$ for all $n > N$.

The following theorem is the operator generalization of Schur's result.

THEOREM 1.2. [29], [20]. *There is a one-to-one correspondence between the Schur class functions* $\mathbf{S}(\mathfrak{M},\mathfrak{N})$ *and the set of all sequences of contractions* $\{\Gamma_n\}_{n\geqslant 0}$ *such that*

$$\Gamma_0 \in \mathbf{L}(\mathfrak{M},\mathfrak{N}),\ \Gamma_n \in \mathbf{L}(\mathfrak{D}_{\Gamma_{n-1}},\mathfrak{D}_{\Gamma^*_{n-1}}),\ n \geqslant 1. \tag{1.3}$$

Notice that a sequence of contractions of the form (1.3) is called the *choice sequence* [28].

It is known [37], [27], [13], [8], [21] that every $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M},\mathfrak{N})$ can be realized as the transfer function

$$\Theta(\lambda) = D + \lambda C(I_{\mathfrak{H}} - \lambda A)^{-1}B$$

of a linear conservative and simple discrete time-invariant system (see Section 4)

$$\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathfrak{M},\mathfrak{N},\mathfrak{H} \right\}$$

with the state space $\mathfrak{H}$ and the input and output spaces $\mathfrak{M}$ and $\mathfrak{N}$, respectively. In this paper we study the problem of the conservative realizations of the Schur iterates of the function $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M},\mathfrak{N})$ by means of the conservative realization of Θ.

In this connection it should be pointed out that the similar problem for a scalar generalized Schur and Nevanlinna classes functions has been studied in [1], [2], [3], [4], [5], [6]. For a scalar finite Blaschke product the realizations of the Schur iterates are constructed in [36].

Here we describe our main results. Let A be a completely non-unitary contraction [47] in a separable Hilbert space $\mathfrak{H}$. Define the subspaces and operators

$$\begin{aligned}
&\mathfrak{H}_{m,0} = \ker D_{A^m},\ \mathfrak{H}_{0,l} = \ker D_{A^{*l}},\\
&\mathfrak{H}_{m,l} = \ker D_{A^m} \cap \ker D_{A^{*l}},\ m,l \in \mathbb{N},\\
&A_{m,l} = P_{m,l}A\upharpoonright \mathfrak{H}_{m,l},
\end{aligned}$$

where $P_{m,l}$ is the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{H}_{m,l}$.

We prove that

1) if A is a completely non-unitary contraction in a Hilbert space then for every $n\in\mathbb{N}$ the operators

$$A_{n,0},\ A_{n-1,1},\dots,A_{0,n}$$

are unitary equivalent completely non-unitary contractions and their Sz.-Nagy– Foias characteristic functions [47] coincide with the pure contractive part [47], [20] for the n-th Schur iterate $\Phi_n(\lambda)$ of the characteristic function $\Phi(\lambda)$ of A;

2) if $\Theta(\lambda)\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$ is the transfer function of a simple conservative system

$$\tau=\left\{\begin{bmatrix}\Gamma_0 & C\\ B & A\end{bmatrix};\mathfrak{M},\mathfrak{N},\mathfrak{H}\right\},$$

then the Schur parameters of Θ take the form

$$\begin{aligned}&\Gamma_1=D^{-1}_{\Gamma_0^*}C\left(D^{-1}_{\Gamma_0}B^*\right)^*,\ \Gamma_2=D^{-1}_{\Gamma_1^*}D^{-1}_{\Gamma_0^*}CA\left(D^{-1}_{\Gamma_1}D^{-1}_{\Gamma_0}\left(B^*\restriction\mathfrak{H}_{1,0}\right)\right)^*,\dots,\\ &\Gamma_n=D^{-1}_{\Gamma_{n-1}^*}\cdots D^{-1}_{\Gamma_0^*}CA^{n-1}\left(D^{-1}_{\Gamma_{n-1}}\cdots D^{-1}_{\Gamma_0}\left(B^*\restriction\mathfrak{H}_{n-1,0}\right)\right)^*,\dots,\end{aligned}$$

and the n-th Schur iterate $\Theta_n(\lambda)$ of Θ is the transfer function of the simple conservative and unitarily equivalent systems

$$\tau_n^{(k)}=\left\{\begin{bmatrix}\Gamma_n & D^{-1}_{\Gamma_{n-1}^*}\cdots D^{-1}_{\Gamma_0^*}(CA^{n-k})\\ A^k\left(D^{-1}_{\Gamma_{n-1}}\cdots D^{-1}_{\Gamma_0}\left(B^*\restriction\mathfrak{H}_{n,0}\right)\right)^* & A_{n-k,k}\end{bmatrix};\mathfrak{D}_{\Gamma_{n-1}},\mathfrak{D}_{\Gamma_{n-1}^*},\mathfrak{H}_{n-k,k}\right\}$$

for $k=0,\dots,n$. Here $D^{-1}_{\Gamma_m}$ and $D^{-1}_{\Gamma_m^*}$ are the Moore–Penrose pseudo-inverses. For a completely non-unitary contraction A with rank one defect operators it was proved in [12] that the characteristic functions of the operators $A_{1,0}=P_{\ker D_A}A\restriction\ker D_A$ and $A_{0,1}=P_{\ker D_{A^*}}A\restriction\ker D_{A^*}$ coincide with the first Schur iterate of the characteristic function of A. This result has been established using the model of A given by a truncated CMV matrix. Here we use another approach based on the parametrization of a contractive block-operator matrix

$$T=\begin{bmatrix}D & C\\ B & A\end{bmatrix}:\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{H}\end{matrix}\to\begin{matrix}\mathfrak{N}\\ \oplus\\ \mathfrak{K}\end{matrix}$$

given in [19], [31], [43], and the construction of the passive realization for the Möbius parameter $Z(\lambda)$ of $\Theta(\lambda)$ obtained in [10] by means of a passive realization of Θ.

2. Completely non-unitary contractions

Let S be an isometry in a separable Hilbert space H. A subspace Ω in H is called wandering for V if $S^p\Omega\perp S^q\Omega$ for all $p,q\in\mathbb{Z}_+$, $p\neq q$. Since S is an isometry, the latter is equivalent to $S^n\Omega\perp\Omega$ for all $n\in\mathbb{N}$. If $H=\sum_{n=0}^\infty\oplus S^n\Omega$ then S is called a *unilateral shift* and Ω is called the generating subspace. The dimension of Ω is called the multiplicity of the unilateral shift S. It is well known [47, Theorem I.1.1] that the isometry S is a unilateral shift if and only if

$$\bigcap_{n=0}^{\infty} S^n H = \bigcap_{k\geqslant 1} \ker D_{S^{*k}} = \{0\}.$$

Clearly, if an isometry S is the unilateral shift in H, then $\Omega = H \ominus SH = \mathfrak{D}_{S^*}$ is the generating subspace for S. An operator is called *co-shift* if its adjoint is a unilateral shift.

A contraction A acting in a Hilbert space $\mathfrak{H}$ is called *completely non-unitary* if there is no nontrivial reducing subspace of A, on which A generates a unitary operator. Given a contraction A in $\mathfrak{H}$, then there is a canonical orthogonal decomposition [47, Theorem I.3.2]

$$\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1, \qquad A = A_0 \oplus A_1, \quad A_j = A{\upharpoonright}\mathfrak{H}_j, \quad j = 0, 1,$$

where $\mathfrak{H}_0$ and $\mathfrak{H}_1$ reduce A, the operator A_0 is a completely non-unitary contraction, and A_1 is a unitary operator. Moreover,

$$\mathfrak{H}_1 = \left(\bigcap_{n\geqslant 1} \ker D_{A^n}\right) \bigcap \left(\bigcap_{n\geqslant 1} \ker D_{A^{*n}}\right).$$

Since

$$\bigcap_{k=0}^{n-1} \ker(D_A A^k) = \ker D_{A^n}, \ \bigcap_{k=0}^{n-1} \ker(D_{A^*} A^{*k}) = \ker D_{A^{*n}},$$

we get

$$\begin{aligned} \bigcap_{n\geqslant 1} \ker D_{A^n} &= \mathfrak{H} \ominus \overline{\operatorname{span}}\, \{A^{*n} D_A \mathfrak{H},\ n = 0, 1, \ldots\}, \\ \bigcap_{n\geqslant 1} \ker D_{A^{*n}} &= \mathfrak{H} \ominus \overline{\operatorname{span}}\, \{A^{n} D_{A^*} \mathfrak{H},\ n = 0, 1, \ldots\}. \end{aligned} \tag{2.1}$$

It follows that

$$\begin{aligned} &A \text{ is completely non-unitary} \iff \left(\bigcap_{n\geqslant 1} \ker D_{A^n}\right) \bigcap \left(\bigcap_{n\geqslant 1} \ker D_{A^{*n}}\right) = \{0\} \iff \\ &\iff \overline{\operatorname{span}}\, \{A^{*n} D_A,\ A^m D_{A^*},\ n, m \geqslant 0\} = \mathfrak{H}. \end{aligned} \tag{2.2}$$

Note that

$$\ker D_A \supset \ker D_{A^2} \supset \cdots \supset \ker D_{A^n} \supset \cdots,$$

$$A \ker D_{A^n} \subset \ker D_{A^{n-1}},\ n = 2, 3, \ldots.$$

From (2.1) we get that the subspaces $\bigcap_{n\geqslant 1} \ker D_{A^n}$ and $\bigcap_{n\geqslant 1} \ker D_{A^{*n}}$ are invariant with respect to A and A^*, respectively, and the operators $A{\upharpoonright} \bigcap_{n\geqslant 1} \ker D_{A^n}$ and $A^*{\upharpoonright} \bigcap_{n\geqslant 1} \ker D_{A^{*n}}$

are unilateral shifts, moreover, these operators are the maximal unilateral shifts contained in A and A^*, respectively [35, Theorem 1.1, Corollary 1]. By definition [35] the operator A contains a co-shift V if the operator A^* contains the unilateral shift V^*. In accordance with the terminology of [21], a contraction A in $\mathfrak{H}$ is called *completely non-isometric* (*c.n.i.*) if there is no nonzero invariant subspace for A on which A is isometric. This equivalent to (see [21])

$$\bigcap_{n\geqslant 1} \ker D_{A^n} = \{0\}.$$

A contraction A is called *completely non-co-isometric* (c.n.c.-i.) if A^* is completely non-isometric. Thus, for a completely non-unitary contraction A we have

$$\begin{aligned} &\bigcap_{n\geqslant 1} \ker D_{A^n} = \{0\} \iff A \text{ is c.n.i.} \iff A \text{ does not contain a unilateral shift,}\\ &\bigcap_{n\geqslant 1} \ker D_{A^{*n}} = \{0\} \iff A \text{ is c.n.c.-i.} \iff A^* \text{ does not contain a unilateral shift.} \end{aligned} \tag{2.3}$$

The function (see [47, Chapter VI])

$$\Phi_A(\lambda) = \left(-A + \lambda D_{A^*}(I - \lambda A^*)^{-1} D_A\right) \restriction \mathfrak{D}_A \tag{2.4}$$

is known as the Sz.-Nagy – Foias *characteristic function* of a contraction A [47]. This function belongs to the Schur class $\mathbf{S}(\mathfrak{D}_A, \mathfrak{D}_{A^*})$ and $\Theta_A(0)$ is a pure contraction. The characteristic functions of A and A^* are connected by the relation

$$\Phi_{A^*}(\lambda) = \Phi_A^*(\bar{\lambda}), \quad \lambda \in \mathbb{D}.$$

Two operator-valued functions $\Theta_1 \in \mathbf{S}(\mathfrak{M}_1, \mathfrak{N}_1)$ and $\Theta_2 \in \mathbf{S}(\mathfrak{M}_2, \mathfrak{N}_2)$ *coincide* [47] if there are two unitary operators $V : \mathfrak{N}_1 \to \mathfrak{N}_2$ and $W : \mathfrak{M}_2 \to \mathfrak{M}_1$ such that

$$V\Theta_1(\lambda)W = \Theta_2(\lambda), \quad \lambda \in \mathbb{D}.$$

The result of Sz.-Nagy–Foias [47, Theorem VI.3.4] states that two completely non-unitary contractions A_1 and A_2 are unitary equivalent if and only if their characteristic functions Φ_{A_1} and Φ_{A_2} coincide.

It is well known that a function $\Theta(\lambda)$ from the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$ has almost everywhere non-tangential strong limit values $\Theta(\xi)$, $\xi \in \mathbb{T}$, where $\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$ stands for the unit circle; cf. [47]. A function $\Theta \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is called *inner* if $\Theta^*(\xi)\Theta(\xi) = I_{\mathfrak{M}}$ and *co-inner* if $\Theta(\xi)\Theta^*(\xi) = I_{\mathfrak{N}}$ almost everywhere on $\xi \in \mathbb{T}$. A function $\Theta \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is called *bi-inner*, if it is both inner and co-inner. A contraction T on a Hilbert space $\mathfrak{H}$ belongs to the class $C_{0\cdot}$ ($C_{\cdot 0}$), if

$$s - \lim_{n\to\infty} A^n = 0 \qquad (s - \lim_{n\to\infty} A^{*n} = 0),$$

respectively. By definition $C_{00} := C_{0\cdot} \cap C_{\cdot 0}$. A completely non-unitary contraction A belongs to the class $C_{\cdot 0}$, $C_{0\cdot}$, or C_{00} if and only if its characteristic function $\Phi_A(\lambda)$ is inner, co-inner, or bi-inner, respectively (cf. [47, Section VI.2]). Note that for a completely non-unitary contraction A the equality $\ker D_A=\ker D_{A^*}\neq\{0\}$ is impossible because otherwise the subspace $\ker D_A$ reduces A and $A\upharpoonright\ker D_A$ is a unitary operator.

We complete this section by a description of completely non-unitary contractions with constant characteristic functions. Note that $\Phi_A(\lambda)=0\in\mathbf{S}(\{0\},\mathfrak{D}_{A^*})\iff A$ is a unilateral shift, and $\Phi_A(\lambda)=0\in\mathbf{S}(\mathfrak{D}_A,\{0\})\iff A$ is a co-shift.

THEOREM 2.1. *Let $\mathfrak{H}$ be a separable Hilbert space. A completely non-unitary contraction A with nonzero defect operators has a constant characteristic function if and only if $\mathfrak{H}$ is the orthogonal sum*

$$\mathfrak{H}=\mathcal{H}_1\oplus\mathcal{H}_2$$

and A takes the operator matrix form

$$A=\begin{bmatrix}S_1&\Gamma\\0&S_2^*\end{bmatrix}:\begin{array}{c}\mathcal{H}_1\\\oplus\\\mathcal{H}_2\end{array}\to\begin{array}{c}\mathcal{H}_1\\\oplus\\\mathcal{H}_2\end{array},\tag{2.5}$$

where S_1 and S_2 are unilateral shifts in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and Γ is a contraction such that

$$\begin{cases}\operatorname{ran}\Gamma\subset\mathfrak{D}_{S_1^*},\operatorname{ran}\Gamma^*\subset\mathfrak{D}_{S_2^*},\\||\Gamma f||<||f||,\ f\in\mathfrak{D}_{S_2^*}\setminus\{0\},\\||\Gamma^*h||<||h||,\ h\in\mathfrak{D}_{S_1^*}\setminus\{0\}.\end{cases}\tag{2.6}$$

In particular, the characteristic function of A is identically equal zero if and only if A is the orthogonal sum of a shift and co-shift.

Proof. Suppose that the contraction A takes the form (2.5) with unilateral shifts S_1 and S_2, and the contraction Γ with the properties (2.6). Then

$$D_A^2=\begin{bmatrix}0&0\\0&D_{S_2^*}-\Gamma^*\Gamma\end{bmatrix}:\begin{array}{c}\mathcal{H}_1\\\oplus\\\mathcal{H}_2\end{array}\to\begin{array}{c}\mathcal{H}_1\\\oplus\\\mathcal{H}_2\end{array},\tag{2.7}$$

and

$$D_{A^*}^2=\begin{bmatrix}D_{S_1^*}-\Gamma\Gamma^*&0\\0&0\end{bmatrix}:\begin{array}{c}\mathcal{H}_1\\\oplus\\\mathcal{H}_2\end{array}\to\begin{array}{c}\mathcal{H}_1\\\oplus\\\mathcal{H}_2\end{array}.\tag{2.8}$$

Since $\mathfrak{D}_{S_1^*}=\ker S_1^*$, $\mathfrak{D}_{S_2^*}=\ker S_2^*$, and $D_{S_1^*}$ and $D_{S_2^*}$ are the orthogonal projections in $\mathfrak{H}$ onto $\mathfrak{D}_{S_1^*}$ and $\mathfrak{D}_{S_2^*}$, respectively, we get from (2.6) the relations

$$\mathfrak{D}_A=\mathfrak{D}_{S_2^*},\ \mathfrak{D}_{A^*}=\mathfrak{D}_{S_1^*}.\tag{2.9}$$

Taking into account that $\mathcal{H}_2$ is an invariant subspace for A^*, we have

$$D_{A^*}(I_{\mathfrak{H}}-\lambda A^*)^{-1}D_A=0.$$

Hence $\Phi_A(\lambda) = \Gamma\upharpoonright\mathfrak{D}_{S_2^*} = const$.

Because S_1 and S_2 are unilateral shifts, we get

$$\mathscr{H}_1 = \sum_{n\geqslant 0} \oplus S_1^n \mathfrak{D}_{S_1^*},\ \mathscr{H}_2 = \sum_{n\geqslant 0} \oplus S_2^n \mathfrak{D}_{S_2^*}.$$

Since $\mathfrak{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, the operator A is completely non-unitary. If $\Gamma = 0$ then A is the orthogonal sum of a shift and co-shift.

Now suppose that the characteristic function of A is a constant. From (2.4) we get

$$D_{A^*}A^{*n}D_A = 0,\ D_A A^n D_{A^*} = 0,\ n = 0,1,2,\ldots.$$

It follows

$$\overline{\text{span}}\{D_{A^{*n}}\mathfrak{D}_A,\ n=0,1,\ldots\} \subset \ker D_{A^*} \iff \bigcap_{n\geqslant 1} \ker D_{A^n} \supset \mathfrak{D}_{A^*},$$
$$\overline{\text{span}}\{D_{A^n}\mathfrak{D}_{A^*},\ n=0,1,\ldots\} \subset \ker D_A \iff \bigcap_{n\geqslant 1} \ker D_{A^{*n}} \supset \mathfrak{D}_A.$$

Let

$$\mathscr{H}_1 = \bigcap_{n\geqslant 1} \ker D_{A^n},\ \mathscr{H}_2 = \bigcap_{n\geqslant 1} \ker D_{A^{*n}}.$$

Since

$$A\mathscr{H}_1 \subset \mathscr{H}_1 \quad \text{and} \quad A\mathscr{H}_1 \perp \mathfrak{D}_{A^*},$$

we get $\mathscr{H}_1 \ominus A\mathscr{H}_1 \supset \mathfrak{D}_{A^*}$ and similarly $\mathscr{H}_2 \ominus A^*\mathscr{H}_2 \supset \mathfrak{D}_A$. Let $h \in \mathscr{H}_1$ and $h \perp \mathfrak{D}_{A^*}$. It follows

$$h \in \ker D_{A^*} \bigcap \left(\bigcap_{n\geqslant 1} \ker D_{A^n} \right).$$

Then $h = Ag$, $g \in \ker D_A$. Hence $g \in \bigcap_{n\geqslant 1} \ker D_{A^n} = \mathscr{H}_1$, i.e., $\mathscr{H}_1 \ominus A\mathscr{H}_1 = \mathfrak{D}_{A^*}$. Similarly $\mathscr{H}_2 \ominus A^*\mathscr{H}_2 = \mathfrak{D}_A$.

Since A is completely non-unitary contraction, the operators $A\upharpoonright\mathscr{H}_1$ and $A^*\upharpoonright\mathscr{H}_2$ are unilateral shifts. Therefore

$$\mathscr{H}_1 = \sum_{n=0}^{\infty} \oplus A^n \mathfrak{D}_{A^*},\ \mathscr{H}_2 = \sum_{n=0}^{\infty} \oplus A^{*n} \mathfrak{D}_A. \tag{2.10}$$

Note that for all $\varphi, \psi \in \mathfrak{H}$

$$(A^m D_{A^*}\varphi, A^{*k} D_A \psi) = (D_A A^{m+k} D_{A^*}\varphi, \psi) = 0,\ m,k = 0,1,2\ldots.$$

Hence $\mathscr{H}_1 \perp \mathscr{H}_2$. Taking into account (2.10) and the relation

$$\mathfrak{H} \ominus \mathscr{H}_1 = \overline{\text{span}}\{A^{*n}\mathfrak{D}_A,\ n = 0,1,2\ldots\},$$

we get $\mathfrak{H} \ominus \mathscr{H}_1 = \mathscr{H}_2$. Because $\mathscr{H}_1$ is invariant with respect to A, the matrix form of A is of the form (2.5) with unilateral shifts

$$S_1 := A\upharpoonright\mathscr{H}_1,\ S_2 := A^*\upharpoonright\mathscr{H}_2,$$

and some operator $\Gamma \in \mathbf{L}(\mathscr{H}_2, \mathscr{H}_1)$. Since A is a contraction, we have

$$||\Gamma f||^2 \leqslant ||D_{S_2^*} f||^2,\ f \in \mathscr{H}_2,$$
$$||\Gamma^* h||^2 \leqslant ||D_{S_1^*} h||^2,\ h \in \mathscr{H}_1.$$

From (2.7) and (2.8) we get

$$\overline{\operatorname{ran}}\,(D_{S_2^*} - \Gamma^*\Gamma) = \mathfrak{D}_A,\ \overline{\operatorname{ran}}\,(D_{S_1^*} - \Gamma\Gamma^*) = \mathfrak{D}_{A^*}.$$

It follows that (2.6) holds true and $\Phi_A(\lambda) = \Gamma$.

If A is the orthogonal sum of a shift and co-shift, then, clearly the characteristic function of A is identically zero.

3. Contractions generated by a contraction

In this section we define and study the subspaces and the corresponding operators obtained from a contraction A in a separable Hilbert space $\mathfrak{H}$.

Suppose $\ker D_A \neq \{0\}$. Define the subspaces

$$\begin{cases} \mathfrak{H}_{0,0} := \mathfrak{H} \\ \mathfrak{H}_{n,0} = \ker D_{A^n},\ \mathfrak{H}_{0,m} := \ker D_{A^{*m}}, \\ \mathfrak{H}_{n,m} := \ker D_{A^n} \cap \ker D_{A^{*m}},\ m,n \in \mathbb{N}. \end{cases} \tag{3.1}$$

Let $P_{n,m}$ be the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{H}_{n,m}$. Define the contractions

$$A_{n,m} := P_{n,m} A \upharpoonright \mathfrak{H}_{n,m} \in \mathbf{L}(\mathfrak{H}_{n,m}) \tag{3.2}$$

and

$$\mathscr{A}_{n,m} := A_{n,m} P_{n+1,m} \upharpoonright \mathfrak{H}_{n,m} \in \mathbf{L}(\mathfrak{H}_{n,m}). \tag{3.3}$$

In the next theorem we establish the main properties of $A_{n,m}$ and $\mathscr{A}_{n,m}$.

THEOREM 3.1. *Let A be a completely non-unitary contraction. Then*

1. *the following equalities are valid:*

$$\begin{cases} \ker D_{A^k_{n,m}} = \mathfrak{H}_{n+k,m} \\ \ker D_{A^{*k}_{n,m}} = \mathfrak{H}_{n,m+k} \end{cases},\ k = 1, 2, \ldots, \tag{3.4}$$

$$\begin{cases} \mathfrak{D}_{A_{n,m}} = \overline{\operatorname{ran}}\,(P_{n,m} D_{A^{n+1}}), \\ \mathfrak{D}_{A^*_{n,m}} = \overline{\operatorname{ran}}\,(P_{n,m} D_{A^{*m+1}}) \end{cases}, \tag{3.5}$$

$$\begin{cases} A\mathfrak{H}_{n,m} = \mathfrak{H}_{n-1,m+1},\ n \geqslant 1, \\ A^*\mathfrak{H}_{n,m} = \mathfrak{H}_{n+1,m-1},\ m \geqslant 1 \end{cases}, \tag{3.6}$$

$$\begin{cases} \ker D_{\mathscr{A}^k_{n,m}} = \mathfrak{H}_{n+k,m} \\ \ker D_{\mathscr{A}^{*k}_{n,m}} = \mathfrak{H}_{n,m+k} \end{cases} k = 1, 2, \ldots, \tag{3.7}$$

$$\begin{cases} \mathfrak{D}_{\mathscr{A}_{n,m}} = \mathfrak{D}_{A_{n+1,m}} \\ \mathfrak{D}_{\mathscr{A}^*_{n,m}} = \mathfrak{D}_{A^*_{n+1,m}} \end{cases}, \tag{3.8}$$

$$(A_{n,m})_{k,l} = A_{n+k,m+l}; \tag{3.9}$$

2. *The operators* $\{A_{n,m}\}$ *and* $\{\mathscr{A}_{n,m}\}$ *are completely non-unitary contractions.*

3. *The operators*

$$A_{n,0},\ A_{n-1,1},\ \ldots, A_{n-k,k}, \ldots, A_{0,n}$$

are unitarily equivalent and

$$A_{n-1,m+1}Af = AA_{n,m}f,\ f \in \mathfrak{H}_{n,m},\ n \geqslant 1. \tag{3.10}$$

4. *The operators*

$$\mathscr{A}_{n,0},\ \mathscr{A}_{n-1,1},\ \ldots, \mathscr{A}_{n-k,k}, \ldots, \mathscr{A}_{0,n}$$

are unitarily equivalent and

$$\mathscr{A}_{n-1,m+1}Af = A\mathscr{A}_{n,m}f,\ f \in \mathfrak{H}_{n,m},\ n \geqslant 1. \tag{3.11}$$

5. *the following statements are equivalent*

   (a) $\mathscr{A}_{n,0} \in C_{\cdot 0}$ $(\mathscr{A}_{n,0} \in C_{0\cdot})$ *for some* n*,*

   (b) $A_{n+1,0} \in C_{\cdot 0}$ $(A_{n+1,0} \in C_{0\cdot})$.

Proof. It is sufficient to prove the first equality from (3.4). From (3.1) and (3.2) we have

$$f \in \mathfrak{H}_{n,m},\ f \in \ker D_{A^k_{n,m}} \iff \begin{cases} \|f\| = \|A^n f\| = \|A^{*m} f\| \\ \|f\| = \|A^k_{n,m} f\| \end{cases}$$
$$\iff Af, \ldots, A^k f \in \mathfrak{H}_{n,m} \iff f \in \mathfrak{H}_{n+k,m}.$$

This proves (3.4). Hence

$$\begin{aligned} \mathfrak{D}_{A_{n,m}} &= \mathfrak{H}_{n,m} \ominus \mathfrak{H}_{n+1,m} = \mathfrak{H}_{n,m} \ominus (\ker D_{A^{n+1}} \cap \ker D_{A^{*m}}) = \\ &= \mathfrak{H}_{n,m} \cap \overline{\mathfrak{D}_{A^{n+1}} + \mathfrak{D}_{A^{*m}}} = \overline{\mathrm{ran}}\,(P_{n,m} D_{A^{n+1}}), \\ \mathfrak{D}_{A^*_{n,m}} &= \mathfrak{H}_{n,m} \ominus \mathfrak{H}_{n,m+1} = \mathfrak{H}_{n,m} \ominus (\ker D_{A^n} \cap \ker D_{A^{*m+1}}) = \\ &= \mathfrak{H}_{n,m} \cap \overline{\mathfrak{D}_{A^n} + \mathfrak{D}_{A^{*m+1}}} = \overline{\mathrm{ran}}\,(P_{n,m} D_{A^{*m+1}}), \end{aligned}$$

i.e., relations (3.5) are valid. Furthermore, if $n \geqslant 2$, then

$$f \in \mathfrak{H}_{n,m} \iff \begin{cases} Af \in \ker D_{A^{n-1}}, \\ A^*Af = f, \\ f \in \ker D_{A^{*m}} \text{ (for } m \geqslant 1) \end{cases} \iff Af \in \ker D_{A^{n-1}} \cap \ker D_{A^{*m+1}} = \mathfrak{H}_{n-1,m+1}.$$

If $n = 1$, then

$$f \in \mathfrak{H}_{1,m} \iff \begin{cases} A^*Af = f, \\ f \in \ker D_{A^{*m}} \end{cases} \iff Af \in \ker D_{A^{*m+1}} = \mathfrak{H}_{0,m+1}.$$

Similarly $A^*\mathfrak{H}_{n,m}=\mathfrak{H}_{n+1,m-1}$, $m\geqslant 1$. Therefore relations (3.6) hold true.

Let $\varphi\in\mathfrak{H}$, $\psi\in\mathfrak{H}_{n-1,m+1}$. Then $A^*\psi\in\mathfrak{H}_{n,m}$ and

$$(AP_{n,m}\varphi,\psi)=(P_{n,m}\varphi,A^*\psi)=(\varphi,A^*\psi)=(A\varphi,\psi)=(P_{n-1,m+1}A\varphi,\psi).$$

Hence

$$AP_{n,m}=P_{n-1,m+1}A. \tag{3.12}$$

Taking into account (3.6), we get

$$AP_{n,m}Ah=P_{n-1,m+1}AAh,\ h\in\mathfrak{H}_{n,m}.$$

This proves (3.10). Since A isometrically maps $\mathfrak{H}_{n,m}$ onto $\mathfrak{H}_{n-1,m+1}$ for $n\geqslant 1$, the operators $A_{n-1,m+1}$ and $A_{n,m}$ are unitarily equivalent, and therefore the operators

$$A_{n,0},\ A_{n-1,1},\ \ldots,A_{n-k,k},\ldots,A_{0,n}$$

are unitarily equivalent.

Note that (3.4) and (3.6) yield the equalities

$$\begin{aligned}\bigcap_{k\geqslant 1}\ker D_{A^k_{n,m}}&=\ker D_{A^{*m}}\bigcap\left(\bigcap_{j\geqslant 1}\ker D_{A^j}\right)=A^m\left(\bigcap_{j\geqslant 1}\ker D_{A^j}\right),\\ \bigcap_{k\geqslant 1}\ker D_{A^{*k}_{n,m}}&=\ker D_{A^n}\bigcap\left(\bigcap_{j\geqslant 1}\ker D_{A^{*j}}\right)=A^{*n}\left(\bigcap_{j\geqslant 1}\ker D_{A^{*j}}\right).\end{aligned} \tag{3.13}$$

Since A is a completely non-unitary contraction, we get

$$\left(\bigcap_{k\geqslant 1}\ker D_{A^k_{n,m}}\right)\bigcap\left(\bigcap_{k\geqslant 1}\ker D_{A^{*k}_{n,m}}\right)=\{0\}.$$

It follows that the contractions $A_{n,m}$ are completely non-unitary.

Note that $\mathfrak{H}_{n-1,m+1}\subset\mathfrak{H}_{n-1,m}$ and $\mathfrak{H}_{n+1,m}\subset\mathfrak{H}_{n,m}$. Using (3.6) we get

$$\begin{aligned}&A_{n-1,m+1}P_{n,m+1}=P_{n-1,m+1}AP_{n,m+1}=AP_{n,m+1},\\ &A_{n,m}P_{n+1,m}=P_{n,m}AP_{n+1,m}=AP_{n+1,m}.\end{aligned}$$

In particular, it follows that the operators $A_{n,m}P_{n+1,m}$ are partial isometries. From (3.12) we obtain

$$AP_{n,m+1}A=A^2P_{n+1,m},$$

i.e.,

$$A_{n-1,m+1}P_{n,m+1}Af=AA_{n,m}P_{n+1,m}f\quad\text{for all}\quad f\in\mathfrak{H}_{n,m}.$$

Because A unitarily maps $\mathfrak{H}_{n,m}$ onto $\mathfrak{H}_{n-1,m+1}$, we get (3.11) and hence, the operators $\mathscr{A}_{n-1,m+1}$ and $\mathscr{A}_{n,m}$ are unitarily equivalent.

By induction it can be easily proved that for every $k\in\mathbb{N}$ holds the equality

$$\mathscr{A}^k_{n,m}f=(AP_{n+1,m})^kf=AA^{k-1}_{n+1,m}P_{n+1,m}f,\ f\in\mathfrak{H}_{n,m}. \tag{3.14}$$

Since $A\upharpoonright\mathfrak{H}_{n+1,m}$ is isometric, relations (3.14) imply

$$||\mathscr{A}_{n,m}^k f|| = ||A_{n+1,m}^{k-1}P_{n+1,m}f||,\ f\in\mathfrak{H}_{n,m},\ k\in\mathbb{N}.$$

It follows the equivalence of the statements (a) and (b), and

$$\ker D_{\mathscr{A}_{n,m}^k} = \ker D_{A_{n+1,m}^{k-1}} = \mathfrak{H}_{n+k,m}.$$

Similarly, since $(A_{n,m}P_{n+1,m})^* = A_{n,m}^*P_{n,m+1}$, we get

$$\ker D_{\mathscr{A}_{n,m}^{*k}} = \ker D_{A_{n,m+1}^{*k-1}} = \mathfrak{H}_{n,m+k}.$$

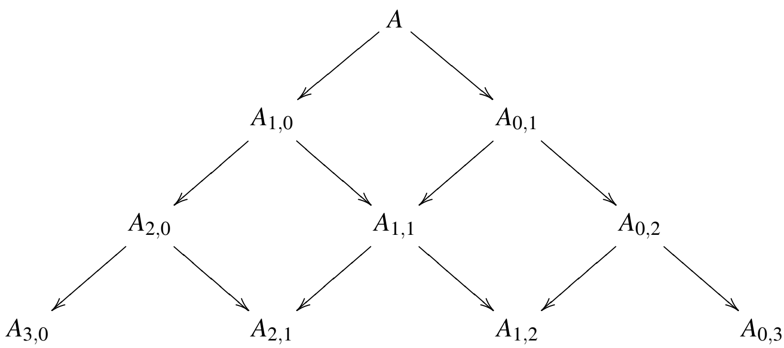
Thus, relations (3.7) are valid. Now we get that the operators $\mathscr{A}_{n,m} = A_{n,m}P_{n+1,m}$ are completely non-unitary. From (3.4) it follows that

$$\begin{aligned}&\ker D_{A_{n,m}^k}\cap\ker D_{A_{n,m}^{*l}} = \mathfrak{H}_{n+k,m}\cap\mathfrak{H}_{n,m+l} =\\ &\ker D_{A^{n+k}}\cap\ker D_{A^{*m}}\cap\ker D_{A^n}\cap\ker D_{A^{*m+l}} = \ker D_{A^{n+k}}\cap\ker D_{A^{*m+l}} = \mathfrak{H}_{n+k,m+l}.\end{aligned}$$

Hence

$$(A_{n,m})_{k,l} = P_{n+k,m+l}P_{n,m}A\upharpoonright\mathfrak{H}_{n+k,m+l} = A_{n+k,m+l}.$$

The relation (3.9) yields the following picture for the creation of the operators $A_{n,m}$:



… … … … … … …

The process terminates on the N-th step if and only if

$$\begin{aligned}&\ker D_{A^N} = \{0\} \iff \ker D_{A^{N-1}}\cap\ker D_{A^*} = \{0\} \iff \dots\\ &\iff \ker D_{A^{N-k}}\cap\ker D_{A^{*k}} = \{0\} \iff \dots\ker D_{A^{*N}} = \{0\}.\end{aligned}$$

Note that from (2.3), (3.7), and (3.13) we get

PROPOSITION 3.2. *Let A be a completely non-unitary contraction. If A does not contain a unilateral shift (respect., co-shift) then the same is true for the operators $\mathscr{A}_{n,m}$ and $A_{n,m}$ for all n and m. Conversely, if for some n and m the operator $\mathscr{A}_{n,m}$ or $A_{n,m}$ does not contain a unilateral shift (respect., co-shift) then the same is valid for the operator A.*

Let $\delta_A = \dim \mathfrak{D}_A$, $\delta_{A^*} = \dim \mathfrak{D}_{A^*}$ be the defect numbers of a completely non-unitary contraction A. For $n = 1, \dots$ denote by δ_n and δ_n^* the defect numbers of unitarily equivalent operators $\{A_{n-m,m}\}_{m=0}^n$. From the relations (3.5) it follows that

$$\begin{aligned}\delta_n &= \dim \mathfrak{D}_{A_{0,n}} = \dim\left(\overline{\mathrm{ran}}\,(P_{0,n}D_A)\right) = \dim\left(\mathfrak{D}_A \ominus (\mathfrak{D}_A \cap \mathfrak{D}_{A^{*n}})\right),\\ \delta_n^* &= \dim \mathfrak{D}_{A^*_{n,0}} = \dim\left(\overline{\mathrm{ran}}\,(P_{n,0}D_{A^*})\right) = \dim\left(\mathfrak{D}_{A^*} \ominus (\mathfrak{D}_{A^*} \cap \mathfrak{D}_{A^n})\right).\end{aligned}$$

Thus

$$\begin{aligned}&\delta_A \geqslant \delta_1 \geqslant \cdots \geqslant \delta_n \geqslant \cdots,\\ &\delta_{A^*} \geqslant \delta_1^* \geqslant \cdots \geqslant \delta_n^* \geqslant \cdots.\end{aligned}$$

Observe also that

$$\delta_1 = \dim\left(\mathfrak{D}_A \ominus (\mathfrak{D}_A \cap \mathfrak{D}_{A^*})\right),\ \delta_1^* = \dim\left(\mathfrak{D}_{A^*} \ominus (\mathfrak{D}_A \cap \mathfrak{D}_{A^*})\right),$$

and by induction

$$\delta_n = \dim\left(\mathfrak{D}_{A_{n-1,0}} \ominus (\mathfrak{D}_{A_{n-1,0}} \cap \mathfrak{D}_{A^*_{n-1,0}})\right),\ \delta_n^* = \dim\left(\mathfrak{D}_{A^*_{n-1,0}} \ominus (\mathfrak{D}_{A_{n-1,0}} \cap \mathfrak{D}_{A^*_{n-1,0}})\right).$$

4. Passive discrete time-invariant linear systems and their transfer functions

4.1. Basic definitions

Let $\mathfrak{M}, \mathfrak{N}$, and $\mathfrak{H}$ be separable Hilbert spaces. A linear system

$$\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\}$$

with bounded linear operators A, B, C, D of the form

$$\begin{cases} \sigma_k = Ch_k + D\xi_k, \\ h_{k+1} = Ah_k + B\xi_k, \end{cases} \qquad k \geqslant 0, \tag{4.1}$$

where $\{h_k\} \subset \mathfrak{H}$, $\{\xi_k\} \subset \mathfrak{M}$, $\{\sigma_k\} \subset \mathfrak{N}$, is called a *discrete time-invariant system*. The Hilbert spaces $\mathfrak{M}$ and $\mathfrak{N}$ are called the input and the output spaces, respectively, and the Hilbert space $\mathfrak{H}$ is called the state space. The operators A, B, C, and D are called the *state space operator*, the *control operator*, the *observation operator*, and the *feedthrough operator* of τ, respectively. If the linear operator T_τ defined by the block form

$$T_\tau = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{H} \end{matrix} \to \begin{matrix} \mathfrak{N} \\ \oplus \\ \mathfrak{H} \end{matrix} \tag{4.2}$$

is contractive, then the corresponding discrete time-invariant system is said to be *passive*. If the block operator matrix T_τ is isometric (respect., co-isometric, unitary), then the system is said to be *isometric* (*respect., co-isometric, conservative*). Isometric, co-isometric, conservative, and passive discrete time-invariant systems have been studied in [25], [26], [8], [47], [37], [27], [21], [7], [13], [14], [15], [16], [17], [18], [45], [46], [11], [10], [36].

The subspaces

$$\mathfrak{H}^c_\tau := \overline{\operatorname{span}}\{A^n B\mathfrak{M} : n = 0,1,\ldots\} \text{ and } \mathfrak{H}^o_\tau = \overline{\operatorname{span}}\{A^{*n}C^*\mathfrak{N} : n = 0,1,\ldots\} \qquad (4.3)$$

are said to be the *controllable* and *observable* subspaces of the system τ, respectively. The system τ is said to be *controllable* (respect., *observable*) if $\mathfrak{H}^c_\tau = \mathfrak{H}$ (respect., $\mathfrak{H}^o_\tau = \mathfrak{H}$), and it is called *minimal* if τ is both controllable and observable. The system τ is said to be *simple* if $\mathfrak{H} = \operatorname{clos}\{\mathfrak{H}^c_\tau + \mathfrak{H}^o_\tau\}$ (the closure of the span). It follows from (4.3) that

$$(\mathfrak{H}^c_\tau)^\perp = \bigcap_{n=0}^{\infty} \ker(B^*A^{*n}), \quad (\mathfrak{H}^o_\tau)^\perp = \bigcap_{n=0}^{\infty} \ker(CA^n), \qquad (4.4)$$

and therefore there are the following alternative characterizations:

(a) τ is controllable $\iff \bigcap_{n=0}^{\infty} \ker(B^*A^{*n}) = \{0\}$;

(b) τ is observable $\iff \bigcap_{n=0}^{\infty} \ker(CA^n) = \{0\}$;

(c) τ is simple $\iff \left(\bigcap_{n=0}^{\infty} \ker(B^*A^{*n})\right) \cap \left(\bigcap_{n=0}^{\infty} \ker(CA^n)\right) = \{0\}$.

The *transfer function*

$$\Theta_\tau(\lambda) := D + \lambda C(I_{\mathfrak{H}} - \lambda A)^{-1}B, \quad \lambda \in \mathbb{D}, \qquad (4.5)$$

of the passive system τ belongs to the Schur class $\mathbf{S}(\mathfrak{M},\mathfrak{N})$ [13]. Conservative systems are also called the *unitary colligations* and their transfer functions are called the characteristic functions [27].

The examples of conservative systems are given by

$$\Sigma = \left\{ \begin{bmatrix} -A & D_{A^*} \\ D_A & A^* \end{bmatrix}; \mathfrak{D}_A, \mathfrak{D}_{A^*}, \mathfrak{H} \right\}, \ \Sigma_* = \left\{ \begin{bmatrix} -A^* & D_A \\ D_{A^*} & A \end{bmatrix}; \mathfrak{D}_{A^*}, \mathfrak{D}_A, \mathfrak{H} \right\}.$$

The transfer functions of these systems

$$\Phi_\Sigma(\lambda) = \left(-A + \lambda D_{A^*}(I_{\mathfrak{H}} - \lambda A^*)^{-1}D_A\right) \upharpoonright \mathfrak{D}_A, \quad \lambda \in \mathbb{D}$$

and

$$\Phi_{\Sigma_*}(\lambda) = \left(-A^* + \lambda D_A(I_{\mathfrak{H}} - \lambda A)^{-1}D_{A^*}\right) \upharpoonright \mathfrak{D}_{A^*}, \quad \lambda \in \mathbb{D}$$

are precisely characteristic functions of A and A^*, correspondingly.

It is well known that every operator-valued function $\Theta(\lambda)$ from the Schur class $\mathbf{S}(\mathfrak{M},\mathfrak{N})$ can be realized as the transfer function of some passive system, which can be chosen as controllable isometric (respect., observable co-isometric, simple conservative, minimal passive); cf. [26], [47], [37], [21], [8] [13], [15], [7]. Moreover, two controllable isometric (respect., observable co-isometric, simple conservative) systems with the same transfer function are unitarily similar: two discrete time-invariant systems

$$\tau_1 = \left\{ \begin{bmatrix} D & C_1 \\ B_1 & A_1 \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}_1 \right\} \quad \text{and} \quad \tau_2 = \left\{ \begin{bmatrix} D & C_2 \\ B_2 & A_2 \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}_2 \right\}$$

are said to be *unitarily similar* if there exists a unitary operator U from $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ such that

$$A_1 = U^{-1}A_2U, \quad B_1 = U^{-1}B_2, \quad C_1 = C_2U;$$

cf. [25], [26], [37], [21], [8], [27], [7]. However, a result of D.Z. Arov [13] states that two minimal passive systems τ_1 and τ_2 with the same transfer function $\Theta(\lambda)$ are only *weakly similar*, i.e., there is a closed densely defined operator $Z : \mathfrak{H}_1 \to \mathfrak{H}_2$ such that Z is invertible, Z^{-1} is densely defined, and

$$ZA_1f = A_2Zf, \quad C_1f = C_2Zf, \quad f \in \operatorname{dom} Z, \quad \text{and} \quad ZB_1 = B_2.$$

4.2. Defect functions of the Schur class functions

The following result [47, Proposition V.4.2] is needed in the sequel.

THEOREM 4.1. *Let $\mathfrak{M}$ be a separable Hilbert space and let $N(\xi)$, $\xi \in \mathbb{T}$, be an $\mathbf{L}(\mathfrak{M})$-valued measurable function such that $0 \leqslant N(\xi) \leqslant I_{\mathfrak{M}}$. Then there exist a Hilbert space $\mathfrak{K}$ and an outer function $\varphi(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{K})$ satisfying the following conditions:*

(i) *$\varphi^*(\xi)\varphi(\xi) \leqslant N^2(\xi)$ a.e. on $\mathbb{T}$;*

(ii) *if $\widetilde{\mathfrak{K}}$ is a Hilbert space and $\widetilde{\varphi}(\lambda) \in \mathbf{S}(\mathfrak{M}, \widetilde{\mathfrak{K}})$ is such that $\widetilde{\varphi}^*(\xi)\widetilde{\varphi}(\xi) \leqslant N^2(\xi)$ a.e. on $\mathbb{T}$, then $\widetilde{\varphi}^*(\xi)\widetilde{\varphi}(\xi) \leqslant \varphi^*(\xi)\varphi(\xi)$ a.e. on $\mathbb{T}$.*

Moreover, the function $\varphi(\lambda)$ is uniquely defined up to a left constant unitary factor.

Assume that $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ and denote by $\varphi_\Theta(\xi)$ and $\psi_\Theta(\xi)$, $\xi \in \mathbb{T}$ the outer functions which are solutions of the factorization problem described in Theorem 4.1 for $N^2(\xi) = I_{\mathfrak{M}} - \Theta^*(\xi)\Theta(\xi)$ and $N^2(\bar{\xi}) = I_{\mathfrak{N}} - \Theta(\bar{\xi})\Theta^*(\bar{\xi})$, respectively. Clearly, if $\Theta(\lambda)$ is inner or co-inner, then $\varphi_\Theta = 0$ or $\psi_\Theta = 0$, respectively. The functions $\varphi_\Theta(\lambda)$ and $\psi_\Theta(\lambda)$ are called the right and left *defect functions* (or the *spectral factors*), respectively, associated with $\Theta(\lambda)$; cf. [20], [22], [23], [24], [35]. The following result has been established in [35, Theorem 1.1, Corollary 1] (see also [23, Theorem 3], [24, Theorem 1.5]).

THEOREM 4.2. *Let $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ and let*

$$\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\}$$

be a simple conservative system with transfer function Θ. Then

1. *the functions* $\varphi_\Theta(\lambda)$ *and* $\psi_\Theta(\lambda)$ *take the form*

$$\begin{aligned}\varphi_\Theta(\lambda) &= P_\Omega(I_{\mathfrak{H}} - \lambda A)^{-1}B,\\ \psi_\Theta(\lambda) &= C(I_{\mathfrak{H}} - \lambda A)^{-1}\restriction\Omega_*,\end{aligned}$$

where

$$\Omega = (\mathfrak{H}^o_\tau)^\perp \ominus A(\mathfrak{H}^o_\tau)^\perp,\ \Omega_* = (\mathfrak{H}^c_\tau)^\perp \ominus A^*(\mathfrak{H}^c_\tau)^\perp$$

and P_Ω *is the orthogonal projector from* $\mathfrak{H}$ *onto* Ω*;*

2. $\varphi_\Theta(\lambda) = 0$ *(* $\psi_\Theta(\lambda) = 0$ *) if and only if the system* τ *is observable (controllable).*

The defect functions play an essential role in the problems of the system theory, in particular, in the problem of similarity and unitary similarity of the minimal passive systems with equal transfer functions [16], [17] and in the problem of *optimal* and $(*)$ *optimal* realizations of the Schur function [14], [15].

4.3. Parametrization of contractive block-operator matrices

Let $\mathfrak{H}$, $\mathfrak{K}$, $\mathfrak{M}$ and $\mathfrak{N}$ be Hilbert spaces. The following theorem goes back to [19], [31], [43]; other proofs of the theorem can be found in [38], [39], [9], [11].

THEOREM 4.3. *Let* $A \in \mathbf{L}(\mathfrak{H},\mathfrak{K})$, $B \in \mathbf{L}(\mathfrak{M},\mathfrak{K})$, $C \in \mathbf{L}(\mathfrak{H},\mathfrak{N})$, *and* $D \in \mathbf{L}(\mathfrak{M},\mathfrak{N})$. *The operator matrix*

$$T = \begin{bmatrix} D & C\\ B & A\end{bmatrix} : \begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{H}\end{matrix} \to \begin{matrix}\mathfrak{N}\\ \oplus\\ \mathfrak{K}\end{matrix}$$

is a contraction if and only if T *is of the form*

$$T = \begin{bmatrix} -KA^*M + D_{K^*}XD_M & KD_A\\ D_{A^*}M & A\end{bmatrix}, \tag{4.6}$$

where $A \in \mathbf{L}(\mathfrak{H},\mathfrak{K})$, $M \in \mathbf{L}(\mathfrak{M},\mathfrak{D}_{A^*})$, $K \in \mathbf{L}(\mathfrak{D}_A,\mathfrak{N})$, *and* $X \in \mathbf{L}(\mathfrak{D}_M,\mathfrak{D}_{K^*})$ *are contractions, all uniquely determined by* T. *Furthermore, the following equality holds for all* $h \in \mathfrak{M}$, $f \in \mathfrak{H}$*:*

$$\begin{aligned}&\left\|\begin{bmatrix} h\\ f\end{bmatrix}\right\|^2 - \left\|\begin{bmatrix} -KA^*M + D_{K^*}XD_M & KD_A\\ D_{A^*}M & A\end{bmatrix}\begin{bmatrix} h\\ f\end{bmatrix}\right\|^2\\ &= \|D_K(D_Af - A^*Mh) - K^*XD_Mh\|^2 + \|D_XD_Mh\|^2.\end{aligned} \tag{4.7}$$

COROLLARY 4.4. *Let*

$$T = \begin{bmatrix} -KA^*M + D_{K^*}XD_M & KD_A\\ D_{A^*}M & A\end{bmatrix} : \begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{H}\end{matrix} \to \begin{matrix}\mathfrak{N}\\ \oplus\\ \mathfrak{K}\end{matrix}$$

be a contraction. Then

1. T is isometric if and only if

$$D_K D_A = 0,\ D_X D_M = 0,$$

2. T is co-isometric if and only if

$$D_{M^*} D_{A^*} = 0,\ D_{X^*} D_{K^*} = 0.$$

If T given by (4.6) *is unitary, then* $D_{K^*} = 0 \iff D_M = 0$.

Note that the relation $D_Y D_Z = 0$ for contractions Y and Z means that either Z is an isometry and $Y = 0$ or $\mathfrak{D}_Z \neq \{0\}$ and Y is an isometry.

Let $\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\}$ be a conservative system. Then from Corollary 4.4 we get

$$\begin{aligned} (\mathfrak{H}_\tau^c)^\perp &= \bigcap_{n\geqslant 0} \ker(D_{A^*} A^{*n}) = \bigcap_{n\geqslant 1} \ker(D_{A^{*n}}), \\ (\mathfrak{H}_\tau^o)^\perp &= \bigcap_{n\geqslant 0} \ker(D_A A^n) = \bigcap_{n\geqslant 1} \ker(D_{A^n}), \end{aligned} \tag{4.8}$$

$$\begin{aligned} &\tau \text{ is controllable} \iff \bigcap_{n\geqslant 1} \ker(D_{A^{*n}}) = \{0\} \iff \\ &\iff \text{the operator } A \text{ is completely non-co-isometric} \iff \\ &\iff \text{the operator } A^* \text{ does not contain a shift,} \\ &\tau \text{ is observable} \iff \bigcap_{n\geqslant 1} \ker(D_{A^n}) = \{0\} \iff \\ &\iff \text{the operator } A \text{ is completely non-isometric} \iff \\ &\iff \text{the operator } A \text{ does not contain a shift,} \\ &\tau \text{ is simple} \iff \text{the operator } A \text{ is completely non-unitary.} \end{aligned}$$

In [11] we used Theorem 4.3 for connections between transfer function $\Theta_\tau(\lambda)$ of the passive system

$$\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\},$$

and the characteristic function of A. In particular, an immediate consequence of (4.6) is the following relation

$$\Theta_\tau(\lambda) = K\Phi_{A^*}(\lambda)M + D_{K^*} X D_M, \quad \lambda \in \mathbb{D}, \tag{4.9}$$

where $\Phi_{A^*}(\lambda)$ is the characteristic function of A^*.

Recall that if $\Theta(\lambda) \in \mathbf{S}(\mathfrak{H}_1, \mathfrak{H}_2)$, then there is a uniquely determined decomposition [47, Proposition V.2.1]

$$\Theta(\lambda) = \begin{bmatrix} \Theta_p(\lambda) & 0 \\ 0 & \Theta_u \end{bmatrix} : \begin{matrix} \mathfrak{D}_{\Theta(0)} \\ \oplus \\ \ker D_{\Theta(0)} \end{matrix} \to \begin{matrix} \mathfrak{D}_{\Theta^*(0)} \\ \oplus \\ \ker D_{\Theta^*(0)} \end{matrix},$$

where $\Theta_p(\lambda) \in \mathbf{S}(\mathfrak{D}_{\Theta(0)}, \mathfrak{D}_{\Theta^*(0)})$, $\Theta_p(0)$ is a pure contraction and Θ_u is a unitary constant. The function $\Theta_p(\lambda)$ is called the *pure part* of $\Theta(\lambda)$ (see [20]). If $\Theta(0)$ is isometric (respect., co-isometric), then the pure part is of the form $\Theta_p(\lambda) = 0 \in \mathbf{S}(\{0\}, \mathfrak{D}_{\Theta^*(0)})$ (respect., $\Theta_p(\lambda) = 0 \in \mathbf{S}(\mathfrak{D}_{\Theta(0)}, \{0\})$).

From (4.6) and (4.9) we get the following statement.

PROPOSITION 4.5. *Let*

$$\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\}$$

be a a simple conservative system and let $\Theta(\lambda)$ *be its transfer function. Then*

$$\begin{aligned} \dim \mathfrak{D}_A &= \dim \mathfrak{D}_{\Theta^*(0)} = \dim(\mathfrak{N} \ominus \ker C^*), \\ \dim \mathfrak{D}_{A^*} &= \dim \mathfrak{D}_{\Theta(0)} = \dim(\mathfrak{M} \ominus \ker B), \end{aligned} \tag{4.10}$$

and the pure part of Θ *coincides with the Sz.-Nagy–Foias characteristic function of* A^*.

In addition

1) if $\Theta(0)$ *is isometric then* $B = 0$, A *is a co-shift of multiplicity* $\dim \mathfrak{D}_{\Theta^*(0)}$, *and the system* τ *is observable;*

2) if $\Theta(0)$ *is co-isometric then* $C = 0$, A *is a unilateral shift of multiplicity* $\dim \mathfrak{D}_{\Theta(0)}$, *and the system* τ *is controllable.*

Proof. According to Theorem 4.3 the operator

$$T = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{H} \end{matrix} \to \begin{matrix} \mathfrak{N} \\ \oplus \\ \mathfrak{H} \end{matrix}$$

takes the form (4.6). Since T is unitary, from (4.12) we get that the operators $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$ and $M^* \in \mathbf{L}(\mathfrak{D}_{A^*}, \mathfrak{M})$ are isometries and the operator $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$ is unitary. From (4.9) it follows that the pure part of Θ is given by

$$\Theta(\lambda) \upharpoonright \operatorname{ran} M^* = K\Phi_{A^*}(\lambda) M \upharpoonright \operatorname{ran} M^* : \operatorname{ran} M^* \to \operatorname{ran} K.$$

Thus, the pure part of Θ coincides with Φ_{A^*}. Since $\operatorname{ran} M^* = \mathfrak{D}_{A^*}$, $\operatorname{ran} K^* = \mathfrak{D}_A$,

$$\begin{aligned} &D = \Theta(0) = K\Phi_{A^*}(0)M^* = -KA^*M^*, \; D^* = \Theta^*(0) = -MAK^*, \\ &\operatorname{ran} K = \mathfrak{N} \ominus \ker K^* = \mathfrak{N} \ominus \ker C^*, \\ &\operatorname{ran} M^* = \mathfrak{M} \ominus \ker M = \mathfrak{M} \ominus \ker B, \end{aligned}$$

we get (4.10).

Suppose $D = \Theta(0)$ is an isometry. Then the pure part of Θ is $0 \in \mathbf{S}(\{0\}, \mathfrak{D}_{D^*})$. It follows that $M = B = 0$ and $\mathfrak{D}_{A^*} = \{0\}$. Hence, A is co-isometric and since A is a completely non-unitary contraction, it is a co-shift of multiplicity $\dim \mathfrak{D}_A = \dim \mathfrak{D}_{\Theta^*(0)}$, and the system τ is observable. Similarly the statement 2) holds.

In this paper we will use a parametrization of a contractive block- operator matrix based on a fixed upper left block $D \in \mathbf{L}(\mathfrak{M}, \mathfrak{N})$. With this aim we reformulate Theorem 4.3 and Corollary 4.4.

THEOREM 4.6. *The operator matrix*

$$T=\begin{bmatrix}D & C\\ B & A\end{bmatrix}:\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{H}\end{matrix}\to\begin{matrix}\mathfrak{N}\\ \oplus\\ \mathfrak{K}\end{matrix}$$

is a contraction if and only if $D\in\mathbf{L}(\mathfrak{M},\mathfrak{N})$ *is a contraction and the entries* A,B, *and* C *take the form*

$$\begin{aligned}&B=FD_D,\ C=D_{D^*}G,\\ &A=-FD^*G+D_{F^*}LD_G,\end{aligned}\tag{4.11}$$

where the operators $F\in\mathbf{L}(\mathfrak{D}_D,\mathfrak{K})$, $G\in\mathbf{L}(\mathfrak{H},\mathfrak{D}_{D^*})$ *and* $L\in\mathbf{L}(\mathfrak{D}_G,\mathfrak{D}_{F^*})$ *are contractions. Moreover, operators* F,G, *and* L *are uniquely determined. Furthermore,the following equality holds*

$$\begin{aligned}&\left\|D_T\begin{bmatrix}h\\ f\end{bmatrix}\right\|^2=||D_F\left(D_Dh-D^*Gf\right)-F^*LD_Gf||^2+||D_LD_Gf||^2,\\ &h\in\mathfrak{M},\ f\in\mathfrak{H}\end{aligned}\tag{4.12}$$

and

$$\begin{aligned}&\left\|D_{T^*}\begin{bmatrix}\varphi\\ g\end{bmatrix}\right\|^2=||D_{G^*}\left(D_{D^*}\varphi-DF^*g\right)-GL^*D_{F^*}g||^2+||D_{L^*}D_{F^*}g||^2,\\ &\varphi\in\mathfrak{N},\ g\in\mathfrak{K}.\end{aligned}\tag{4.13}$$

1. *the operator* T *is isometric if and only if*

$$D_FD_D=0,\ D_LD_G=0,$$

2. *the operator* T *is co-isometric if and only if*

$$D_{G^*}D_{D^*}=0,\ D_{L^*}D_{F^*}=0,$$

3. *if* T *is unitary, then* $D_{F^*}=0\iff D_G=0$.

Let us give connections between the parametrization of a unitary block-operator matrix T given by (4.6) and (4.11).

PROPOSITION 4.7. *Let*

$$\begin{aligned}T&=\begin{bmatrix}-KA^*M+D_{K^*}XD_M & KD_A\\ D_{A^*}M & A\end{bmatrix}=\\ &=\begin{bmatrix}D & D_{D^*}G\\ FD_D & -FD^*G+D_{F^*}LD_G\end{bmatrix}:\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{H}\end{matrix}\to\begin{matrix}\mathfrak{N}\\ \oplus\\ \mathfrak{H}\end{matrix}\end{aligned}$$

be a unitary operator matrix. Then

$$\mathfrak{D}_D=\operatorname{ran}M^*,\ \mathfrak{D}_{D^*}=\operatorname{ran}K,$$

$$F^* = M^* P_{\mathfrak{D}_{A^*}},\ G = KP_{\mathfrak{D}_A},$$
$$GFf = KP_{\mathfrak{D}_A} Mf,\ f \in \mathfrak{D}_D,$$
$$L = A \upharpoonright \ker D_A.$$

Proof. Since $FD_D = D_{A^*}M$, $KD_A = D_{D^*}G$, and by Corollary 4.4 the operators F, G^*, K, and M^* are isometries, we get $D_D^2 = M^* D_{A^*}^2 M$, $D_{D^*}^2 = KD_A^2 K^*$, and

$$D_D = M^* D_{A^*} M,\ D_{D^*} = KD_A K^*.$$

It follows that $\mathfrak{D}_D = \operatorname{ran} M^*$, $\mathfrak{D}_{D^*} = \operatorname{ran} K$, $D_{A^*}M = FM^* D_{A^*} M$, and $D_A K^* = G^* K D_A K^*$. Therefore,

$$FM^* = I_{\mathfrak{D}_{A^*}},\ G^* K = I_{\mathfrak{D}_A}.$$

It follows

$$F = M \upharpoonright \mathfrak{D}_D,\ G^* = K^* \upharpoonright \mathfrak{D}_{D^*}.$$

Hence, $F^* = M^* P_{\mathfrak{D}_{A^*}}$ and $G = KP_{\mathfrak{D}_A}$. In addition

$$D_{F^*}^2 = I_{\mathfrak{H}} - MM^* P_{\mathfrak{D}_{A^*}} = P_{\ker D_{A^*}},\ D_G^2 = I_{\mathfrak{H}} - K^* K P_{\mathfrak{D}_A} = P_{\ker D_A},$$

$$\begin{aligned} -FD^*G &= -F(-M^*AK^* + D_M X^* D_{K^*})KP_{\mathfrak{D}_A} = AP_{\mathfrak{D}_A}, \\ A &= -FD^*G + D_{F^*} L D_G = AP_{\mathfrak{D}_A} + P_{\ker D_{A^*}} L P_{\ker D_A}. \end{aligned}$$

On the other hand

$$A = AP_{\mathfrak{D}_A} + AP_{\ker D_A}.$$

Hence $L = A \upharpoonright \ker D_A$.

Let $D : \mathfrak{M} \to \mathfrak{N}$ be a contraction with nonzero defect operators and let

$$Q = \begin{bmatrix} 0 & G \\ F & S \end{bmatrix} : \begin{matrix} \mathfrak{D}_D \\ \oplus \\ \mathfrak{H} \end{matrix} \to \begin{matrix} \mathfrak{D}_{D^*} \\ \oplus \\ \mathfrak{K} \end{matrix}$$

be a bounded operator. Define the transformation (see[10])

$$\mathscr{M}_D(Q) = \begin{bmatrix} D & 0 \\ 0 & -FD^*G \end{bmatrix} + \begin{bmatrix} D_{D^*} & 0 \\ 0 & I_{\mathfrak{K}} \end{bmatrix} \begin{bmatrix} 0 & G \\ F & S \end{bmatrix} \begin{bmatrix} D_D & 0 \\ 0 & I_{\mathfrak{H}} \end{bmatrix}. \tag{4.14}$$

Clearly, the operator $T = \mathscr{M}_D(Q)$ has the following matrix form

$$T = \begin{bmatrix} D & D_{D^*}G \\ FD_D & S - FD^*G \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{H} \end{matrix} \to \begin{matrix} \mathfrak{N} \\ \oplus \\ \mathfrak{K} \end{matrix}.$$

PROPOSITION 4.8. [10]. *Let $\mathfrak{H}, \mathfrak{M}, \mathfrak{N}$ be separable Hilbert spaces and let $D : \mathfrak{M} \to \mathfrak{N}$ be a contraction with nonzero defect operators. Let $Q = \begin{bmatrix} 0 & G \\ F & S \end{bmatrix} : \begin{matrix} \mathfrak{D}_D \\ \oplus \\ \mathfrak{H} \end{matrix} \to \begin{matrix} \mathfrak{D}_{D^*} \\ \oplus \\ \mathfrak{H} \end{matrix}$ be a bounded operator. Then*

1.

$$T = \mathscr{M}_D(Q) = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \begin{matrix} \mathfrak{D}_D \\ \oplus \\ \mathfrak{H} \end{matrix} \to \begin{matrix} \mathfrak{D}_{D^*} \\ \oplus \\ \mathfrak{H} \end{matrix}$$

is a contraction if and only if Q is a contraction. T is isometric (respect., co-isometric) if and only if Q is isometric (respect., co-isometric);

2. the relations

$$\begin{cases} \bigcap\limits_{n=0}^{\infty} \ker(B^*A^{*n}) = \bigcap\limits_{n=0}^{\infty} \ker(F^*S^{*n}), \\ \bigcap\limits_{n=0}^{\infty} \ker(CA^n) = \bigcap\limits_{n=0}^{\infty} \ker(GS^n) \end{cases} \tag{4.15}$$

hold.

5. The Möbius representations

Let $T : \mathfrak{H}_1 \to \mathfrak{H}_2$ be a contraction. In [44] and [41] were studied the fractional-linear transformations of the form

$$Z \to Q = T + D_{T^*}Z(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_T = T + D_{T^*}(I_{\mathfrak{D}_{T^*}} + ZT^*)^{-1}ZD_T \tag{5.1}$$

defined on the set $\mathscr{V}_{T^*}$ of all contractions $Z \in \mathbf{L}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ such that $-1 \in \rho(T^*Z)$. The following result holds.

THEOREM 5.1. [41]. *Let the $T \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ be a contraction and let $Z \in \mathscr{V}_{T^*}$. Then $Q = T + D_{T^*}Z(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_T$ is a contraction,*

$$||D_Q f||^2 = ||D_Z(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_T f||, \; f \in \mathfrak{H}_1, \tag{5.2}$$

$\operatorname{ran} D_Q \subseteq \operatorname{ran} D_T$*, and* $\operatorname{ran} D_Q = \operatorname{ran} D_T$ *if and only if* $||Z|| < 1$. *Moreover, if $Q \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ is a contraction and $Q = T + D_{T^*}XD_T$, where $X \in \mathbf{L}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ then $X \in \mathscr{V}_{T^*}$,*

$$Z = X(I_{\mathfrak{D}_T} - T^*X)^{-1} \in \mathscr{V}_{T^*},$$

and the operator Q takes the form $Q = T + D_{T^*}Z(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_T$.

Observe that from (5.1) one can derive the equalities

$$\begin{aligned} I_{\mathfrak{H}_2} - QT^* &= D_{T^*}(I_{\mathfrak{D}_{T^*}} + ZT^*)^{-1}D_{T^*}, \\ Z{\upharpoonright}\operatorname{ran} D_T &= D_{T^*}(I_{\mathfrak{H}_2} - QT^*)^{-1}(Q-T)D_T^{-1}. \end{aligned}$$

The transformation (5.1) is called in [41] the unitary linear-fractional transformation. It is easy to see that if $||T|| < 1$ then the closed unit operator ball in $\mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ belongs to the set $\mathscr{V}_{T^*}$ and, moreover

$$\begin{aligned} T + D_{T^*}Z(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_T &= D_{T^*}^{-1}(Z+T)(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_T = \\ &= D_{T^*}(I_{\mathfrak{D}_{T^*}} + ZT^*)^{-1}(Z+T)D_T^{-1} \end{aligned}$$

for all $Z \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$, $||Z|| \leqslant 1$. Thus, the transformation (5.1) is an operator analog of a well known Möbius transformation of the complex plane

$$z \to \frac{z+t}{1+\bar{t}z}, \ |t| \leqslant 1.$$

The next theorem is a version of a more general result established by Yu.L. Shmul'yan in [42].

THEOREM 5.2. [42]. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be Hilbert spaces and let the function $\Theta(\lambda)$ be from the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$. Then*

1. *the linear manifolds $\operatorname{ran} D_{\Theta(\lambda)}$ and $\operatorname{ran} D_{\Theta^*(\lambda)}$ do not depend on $\lambda \in \mathbb{D}$,*
2. *for arbitrary λ_1, λ_2, λ_3 in $\mathbb{D}$ the function $\Theta(\lambda)$ admits the representation*

$$\Theta(\lambda) = \Theta(\lambda_1) + D_{\Theta^*(\lambda_2)}\Psi(\lambda)D_{\Theta(\lambda_3)},$$

where $\Psi(\lambda)$ is a holomorphic in $\mathbb{D}$ and $\mathbf{L}\left(\mathfrak{D}_{\Theta(\lambda_3)}, \mathfrak{D}_{\Theta^*(\lambda_2)}\right)$-valued function.

Now using Theorems 5.1 and 5.2 we get Theorem 1.1. Recall that the representation (1.2) of a function $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is called the Möbius representation of Θ and the function $Z(\lambda) \in \mathbf{S}(\mathfrak{D}_{\Theta(0)}, \mathfrak{D}_{\Theta^*(0)})$ is called the Möbius parameter of Θ.

The next result established in [10] provides connections between the realizations of $\Theta(\lambda)$ and $Z(\lambda)$ as transfer functions of passive systems.

THEOREM 5.3. [10].

1. *Let $\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\}$ be a passive system and let*

$$T = \begin{bmatrix} D & C \\ B & A \end{bmatrix} = \begin{bmatrix} D & D_{D^*}G \\ FD_D & -FD^*G + D_{F^*}LD_G \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{H} \end{matrix} \to \begin{matrix} \mathfrak{N} \\ \oplus \\ \mathfrak{H} \end{matrix}.$$

Let $\Theta(\lambda)$ be the transfer function of τ. Then

   (a) *the Möbius parameter $Z(\lambda)$ of the function $\Theta(\lambda)$ is the transfer function of the passive system*

$$\nu = \left\{ \begin{bmatrix} 0 & G \\ F & D_{F^*}LD_G \end{bmatrix}; \mathfrak{D}_D, \mathfrak{D}_{D^*}, \mathfrak{H} \right\};$$

   (b) *the system τ isometric (respect., co-isometric) $\Rightarrow$ the system ν isometric (respect., co-isometric);*

   (c) *the equalities $\mathfrak{H}^c_\nu = \mathfrak{H}^c_\tau$, $\mathfrak{H}^o_\nu = \mathfrak{H}^o_\tau$ hold and hence the system τ is controllable (respect., observable) $\Rightarrow$ the system ν is controllable (respect., observable), the system τ is simple (respect., minimal) $\Rightarrow$ the system ν is simple (respect., minimal).*

2. *Let* $\Theta(\lambda)\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$ *and let* $Z(\lambda)$ *be the Möbius parameter of* $\Theta(\lambda)$. *Suppose that the transfer function of the linear system*

$$\nu'=\left\{\begin{bmatrix}0 & G\\ F & S\end{bmatrix};\mathfrak{D}_{\Theta(0)},\mathfrak{D}_{\Theta^*(0)},\mathfrak{H}\right\}$$

coincides with $Z(\lambda)$ *in a neighborhood of the origin. Then the transfer function of the linear system*

$$\tau'=\left\{\begin{bmatrix}\Theta(0) & D_{\Theta^*(0)}G\\ FD_{\Theta(0)} & -F\Theta^*(0)G+S\end{bmatrix};\mathfrak{M},\mathfrak{N},\mathfrak{H}\right\}$$

coincides with $\Theta(\lambda)$ *in a neighborhood of the origin. Moreover*

(a) *the equalities* $\mathfrak{H}^c_{\tau'}=\mathfrak{H}^c_{\nu'}$, $\mathfrak{H}^o_{\tau'}=\mathfrak{H}^o_{\nu'}$ *hold, and hence the system* ν' *is controllable (respect., observable)* $\Rightarrow$ *the system* τ' *is controllable (respect., observable), the system* ν' *is simple* $\Rightarrow$ *the system* τ' *is simple (respect., minimal),*

(b) *the system* ν' *is passive* $\Rightarrow$ *the system* τ' *is passive (respect., minimal),*

(c) *the system* ν' *isometric (respect., co-isometric)* $\Rightarrow$ *the system* τ' *isometric (respect., co-isometric).*

COROLLARY 5.4. *1) The equivalences*

$$\varphi_\Theta(\lambda)=0\iff\varphi_Z(\lambda)=0,$$
$$\psi_\Theta(\lambda)=0\iff\psi_Z(\lambda)=0$$

hold.

2) Let $||\Theta(0)\upharpoonright\mathfrak{D}_{\Theta(0)}||<1$. *Suppose* $\varphi(\lambda)\in\mathbf{S}(\mathfrak{M},\mathfrak{L})$ ($\psi(\lambda)\in\mathbf{S}(\mathfrak{K},\mathfrak{N})$) *and*

$$\varphi^*(\xi)\varphi(\xi)=D^2_{\Theta(\xi)}\quad\text{for almost all}\quad\xi\in\mathbb{T}$$
$$\left(\psi(\xi)\psi^*(\xi)=D^2_{\Theta^*(\xi)}\quad\text{for almost all}\quad\xi\in\mathbb{T}\right).$$

Then

$$\widetilde{\varphi}(\lambda):=\varphi(\lambda)D^{-1}_{\Theta(0)}(I_{\mathfrak{D}_{\Theta(0)}}+\Theta^*(0)Z(\lambda))\in\mathbf{S}(\mathfrak{D}_{\Theta(0)},\mathfrak{L})$$
$$\left(\widetilde{\psi}(\lambda):=(I_{\mathfrak{D}_{\Theta^*(0)}}+Z(\lambda)\Theta^*(0))D^{-1}_{\Theta^*(0)}P_{\mathfrak{D}_{\Theta^*(0)}}\psi(\lambda)\in\mathbf{S}(\mathfrak{K},\mathfrak{D}_{\Theta^*(0)})\right)$$

and

$$\widetilde{\varphi}^*(\xi)\widetilde{\varphi}(\xi)=D^2_{Z(\xi)}\quad\text{for almost all}\quad\xi\in\mathbb{T}$$
$$\left(\widetilde{\psi}(\xi)\widetilde{\psi}^*(\xi)=D^2_{Z^*(\xi)}\quad\text{for almost all}\quad\xi\in\mathbb{T}\right).$$

In particular,

$$\Theta(\lambda)\quad\text{is inner (respect., co-inner)}\iff Z(\lambda)\quad\text{is inner (respect., co-inner).}$$

Proof. 1) Let $\varphi_\Theta(\lambda)=0$ ($\psi_\Theta(\lambda)=0$) and let $\tau=\left\{\begin{bmatrix} D & C\\ B & A\end{bmatrix};\mathfrak{M},\mathfrak{N},\mathfrak{H}\right\}$ be a simple conservative system with transfer function $\Theta(\lambda)$. By Theorem 4.2 the system τ is observable (controllable). As it is proved above the corresponding system ν with transfer function $Z(\lambda)$ is conservative and observable (controllable). Theorem 4.2 yields that $\varphi_Z(\lambda)=0$ ($\psi_Z(\lambda)=0$).

Conversely. Let $\varphi_Z(\lambda)=0$ ($\psi_Z(\lambda)=0$) and let ν' be a simple conservative system with transfer function $Z(\lambda)$. Again by Theorem 4.2 the system ν' is observable (controllable). As it is already proved the corresponding system τ' with transfer function $\Theta(\lambda)$ is conservative and observable (controllable) as well. Now Theorem 4.2 yields that $\varphi_\Theta(\lambda)=0$ ($\psi_\Theta(\lambda)=0$).

2) Let $||\Theta(0)\upharpoonright\mathfrak{D}_{\Theta(0)}||<1$. Since

$$\Theta^*(0)\upharpoonright\mathfrak{D}_{\Theta^*(0)}=\left(\Theta(0)\upharpoonright\mathfrak{D}_{\Theta(0)}\right)^*,$$

we get $||\Theta^*(0)\upharpoonright\mathfrak{D}_{\Theta^*(0)}||<1$. It follows that the operators $D_{\Theta(0)}\upharpoonright\mathfrak{D}_{\Theta(0)}$ and $D_{\Theta^*(0)}\upharpoonright\mathfrak{D}_{\Theta^*(0)}$ have bounded inverses. From (5.2) we obtain the relation

$$||D_{\Theta(\lambda)}D^{-1}_{\Theta(0)}(I_{\mathfrak{D}_{\Theta(0)}}+\Theta^*(0)Z(\lambda))f||^2=||D_{Z(\lambda)}f||^2,\ \lambda\in\mathbb{D},\ f\in\mathfrak{D}_{\Theta(0)}.$$

The non-tangential limits $\Theta(\xi)$ and $Z(\xi)$ exist for almost all $\xi\in\mathbb{T}$. It follows the relation

$$||D_{\Theta(\xi)}D^{-1}_{\Theta(0)}(I_{\mathfrak{D}_{\Theta(0)}}+\Theta^*(0)Z(\xi))f||^2=||D_{Z(\xi)}f||^2,\ f\in\mathfrak{D}_{\Theta(0)}.$$

for almost all $\xi\in\mathbb{T}$. This completes the proof.

THEOREM 5.5. *Let A be a completely non-unitary contraction in the Hilbert space $\mathfrak{H}$ and let $Z(\lambda)$ be the Möbius parameter of the Sz.Nagy–Foias characteristic function of A. Then $Z(\lambda)$ is the characteristic function of the operator $\mathscr{A}_{1,0}=AP_{\ker D_A}$ (see* (3.2) *and* (3.3)*). Moreover, the following statements are equivalent*

(i) *the unitary equivalent operators $A_{1,0}$ and $A_{0,1}$ are unilateral shifts (co-shifts),*

(ii) $\mathfrak{D}_A\subset\mathfrak{D}_{A^*}$ ($\mathfrak{D}_{A^*}\subset\mathfrak{D}_A$),

(iii) *the Möbius parameter takes the form $Z(\lambda)=\lambda I_{\mathfrak{D}_A}$ ($Z^*(\bar{\lambda})=\lambda I_{\mathfrak{D}_{A^*}}$).*

Proof. The system

$$\Sigma=\left\{\begin{bmatrix} -A & D_{A^*}\\ D_A & A^*\end{bmatrix};\mathfrak{D}_A,\mathfrak{D}_{A^*},\mathfrak{H}\right\}$$

is conservative and simple and its transfer function

$$\Phi(\lambda)=\left(-A+\lambda D_{A^*}(I_{\mathfrak{H}}-\lambda A^*)^{-1}D_A\right)\upharpoonright\mathfrak{D}_A$$

is the characteristic function of A. Let F and G^* be the embedding of the subspaces $\mathfrak{D}_A$ and $\mathfrak{D}_{A^*}$ into $\mathfrak{H}$, respectively. It follows that

$$D_{F^*} = P_{\ker D_A},\ D_G = P_{\ker D_{A^*}}.$$

Let $L = A^* \upharpoonright \ker D_{A^*}$. Then

$$A^* = A^* P_{\mathfrak{D}_{A^*}} + A^* P_{\ker D_{A^*}} = -F(-A^*)G + D_{F^*} L D_G$$

Let

$$\Phi(\lambda) = \Phi(0) + D_{\Phi^*(0)} Z(\lambda)(I + \Phi^*(0)Z(\lambda))^{-1} D_{\Phi(0)},\ \lambda \in \mathbb{D}$$

be the Möbius representation of the function $\Phi(\lambda)$. By Theorem 5.3 the system

$$\nu = \left\{ \begin{bmatrix} 0 & P_{\mathfrak{D}_{A^*}} \\ I_{\mathfrak{D}_A} & A^* P_{\ker D_{A^*}} \end{bmatrix}; \mathfrak{D}_A, \mathfrak{D}_{A^*}, \mathfrak{H} \right\}$$

is conservative and simple and its transfer function is the function $Z(\lambda)$, i.e.,

$$Z(\lambda) = \lambda P_{\mathfrak{D}_{A^*}} \left(I_{\mathfrak{H}} - \lambda A^* P_{\ker D_{A^*}} \right)^{-1} \upharpoonright \mathfrak{D}_A,\ |\lambda| < 1.$$

This function is precisely the Sz.-Nagy–Foias characteristic function of the partial isometry $\mathscr{A}_{1,0} = A P_{\ker \mathfrak{D}_A}$.

Suppose $A_{1,0} = P_{\ker D_A} A \upharpoonright \ker D_A$ is a unilateral shift. Since $A \ker D_A = \ker D_{A^*}$, we have $\ker D_{A^*} \subset \ker D_A$. Equivalently $\mathfrak{D}_A \subset \mathfrak{D}_{A^*}$. Hence,

$$P_{\ker D_{A^*}} \upharpoonright \mathfrak{D}_A = 0 \quad \text{and} \quad (A^* P_{\ker D_{A^*}})^n \upharpoonright \mathfrak{D}_A = 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Therefore,

$$Z(\lambda) = \lambda P_{\mathfrak{D}_{A^*}} \upharpoonright \mathfrak{D}_A = \lambda I_{\mathfrak{D}_A}.$$

Conversely, suppose $Z(\lambda) = \lambda I_{\mathfrak{D}_A}$. Then $\mathfrak{D}_A \subset \mathfrak{D}_{A^*} \Rightarrow \ker D_A \supset \ker D_{A^*}$. It follows

$$A \ker D_A \subset \ker D_A \Rightarrow A_{1,0} \quad \text{is isometry.}$$

Since the operator $A_{1,0}$ is completely non-unitary, it is a unilateral shift.

COROLLARY 5.6. *Let A be a completely non-unitary contraction in a separable Hilbert space $\mathfrak{H}$ and let $||A \upharpoonright \mathfrak{D}_A|| < 1 (\iff \operatorname{ran} D_A = \mathfrak{D}_A)$. Then the following statements are equivalent*

(i) $A \in C_{\cdot 0}$ *(respect., $A \in C_{0 \cdot}$),*

(ii) $\mathscr{A}_{1,0} \in C_{\cdot 0}$ *(respect., $\mathscr{A}_{1,0} \in C_{0 \cdot}$).*

Proof. By (2.4) we have $\Phi_A(0) = -A \upharpoonright \mathfrak{D}_A$. Then in accordance with [47], Corollary 5.4, and Theorem 5.5 we get the equivalences

$$A \in C_{\cdot 0}\ (C_{0 \cdot}) \iff \Phi_A(\lambda) \quad \text{is inner (co-inner)} \quad \iff Z(\lambda) \quad \text{is inner (co-inner)} \iff \mathscr{A}_{1,0} \in C_{\cdot 0}\ (C_{0 \cdot}).$$

6. Realizations of the Schur iterates

6.1. Realizations of the first Schur iterate

PROPOSITION 6.1. *Let $\mathfrak{H}$, $\mathfrak{L}$, $\mathfrak{K}$ be Hilbert spaces and let $F \in \mathbf{L}(\mathfrak{L},\mathfrak{H})$, $G \in \mathbf{L}(\mathfrak{H},\mathfrak{K})$ and $L \in \mathbf{L}(\mathfrak{D}_G,\mathfrak{D}_{F^*})$ be contractions. Let $Z_\nu(\lambda)$ be the transfer function of the system*

$$\nu = \left\{ \begin{bmatrix} 0 & G \\ F & D_{F^*}LD_G \end{bmatrix}; \mathfrak{L},\mathfrak{K},\mathfrak{H} \right\} \tag{6.1}$$

Then the function $\Gamma(\lambda) = \lambda^{-1}Z_\nu(\lambda)$ is the transfer function of the passive systems

$$\eta_1 = \left\{ \begin{bmatrix} GF & GD_{F^*} \\ LD_GF & LD_GD_{F^*} \end{bmatrix}; \mathfrak{L},\mathfrak{K},\mathfrak{H} \right\},\ \eta_2 = \left\{ \begin{bmatrix} GF & GD_{F^*}\widetilde{L} \\ D_GF & D_GD_{F^*}\widetilde{L} \end{bmatrix}; \mathfrak{L},\mathfrak{K},\mathfrak{H} \right\},$$

where $\widetilde{L} = LP_{\mathfrak{D}_G}$.

Suppose that the subspaces $\mathfrak{D}_{F^*}$ and $\mathfrak{D}_G$ are nontrivial. Then the transfer functions of the passive systems

$$\zeta_1 = \left\{ \begin{bmatrix} GF & GD_{F^*} \\ LD_GF & LD_GD_{F^*} \end{bmatrix}; \mathfrak{L},\mathfrak{K},\mathfrak{D}_{F^*} \right\},\ \zeta_2 = \left\{ \begin{bmatrix} GF & GD_{F^*}\widetilde{L} \\ D_GF & D_GD_{F^*}\widetilde{L} \end{bmatrix}; \mathfrak{L},\mathfrak{K},\mathfrak{D}_G \right\} \tag{6.2}$$

are equal to $\Gamma(\lambda)$. Moreover, for the orthogonal complements to the controllable and observable subspaces of the systems ν, ζ_1, and ζ_2 hold the following relations

$$\begin{aligned} &(\mathfrak{H}^c_\nu)^\perp = \left(\mathfrak{H}^c_{\zeta_1}\right)^\perp \cap \ker F^*,\ (\mathfrak{H}^o_\nu)^\perp = \left(\mathfrak{H}^o_{\zeta_2}\right)^\perp \cap \ker G, \\ &D_G\left(\mathfrak{H}^c_{\zeta_2}\right)^\perp \subset (\mathfrak{H}^c_\nu)^\perp,\ D_{F^*}\left(\mathfrak{H}^o_{\zeta_1}\right)^\perp \subset (\mathfrak{H}^o_\nu)^\perp. \end{aligned} \tag{6.3}$$

If the operators G^* and F are isometries, then

$$\left(\mathfrak{H}^o_{\zeta_1}\right)^\perp = (\mathfrak{H}^o_\nu)^\perp \cap \ker F^*,\ \left(\mathfrak{H}^c_{\zeta_2}\right)^\perp = (\mathfrak{H}^c_\nu)^\perp \cap \ker G. \tag{6.4}$$

Proof. We have

$$Z_\nu(\lambda) = \lambda G(I_{\mathfrak{H}} - \lambda D_{F^*}LD_G)^{-1}F.$$

Hence

$$\Gamma(\lambda) = \frac{Z_\nu(\lambda)}{\lambda} = G(I_{\mathfrak{H}} - \lambda D_{F^*}LD_G)^{-1}F$$

and $\Gamma(0) = GF$. It follows that

$$\begin{aligned} &\Gamma(\lambda) - \Gamma(0) = G(I_{\mathfrak{H}} - \lambda D_{F^*}LD_G)^{-1}F - GF = \lambda GD_{F^*}LD_G(I_{\mathfrak{H}} - \lambda D_{F^*}LD_G)^{-1}F \\ &= \lambda GD_{F^*}(I_{\mathfrak{H}} - \lambda LD_GD_{F^*})^{-1}LD_GF = \lambda GD_{F^*}(I_{\mathfrak{H}} - \lambda\widetilde{L}D_GD_{F^*})^{-1}\widetilde{L}D_GF \\ &= \lambda GD_{F^*}\widetilde{L}(I_{\mathfrak{H}} - \lambda D_GD_{F^*}\widetilde{L})^{-1}D_GF, \end{aligned}$$

$$\begin{aligned}\Gamma(\lambda) &= GF + \lambda GD_{F^*}(I_{\mathfrak{H}} - \lambda LD_G D_{F^*})^{-1} LD_G F \\ &= GF + \lambda GD_{F^*}\widetilde{L}(I_{\mathfrak{H}} - \lambda D_G D_{F^*}\widetilde{L})^{-1} D_G F.\end{aligned} \tag{6.5}$$

The operators

$$K_1 = \begin{bmatrix} GF & GD_{F^*} \\ LD_G F & LD_G D_{F^*}\end{bmatrix} : \begin{matrix}\mathfrak{L} \\ \oplus \\ \mathfrak{H}\end{matrix} \to \begin{matrix}\mathfrak{K} \\ \oplus \\ \mathfrak{H}\end{matrix}$$

and

$$K_2 = \begin{bmatrix} GF & GD_{F^*}\widetilde{L} \\ D_G F & D_G D_{F^*}\widetilde{L}\end{bmatrix} : \begin{matrix}\mathfrak{L} \\ \oplus \\ \mathfrak{H}\end{matrix} \to \begin{matrix}\mathfrak{K} \\ \oplus \\ \mathfrak{H}\end{matrix}$$

are contraction. Actually, let $f \in \mathfrak{H}$ and $h \in \mathfrak{L}$ then one can check that

$$\left\|\begin{bmatrix} h \\ f\end{bmatrix}\right\|^2 - \left\|K_1\begin{bmatrix} h \\ f\end{bmatrix}\right\|^2 = ||F^*f - D_F h||^2_{\mathfrak{L}} + ||D_L D_G(D_{F^*}f + Fh)||^2_{\mathfrak{H}} \geqslant 0,$$

$$\left\|\begin{bmatrix} h \\ f\end{bmatrix}\right\|^2 - \left\|K_2\begin{bmatrix} h \\ f\end{bmatrix}\right\|^2 = ||F^*\widetilde{L}f - D_F h||^2_{\mathfrak{L}} + ||D_{\widetilde{L}}f||^2_{\mathfrak{H}} \geqslant 0.$$

Thus, the systems η_1, η_2, ζ_1, and ζ_2 are passive and their transfer functions are precisely $\Gamma(\lambda)$.

Since $\widetilde{L}^* \upharpoonright \ker D_{F^*} = 0$ and $F^*f = 0 \iff D_{F^*}f = f$, $Gh = 0 \iff D_G h = h$, by induction one can derive the following equalities

$$\begin{cases} \bigcap\limits_{n\geqslant 0} \ker\left(F^*(D_G L^* D_{F^*})^n\right) = \bigcap\limits_{n\geqslant 0} \ker\left(F^*(D_G\widetilde{L}^*)^n)\right), \\ \bigcap\limits_{n\geqslant 0} \ker\left(G(D_{F^*}LD_G)^n\right) = \bigcap\limits_{n\geqslant 0} \ker\left(G(D_{F^*}\widetilde{L})^n\right), \\ \bigcap\limits_{n\geqslant 0} \ker\left(F^*D_G\widetilde{L}^*(D_{F^*}D_G\widetilde{L}^*)^n\right) = \bigcap\limits_{n\geqslant 1} \ker\left(F^*(D_G\widetilde{L}^*)^n)\right), \\ \bigcap\limits_{n\geqslant 0} \ker\left(GD_{F^*}(\widetilde{L}D_G D_{F^*})^n\right) = \bigcap\limits_{n\geqslant 0} \ker\left(G(D_{F^*}\widetilde{L})^n\right), \\ \bigcap\limits_{n\geqslant 0} \ker\left(F^*D_G(\widetilde{L}^*D_{F^*}D_G)^n\right) = \bigcap\limits_{n\geqslant 0} \ker\left(F^*(D_G\widetilde{L}^*)^n D_G\right), \\ \bigcap\limits_{n\geqslant 0} \ker\left(GD_{F^*}\widetilde{L}(D_G D_{F^*}\widetilde{L})^n\right) = \bigcap\limits_{n\geqslant 1} \ker\left(G(D_{F^*}\widetilde{L})^n\right). \end{cases} \tag{6.6}$$

From (6.6) follow the relations (6.3) and (6.4).

THEOREM 6.2. *Let the system*

$$\tau = \left\{\begin{bmatrix} D & D_{D^*}G \\ FD_D & -FD^*G + D_{F^*}LD_G\end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\right\}$$

be conservative and simple and let $\Theta(\lambda)$ *be its transfer function. Suppose that the first Schur iterate* $\Theta_1(\lambda)$ *of* Θ *is non-unitary constant. Then the systems*

$$\begin{aligned}\zeta_1 &= \left\{\begin{bmatrix} GF & G \\ LD_G F & LD_G\end{bmatrix}; \mathfrak{D}_D, \mathfrak{D}_{D^*}, \mathfrak{D}_{F^*}\right\}, \\ \zeta_2 &= \left\{\begin{bmatrix} GF & GL \\ D_G F & D_G L\end{bmatrix}; \mathfrak{D}_D, \mathfrak{D}_{D^*}, \mathfrak{D}_G\right\}\end{aligned} \tag{6.7}$$

are conservative and simple and their transfer functions are equal to $\Theta_1(\lambda)$.

Proof. Because the system ν is conservative, the operators F and G^* are isometries. Since $\Theta_1(\lambda)$ is non-unitary constant, from (6.5) it follows that the operator GF is non-unitary. Hence by Theorem 4.6 the subspaces $\mathfrak{D}_{F^*}$ and $\mathfrak{D}_G$ are nontrivial, and the operator $L \in \mathbf{L}(\mathfrak{D}_G, \mathfrak{D}_{F^*})$ is unitary. In addition, $\ker F^* = \mathfrak{D}_{F^*}$, $\ker G = \mathfrak{D}_G$, and the operators D_{F^*} and D_G are orthogonal projections in $\mathfrak{H}$ onto $\ker F^*$ and $\ker G$, respectively. By Theorem 5.3 the system

$$\nu = \left\{ \begin{bmatrix} 0 & G \\ F & D_{F^*}LD_G \end{bmatrix}; \mathfrak{D}_D, \mathfrak{D}_{D^*}, \mathfrak{H} \right\}$$

is conservative and simple. One can directly check that the operators

$$\begin{bmatrix} GF & G \\ LD_GF & LD_G \end{bmatrix} : \begin{matrix} \mathfrak{D}_D \\ \oplus \\ \mathfrak{D}_{F^*} \end{matrix} \to \begin{matrix} \mathfrak{D}_{D^*} \\ \oplus \\ \mathfrak{D}_{F^*} \end{matrix}, \quad \begin{bmatrix} GF & GL \\ D_GF & D_GL \end{bmatrix} : \begin{matrix} \mathfrak{D}_D \\ \oplus \\ \mathfrak{D}_G \end{matrix} \to \begin{matrix} \mathfrak{D}_{D^*} \\ \oplus \\ \mathfrak{D}_G \end{matrix}$$

are unitary. Hence, the systems ζ_1 and ζ_2 given by (6.7) are conservative. In our case relations (6.3) yield equalities

$$(\mathfrak{H}^c_\nu)^\perp = \left(\mathfrak{H}^c_{\zeta_1}\right)^\perp, \ (\mathfrak{H}^o_\nu)^\perp = \left(\mathfrak{H}^o_{\zeta_2}\right)^\perp.$$

Taking into account (6.4) and the simplicity of ν we get that the systems ζ_1 and ζ_2 are simple.

THEOREM 6.3. *Let* $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$, $\Gamma_0 = \Theta(0)$ *and let* $\Theta_1(\lambda)$ *be the first Schur iterate of* Θ. *Suppose*

$$\tau = \left\{ \begin{bmatrix} \Gamma_0 & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\}$$

is a simple conservative system with transfer function Θ. *Then the simple conservative system*

$$\nu = \left\{ \begin{bmatrix} 0 & D^{-1}_{\Gamma^*_0}C \\ D^{-1}_{A^*}B & AP_{\ker D_A} \end{bmatrix}, \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma^*_0}, \mathfrak{H} \right\}$$

has the transfer function $\lambda\Theta_1(\lambda)$ *while the simple conservative systems*

$$\begin{aligned} \zeta_1 &= \left\{ \begin{bmatrix} D^{-1}_{\Gamma^*_0}C(D^{-1}_{\Gamma_0}B^*)^* & D^{-1}_{\Gamma^*_0}C\restriction \ker D_{A^*} \\ AP_{\ker D_A}D^{-1}_{A^*}B & P_{\ker D_{A^*}}A\restriction \ker D_{A^*} \end{bmatrix}; \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma^*_0}, \ker D_{A^*} \right\}, \\ \zeta_2 &= \left\{ \begin{bmatrix} D^{-1}_{\Gamma^*_0}C(D^{-1}_{\Gamma_0}B^*)^* & D^{-1}_{\Gamma^*_0}CA\restriction \ker D_A \\ P_{\ker D_A}D^{-1}_{A^*}B & P_{\ker D_A}A\restriction \ker D_A \end{bmatrix}; \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma^*_0}, \ker D_A \right\} \end{aligned} \tag{6.8}$$

have transfer functions $\Theta_1(\lambda)$. *Here the operators* $D^{-1}_{\Gamma_0}$, $D^{-1}_{\Gamma^*_0}$, *and* $D^{-1}_{A^*}$ *are the Moore–Penrose pseudo-inverses.*

Proof. Let

$$T=\begin{bmatrix}\Gamma_0 & C\\ B & A\end{bmatrix}=\begin{bmatrix}\Gamma_0 & D_{\Gamma_0^*}G\\ FD_{\Gamma_0} & -F\Gamma_0^*G+D_{F^*}LD_G\end{bmatrix}=$$
$$=\begin{bmatrix}-KA^*M+D_{K^*}XD_M & KD_A\\ D_{A^*}M & A\end{bmatrix}:\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{H}\end{matrix}\to\begin{matrix}\mathfrak{N}\\ \oplus\\ \mathfrak{H}\end{matrix}.$$

Then $G=D^{-1}_{\Gamma_0^*}C,\ F^*=D^{-1}_{\Gamma_0}B^*,\ F=M\upharpoonright\mathfrak{D}_{\Gamma_0},\ M=D^{-1}_{A^*}B.$ According to Proposition 4.7 we have

$$D_{F^*}=P_{\ker D_{A^*}},\ D_G=P_{\ker D_A},\ L=A\upharpoonright\ker D_A.$$

Hence

$$\begin{aligned}&GF=D^{-1}_{\Gamma_0^*}C(D^{-1}_{\Gamma_0}B^*)^*,\ D_GD_{F^*}L=P_{\ker D_A}A\upharpoonright\ker D_A,\\ &D_GF=P_{\ker D_A}M=P_{\ker D_A}D^{-1}_{A^*}B,\ GD_{F^*}L=D^{-1}_{\Gamma_0^*}CP_{\mathfrak{D}_A}A\upharpoonright\ker D_A,\\ &LD_G\upharpoonright\ker D_{A^*}=AP_{\ker D_A}\upharpoonright\ker D_{A^*},\ LD_GF=AP_{\ker D_A}D^{-1}_{A^*}B.\end{aligned}$$

Note that if $f\in\ker D_{A^*}$ then

$$AP_{\ker D_A}f=P_{\ker D_{A^*}}AP_{\ker D_A}f=P_{\ker D_{A^*}}Af-P_{\ker D_{A^*}}AP_{\mathfrak{D}_A}f=P_{\ker D_{A^*}}Af.$$

Now the statement of theorem follows from Theorem 5.3 and Theorem 6.2.

REMARK 6.4. Since $F^*=D^{-1}_{\Gamma_0}B^*$, we get $F=\left(D^{-1}_{\Gamma_0}B^*\right)^*\in\mathbf{L}(\mathfrak{D}_{\Gamma_0},\mathfrak{H})$. Hence

$$D^{-1}_{A^*}B\upharpoonright\mathfrak{D}_{\Gamma_0}=\left(D^{-1}_{\Gamma_0}B^*\right)^*.$$

Using the Hilbert spaces and operators defined by (3.1) and (3.2), we get

$$P_{\ker D_A}D^{-1}_{A^*}B\upharpoonright\mathfrak{D}_{\Gamma_0}=P_{1,0}D^{-1}_{A^*}B\upharpoonright\mathfrak{D}_{\Gamma_0}=\left(D^{-1}_{\Gamma_0}\left(B^*\upharpoonright\mathfrak{H}_{1,0}\right)\right)^*\in\mathbf{L}(\mathfrak{D}_{\Gamma_0},\mathfrak{H}_{1,0}).$$

In addition

$$D^{-1}_{\Gamma_0^*}C(D^{-1}_{\Gamma_0}B^*)^*=\Gamma_1\in\mathbf{L}(\mathfrak{D}_{\Gamma_0},\mathfrak{D}_{\Gamma_0^*}).$$

So,

$$\begin{aligned}\zeta_1&=\left\{\begin{bmatrix}\Gamma_1 & D^{-1}_{\Gamma_0^*}C\\ A\left(D^{-1}_{\Gamma_0}\left(B^*\upharpoonright\mathfrak{H}_{1,0}\right)\right)^* & A_{0,1}\end{bmatrix};\mathfrak{D}_{\Gamma_0},\mathfrak{D}_{\Gamma_0^*},\mathfrak{H}_{0,1}\right\},\\ \zeta_2&=\left\{\begin{bmatrix}\Gamma_1 & D^{-1}_{\Gamma_0^*}CA\\ \left(D^{-1}_{\Gamma_0}\left(B^*\upharpoonright\mathfrak{H}_{1,0}\right)\right)^* & A_{1,0}\end{bmatrix};\mathfrak{D}_{\Gamma_0},\mathfrak{D}_{\Gamma_0^*},\mathfrak{H}_{1,0}\right\}.\end{aligned}\tag{6.9}$$

It follows that

$$\begin{aligned}&\operatorname{ran}\left(D^{-1}_{\Gamma_0^*}C\upharpoonright\mathfrak{H}_{1,0}\right)\subset\operatorname{ran}D_{\Gamma_1^*},\\ &\operatorname{ran}\left(D^{-1}_{\Gamma_0}B^*\upharpoonright\mathfrak{H}_{1,0}\right)\subset\operatorname{ran}D_{\Gamma_1}\end{aligned}$$

6.2. Schur iterates of the characteristic function

THEOREM 6.5. *Let A be a completely non-unitary contraction in a separable Hilbert space $\mathfrak{H}$. Assume $\ker D_A \neq \{0\}$ and let the contractions $A_{n,m}$ be defined by* (3.1) *and* (3.2)*. Then the characteristic functions of the operators*

$$A_{n,0}, A_{n-1,1}, \ldots, A_{n-m,m}, \ldots A_{1,n-1}, A_{0,n}$$

coincide with the pure part of the n-th Schur iterate of the characteristic function $\Phi(\lambda)$ of A. Moreover, each operator from the set $\{A_{n-k,k}\}_{k=0}^{n}$ is

1. *a unilateral shift (respect., co-shift) if and only if the n-th Schur parameter Γ_n of Φ is isometric (respect., co-isometric),*
2. *the orthogonal sum of a unilateral shift and co-shift if and only if*

$$\mathfrak{D}_{\Gamma_{n-1}} \neq \{0\},\ \mathfrak{D}_{\Gamma^*_{n-1}} \neq \{0\} \quad \textit{and} \quad \Gamma_m = 0 \quad \textit{for all} \quad m \geqslant n. \tag{6.10}$$

Each subspace from the set $\{\mathfrak{H}_{n-k,k}\}_{k=0}^{n}$ is trivial if and only if Γ_n is unitary.

Proof. We will prove by induction. The system

$$\Sigma = \left\{ \begin{bmatrix} -A & D_{A^*} \\ D_A & A^* \end{bmatrix}; \mathfrak{D}_A, \mathfrak{D}_{A^*}, \mathfrak{H} \right\}$$

is conservative and simple and its transfer function $\Phi(\lambda)$ is Sz.-Nagy–Foias characteristic function of A. As in Theorem 5.5, let F and G^* be the embedding of the subspaces $\mathfrak{D}_A$ and $\mathfrak{D}_{A^*}$ into $\mathfrak{H}$, respectively. Then $D_{F^*} = P_{\ker D_A} = P_{1,0}$, $D_G = P_{\ker D_{A^*}} = P_{0,1}$, and $L = A^* \restriction \ker D_{A^*} \in \mathbf{L}(\mathfrak{D}_{A^*}, \mathfrak{D}_A)$ is unitary operator. The system

$$\nu = \left\{ \begin{bmatrix} 0 & P_{\mathfrak{D}_{A^*}} \\ I_{\mathfrak{D}_A} & A^* P_{\ker D_{A^*}} \end{bmatrix}; \mathfrak{D}_A, \mathfrak{D}_{A^*}, \mathfrak{H} \right\}$$

is conservative and simple and its transfer function $Z(\lambda)$ is the Möbius parameter of $\Phi(\lambda)$. Constructing the systems given by (6.7) in Theorem 6.2 we get

$$\zeta_1 = \left\{ \begin{bmatrix} P_{\mathfrak{D}_{A^*}} \restriction \mathfrak{D}_A & P_{\mathfrak{D}_{A^*}} \restriction \ker D_A \\ A^* P_{\ker D_{A^*}} \restriction \mathfrak{D}_A & A^* P_{\ker D_{A^*}} \restriction \ker D_A \end{bmatrix}; \mathfrak{D}_A, \mathfrak{D}_{A^*}, \ker D_A \right\}$$

and

$$\zeta_2 = \left\{ \begin{bmatrix} P_{\mathfrak{D}_{A^*}} \restriction \mathfrak{D}_A & P_{\mathfrak{D}_{A^*}} A^* \restriction \ker D_{A^*} \\ P_{\ker D_{A^*}} \restriction \mathfrak{D}_A & P_{\ker D_{A^*}} A^* \restriction \ker D_{A^*} \end{bmatrix}; \mathfrak{D}_A, \mathfrak{D}_{A^*}, \ker D_{A^*} \right\}.$$

By Theorem 6.2 the systems ζ_1 and ζ_2 are conservative and simple and their transfer functions are precisely the first Schur iterate $\Phi_1(\lambda)$ of $\Phi(\lambda)$. Note (see (3.1) and (3.2)) that

$$A^* P_{\ker D_{A^*}} \restriction \ker D_A = A^*_{1,0},\ P_{\ker D_{A^*}} A^* \restriction \ker D_{A^*} = A^*_{0,1}.$$

Applying Proposition 4.5 we get that the pure part of $\Phi_1(\lambda)$ coincides with the characteristic functions of the operators $A_{1,0}$ and $A_{0,1}$.

By Theorem 3.1 completely non-unitary contractions $\{A_{n-k,k}\}_{k=0}^{n}$ are unitarily equivalent. Assume that their characteristic functions coincide with the pure part of the n-th Schur iterate $\Phi_n(\lambda)$ of Φ. The first Schur iterate of Φ_n is the function $\Phi_{n+1}(\lambda)$. As is already proved above the pure part of Φ_{n+1} coincides with the characteristic function of the operators $(A_{n-k,k})_{1,0}$ and $(A_{n-k,k})_{0,1}$. From (3.9) it follows

$$(A_{n-k,k})_{1,0} = A_{n+1-k,k},\ (A_{n-k,k})_{0,1} = A_{n-k,k+1} = A_{n+1-(k+1),k+1}.$$

Thus, characteristic functions of the unitarily equivalent completely non-unitary contractions $\{A_{n+1-k,k}\}_{k=0}^{n+1}$ coincide with Φ_{n+1}.

Note that the Möbius parameter of the $n-1$-th Schur iterate Φ_{n-1} is $\lambda\Phi_n(\lambda)$ and by Theorem 5.5 this function coincides with the characteristic function of the operator $\mathscr{A}_{n,0} = A_{n,0}P_{\ker D_{A_{n,0}}}$. Applying Theorem 5.5 once again, we get that $A_{n,0}$ is a unilateral shift if and only if Γ_n is a isometry.

The function $\Phi^*(\bar{\lambda})$ is the characteristic function of the operator A^* and its Schur parameters are adjoint to the corresponding Schur parameters of Φ. In addition if $B = A^*$ then $B_{n,m} = A^*_{m,n}$. Therefore, $A^*_{0,n}$ is a unilateral shift if and only if Γ^*_n is isometric. But $A^*_{0,n}$ is unuitarily equivalent to $A^*_{n,0}$. Hence, $A_{n,0}$ is a co-shift if and only if Γ_n is a co-isometry.

It follows that Γ_n is a unitary if and only if $A_{n,0}$ is a unilateral shift and co-shift in $\mathfrak{H}_{n,0} \iff \mathfrak{H}_{n,0} = \{0\}$.

Condition (6.10) holds true if and only if Φ_n is identically equal zero. This is equivalent to the condition that $A_{n,0}$ (as well and $A_{n-1,1},\ A_{n-2,2},\ \ldots A_{0,n}$) is the orthogonal sum of a shift and co-shift.

REMARK 6.6. Theorem 6.5 and Theorem 3.1 yield the following equivalences:

$$\begin{aligned}&\Gamma_n \text{ is isometry} \iff \ker D_{A^{n+1}} = \ker D_{A^n} \iff \ker D_{A^n}\cap\ker D_{A^*} = \ker D_{A^{n-1}}\cap\ker D_{A^*}\\ &\iff \ldots \iff \ker D_{A^{n+1-k}}\cap\ker D_{A^{*k}} = \ker D_{A^{n-k}}\cap\ker D_{A^{*k}} \iff \ldots\\ &\iff \ker D_{A^{*n}} \subset \ker D_A;\end{aligned}$$

$$\begin{aligned}&\Gamma^*_n \text{ is isometry} \iff \ker D_{A^n} \subset \ker D_{A^*} \iff \ker D_{A^{n-1}}\cap\ker D_{A^{*2}} = \ker D_{A^{n-1}}\cap\ker D_{A^*}\\ &\iff \ldots \iff \ker D_{A^{n-k}}\cap\ker D_{A^{*k+1}} = \ker D_{A^{n-k}}\cap\ker D_{A^{*k}}\\ &\iff \ldots \iff \ker D_{A^{*n+1}} = \ker D_{A^{*n}};\end{aligned}$$

$$\text{conditions (6.10)} \iff \begin{cases}\ker D_{A^n} = \left(\bigcap\limits_{l\geqslant 1}\ker D_{A^l}\right)\oplus\left(\bigcap\limits_{l\geqslant 1}\ker D_{A^{*l}}\right),\\ P_{\ker D_{A^n}}A\left(\bigcap\limits_{l\geqslant 1}\ker D_{A^{*l}}\right)\subset\left(\bigcap\limits_{l\geqslant 1}\ker D_{A^{*l}}\right).\end{cases}$$

In particular, we get the following statement:

if the Schur parameter Γ_n of the characteristic function of a completely non-unitary contraction A is non-unitary, but isometry (respect., co-isometry) for some n, then A is not completely non-isometric, but is completely non-co-isometric (respect., A is not completely non-co-isometric, but is completely non-isometric); *if Γ_n is unitary, then A is completely non-isometric and completely non-co-isometric.*

6.3. Conservative realizations of the Schur iterates

THEOREM 6.7. *Let* $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ *and let*

$$\tau_0 = \left\{ \begin{bmatrix} \Gamma_0 & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\}$$

be a simple conservative realization of Θ. *Then the Schur parameters* $\{\Gamma_n\}_{n\geqslant 1}$ *of* Θ *can be calculated as follows*

$$\begin{aligned} &\Gamma_1 = D^{-1}_{\Gamma_0^*} C \left(D^{-1}_{\Gamma_0} B^* \right)^*, \; \Gamma_2 = D^{-1}_{\Gamma_1^*} D^{-1}_{\Gamma_0^*} CA \left(D^{-1}_{\Gamma_1} D^{-1}_{\Gamma_0} \left(B^* \upharpoonright \mathfrak{H}_{1,0}\right)\right)^*, \ldots, \\ &\Gamma_n = D^{-1}_{\Gamma_{n-1}^*} \cdots D^{-1}_{\Gamma_0^*} CA^{n-1} \left(D^{-1}_{\Gamma_{n-1}} \cdots D^{-1}_{\Gamma_0} \left(B^* \upharpoonright \mathfrak{H}_{n-1,0}\right)\right)^*, \ldots. \end{aligned} \tag{6.11}$$

Here the operators $D^{-1}_{\Gamma_k}$ *and* $D^{-1}_{\Gamma_k^*}, k = 0, 1, \ldots$ *are the Moore-Penrose pseudo inverses, the operator*

$$\left(D^{-1}_{\Gamma_{n-1}} \cdots D^{-1}_{\Gamma_0} \left(B^* \upharpoonright \mathfrak{H}_{n-1,0}\right)\right)^* \in \mathbf{L}(\mathfrak{D}_{\Gamma_{n-1}}, \mathfrak{H}_{n-1,0})$$

is the adjoint to the operator

$$D^{-1}_{\Gamma_{n-1}} \cdots D^{-1}_{\Gamma_0} \left(B^* \upharpoonright \mathfrak{H}_{n-1,0}\right) \in \mathbf{L}(\mathfrak{H}_{n-1,0}, \mathfrak{D}_{\Gamma_{n-1}}),$$

and

$$\begin{aligned} &\operatorname{ran}\left(D^{-1}_{\Gamma_{n-1}} \cdots D^{-1}_{\Gamma_0} \left(B^* \upharpoonright \mathfrak{H}_{n,0}\right)\right) \subset \operatorname{ran} D_{\Gamma_n}, \\ &\operatorname{ran}\left(D^{-1}_{\Gamma_{n-1}^*} \cdots D^{-1}_{\Gamma_0^*} \left(C \upharpoonright \mathfrak{H}_{0,n}\right)\right) \subset \operatorname{ran} D_{\Gamma_n^*} \end{aligned}$$

for every $n \geqslant 1$. *Moreover, for each* $n \geqslant 1$ *the unitarily equivalent simple conservative systems*

$$\tau_n^{(k)} = \left\{ \begin{bmatrix} \Gamma_n & D^{-1}_{\Gamma_{n-1}^*} \cdots D^{-1}_{\Gamma_0^*} (CA^{n-k}) \\ A^k \left(D^{-1}_{\Gamma_{n-1}} \cdots D^{-1}_{\Gamma_0} \left(B^* \upharpoonright \mathfrak{H}_{n,0}\right)\right)^* & A_{n-k,k} \end{bmatrix}; \right. \left. \mathfrak{D}_{\Gamma_{n-1}}, \mathfrak{D}_{\Gamma_{n-1}^*}, \mathfrak{H}_{n-k,k} \right\}, \quad k = 0, 1, \ldots, n \tag{6.12}$$

are realizations of the n*-th Schur iterate* Θ_n *of* Θ. *Here the operator*

$$B_n = \left(D^{-1}_{\Gamma_{n-1}} \cdots D^{-1}_{\Gamma_0} \left(B^* \upharpoonright \mathfrak{H}_{n,0}\right)\right)^* \in \mathbf{L}(\mathfrak{D}_{\Gamma_{n-1}}, \mathfrak{H}_{n,0})$$

is the adjoint to the operator

$$D^{-1}_{\Gamma_{n-1}} \cdots D^{-1}_{\Gamma_0} \left(B^* \upharpoonright \mathfrak{H}_{n,0}\right) \in \mathbf{L}(\mathfrak{H}_{n,0}, \mathfrak{D}_{\Gamma_{n-1}}).$$

Proof. We will prove by induction. For $n=1$ it is already established (see Remark 6.4, (6.8), and (6.9)) that

$$\Gamma_1 = D^{-1}_{\Gamma^*_0} C \left(D^{-1}_{\Gamma_0} B^*\right)^*$$

and the systems

$$\tau_1^{(0)} = \left\{ \begin{bmatrix} \Gamma_1 & D^{-1}_{\Gamma^*_0}(CA) \\ \left(D^{-1}_{\Gamma_0}\left(B^*\upharpoonright \mathfrak{H}_{1,0}\right)\right)^* & A_{1,0} \end{bmatrix}; \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma^*_0}, \mathfrak{H}_{1,0} \right\}$$

and

$$\tau_1^{(1)} = \left\{ \begin{bmatrix} \Gamma_1 & D^{-1}_{\Gamma^*_0}(C) \\ A\left(D^{-1}_{\Gamma_0}\left(B^*\upharpoonright \mathfrak{H}_{1,0}\right)\right)^* & A_{0,1} \end{bmatrix}; \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma^*_0}, \mathfrak{H}_{0,1} \right\}$$

are conservative and simple realizations of Θ_1. Suppose

$$\tau_m^{(0)} = \left\{ \begin{bmatrix} \Gamma_m & D^{-1}_{\Gamma^*_{m-1}} \cdots D^{-1}_{\Gamma^*_0}(CA^m) \\ \left(D^{-1}_{\Gamma_{m-1}} \cdots D^{-1}_{\Gamma_0}\left(B^*\upharpoonright \mathfrak{H}_{m,0}\right)\right)^* & A_{m,0} \end{bmatrix}; \mathfrak{D}_{\Gamma_{m-1}}, \mathfrak{D}_{\Gamma^*_{m-1}}, \mathfrak{H}_{m,0} \right\}$$

is a simple conservative realization of Θ_m. Then

$$\begin{aligned} B_m &= \left(D^{-1}_{\Gamma_{m-1}} \cdots D^{-1}_{\Gamma_0}\left(B^*\upharpoonright \mathfrak{H}_{m,0}\right)\right)^* \in \mathbf{L}(\mathfrak{D}_{\Gamma_{m-1}}, \mathfrak{H}_{m,0}), \\ C_m &= D^{-1}_{\Gamma^*_{m-1}} \cdots D^{-1}_{\Gamma^*_0}(CA^m) \in \mathbf{L}(\mathfrak{H}_{m,0}, \mathfrak{D}_{\Gamma^*_{m-1}}),\ A_{m,0} \in \mathbf{L}(\mathfrak{H}_{m,0}, \mathfrak{H}_{m,0}). \end{aligned}$$

Hence

$$B_m^* = D^{-1}_{\Gamma_{m-1}} \cdots D^{-1}_{\Gamma_0}\left(B^*\upharpoonright \mathfrak{H}_{m,0}\right) \in \mathbf{L}(\mathfrak{H}_{m,0}, \mathfrak{D}_{\Gamma_{m-1}}).$$

The first Schur iterate of $\Theta_m(\lambda)$ is the function $\Theta_{m+1}(\lambda) \in \mathbf{S}(\mathfrak{D}_{\Gamma_m}, \mathfrak{D}_{\Gamma^*_m})$ and the first Schur parameter of Θ_m is Γ_{m+1}. From (3.4) and (3.9) it follows that

$$\ker D_{A_{m,0}} = \mathfrak{H}_{m+1,0},\ (A_{m,0})_{1,0} = A_{m+1,0} \in \mathbf{L}(\mathfrak{H}_{m+1,0}, \mathfrak{H}_{m+1,0}).$$

Hence by (6.8), and (6.9)

$$\Gamma_{m+1} = D^{-1}_{\Gamma^*_m} C_m \left(D^{-1}_{\Gamma_m} B^*_m\right)^* = D^{-1}_{\Gamma^*_m} \cdots D^{-1}_{\Gamma^*_0} CA^m \left(D^{-1}_{\Gamma_m} \cdots D^{-1}_{\Gamma_0}\left(B^*\upharpoonright \mathfrak{H}_{m,0}\right)\right)^*$$

and the system

$$\tau_{m+1}^{(0)} = \left\{ \begin{bmatrix} \Gamma_{m+1} & D^{-1}_{\Gamma^*_m} \cdots D^{-1}_{\Gamma^*_0}(CA^{m+1}) \\ \left(D^{-1}_{\Gamma_m} \cdots D^{-1}_{\Gamma_0}\left(B^*\upharpoonright \mathfrak{H}_{m+1,0}\right)\right)^* & A_{m+1,0} \end{bmatrix}; \mathfrak{D}_{\Gamma_m}, \mathfrak{D}_{\Gamma^*_m}, \mathfrak{H}_{m+1,0} \right\}$$

is a simple conservative realization of Θ_{m+1}. From Theorem 3.1 it follows that the systems

$$\tau_{m+1}^{(k)} = \left\{ \begin{bmatrix} \Gamma_m & D_{\Gamma_m^*}^{-1}\cdots D_{\Gamma_0^*}^{-1}(CA^{m+1-k}) \\ A^k\left(D_{\Gamma_m}^{-1}\cdots D_{\Gamma_0}^{-1}(B^*\restriction \mathfrak{H}_{m+1,0})\right)^* & A_{m+1-k,k} \end{bmatrix}; \right.$$
$$\left. \mathfrak{D}_{\Gamma_m}, \mathfrak{D}_{\Gamma_m^*}, \mathfrak{H}_{m+1-k,k} \right\}$$

are unitarily equivalent to the system $\tau_{m+1}^{(0)}$ for $k=1,\dots,m+1$ and hence their transfer functions are equal to Θ_{m+1}. This completes the proof.

Let us make a few remarks which follow from (4.9), Proposition 4.5, Theorem 6.5, and Remark 6.6.

If $D_{\Gamma_N}=0$ and $D_{\Gamma_N^*}\neq 0$, then $\mathfrak{D}_{\Gamma_n}=0$, $\Gamma_n^*=0\in \mathbf{L}(\mathfrak{D}_{\Gamma_N^*},\{0\})$, $\mathfrak{D}_{\Gamma_n^*}=\mathfrak{D}_{\Gamma_N^*}$, and $\mathfrak{H}_{0,n}=\mathfrak{H}_{0,N}$ for $n\geqslant N$. The conservative systems τ_0, $\tau_n^{(k)}$, for $n=1,\dots,N$, and $k=0,1,\dots,n$, are observable. The unitarily equivalent observable conservative systems $\tau_N^{(k)}$ are of the form

$$\tau_N^{(k)} = \left\{ \begin{bmatrix} \Gamma_N & D_{\Gamma_{N-1}^*}^{-1}\cdots D_{\Gamma_0^*}^{-1}(CA^{N-k}) \\ 0 & A_{N-k,k} \end{bmatrix}; \mathfrak{D}_{\Gamma_{N-1}}, \mathfrak{D}_{\Gamma_{N-1}^*}, \mathfrak{H}_{N-k,k} \right\},\ k=0,1,\dots,N,$$

have transfer functions $\Theta_N(\lambda)=\Gamma_N$ and the operators $A_{N-k,k}$ are unitarily equivalent co-shifts of multiplicity $\dim \mathfrak{D}_{\Gamma_N^*}$, the Schur iterates Θ_n are null operators from $\mathbf{L}(\{0\},\mathfrak{D}_{\Gamma_N^*})$ for $n\geqslant N+1$ and are transfer functions of the conservative observable system

$$\tau_{N+1} = \left\{ \begin{bmatrix} 0 & D_{\Gamma_{N-1}^*}^{-1}\cdots D_{\Gamma_0^*}^{-1}C \\ 0 & A_{0,N} \end{bmatrix}; \{0\}, \mathfrak{D}_{\Gamma_N^*}, \mathfrak{H}_{0,N} \right\}.$$

If $D_{\Gamma_N^*}=0$ and $D_{\Gamma_N}\neq 0$, then $\mathfrak{D}_{\Gamma_n^*}=0$, $\mathfrak{D}_{\Gamma_n}=\mathfrak{D}_{\Gamma_N}$, and $\Gamma_n=0\in\mathbf{L}(\mathfrak{D}_{\Gamma_N},\{0\})$, $\mathfrak{H}_{n,0}=\mathfrak{H}_{N,0}$ for $n\geqslant N$. The conservative systems τ_0, $\tau_n^{(k)}$, for $n=1,\dots,N$, and $k=0,1,\dots,n$, are controllable. The unitarily equivalent controllable conservative systems $\tau_N^{(k)}$ are of the form

$$\tau_N^{(k)} = \left\{ \begin{bmatrix} \Gamma_N & 0 \\ A^k\left(D_{\Gamma_{N-1}}^{-1}\cdots D_{\Gamma_0}^{-1}(B^*\restriction\mathfrak{H}_{N,0})\right)^* & A_{N-k,k} \end{bmatrix}; \mathfrak{D}_{\Gamma_{N-1}}, \mathfrak{D}_{\Gamma_{N-1}^*}, \mathfrak{H}_{N-k,k} \right\},$$
$$k=0,1,\dots,N,$$

have transfer functions $\Theta_N(\lambda)=\Gamma_N$, and the operators $A_{N-k,k}$ are unitarily equivalent unilateral shifts of multiplicity $\dim\mathfrak{D}_{\Gamma_N}$, the Schur iterates Θ_n are null operators from $\mathbf{L}(\mathfrak{D}_{\Gamma_N},\{0\})$ for $n\geqslant N+1$, and are transfer functions of the conservative controllable system

$$\tau_{N+1} = \left\{ \begin{bmatrix} 0 & 0 \\ \left(D_{\Gamma_N}^{-1}\cdots D_{\Gamma_0}^{-1}(B^*\restriction\mathfrak{H}_{N+1,0})\right)^* & A_{N,0} \end{bmatrix}; \mathfrak{D}_{\Gamma_N}, \{0\}, \mathfrak{H}_{N,0} \right\}.$$

If the operator Γ_N is unitary, then the conservative systems τ_0, $\tau_n^{(k)}$, for $n = 1, \ldots, N$, and $k = 0, 1, \ldots, n$, are controllable and observable. The unitarily equivalent controllable and observable conservative systems $\tau_N^{(k)}$ are of the form

$$\tau_N^{(k)} = \left\{ \begin{bmatrix} \Gamma_N & 0 \\ 0 & A_{N-k,k} \end{bmatrix}; \mathfrak{D}_{\Gamma_{N-1}}, \mathfrak{D}_{\Gamma^*_{N-1}}, \mathfrak{H}_{N-k,k} \right\},\ k = 0, 1, \ldots, N,$$

and $\mathfrak{H}_{N+1-k,k} = \{0\}$ for $k = 0, 1, \ldots, N+1$.

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