

## DETERMINANT COMPUTATIONS FOR SOME CLASSES OF TOEPLITZ–HANKEL MATRICES

ESTELLE L. BASOR\* AND TORSTEN EHRHARDT

(Communicated by A. Böttcher)

*Abstract.* The purpose of this paper is to compute the asymptotics of determinants of finite sections of operators that are trace class perturbations of Toeplitz operators. For example, we consider the asymptotics in the case where the matrices are of the form  $(a_{i-j} \pm a_{i+j+1-k})_{i,j=0 \dots N-1}$  with  $k$  fixed. We will show that this example as well as some general classes of operators have expansions that are similar to those that appear in the Strong Szegő Limit Theorem. We also obtain exact identities for some of the determinants that are analogous to the one derived independently by Geronimo and Case and by Borodin and Okounkov for finite Toeplitz matrices. These problems were motivated by certain statistical quantities that appear in random matrix theory.

### 1. Introduction

There is a fundamental connection between determinants of certain matrices and random matrix ensembles. For example, the Circular Unitary Ensemble (CUE) is the set of  $N \times N$  unitary matrices along with the Haar measure as the probability measure. The probability density function of the distribution for the eigenvalues  $e^{i\theta_1}, \dots, e^{i\theta_N}$  of the unitary matrices turns out to be a constant times

$$\prod_{j < k} |e^{i\theta_j} - e^{i\theta_k}|^2.$$

Given a function  $f$ , a linear statistic for this ensemble is a random variable of the form

$$X_f = \sum_{j=1}^N f(e^{i\theta_j}).$$

This quantity is connected to a Toeplitz determinant. More precisely,

$$\frac{1}{(2\pi)^N N!} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \prod_{j=1}^N e^{i\lambda f(e^{i\theta_j})} \prod_{j < k} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 \dots d\theta_N$$

---

*Mathematics subject classification* (2000): 47B35, 47B10, 82B.

*Keywords and phrases:* Toeplitz operator, Hankel operator, determinant asymptotics, random matrix theory.

\* Supported in part by NSF Grant DMS-0500892.

is identically equal to

$$\det \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda f(e^{i\theta})} e^{-i(j-k)\theta} d\theta \right)_{j,k=0,\dots,N-1} .$$

In probability terms this means that the inverse Fourier transform of the probability density of  $X_f$  is a Toeplitz determinant. In the opposite sense, the Toeplitz determinant can be thought of as an average or expectation value with respect to CUE. For a proof of this and for more general facts about random matrices, we refer the reader to [12, 13].

Thus the asymptotics of the determinant gives us information about the linear statistic. This is in particular the case when the function  $f$  is smooth enough, because we may then appeal to the Strong Szegő Limit Theorem to tell us asymptotically the behavior of the probability density function.

Let us recall this theorem. For general Toeplitz determinants we consider

$$\det (a_{j-k})_{j,k=0,\dots,N-1}$$

where  $a_k$  denotes the  $k$ th Fourier coefficients of some function  $a \in L^1(\mathbb{T})$ , and  $\mathbb{T}$  stands for the unit circle in the complex plane. Under appropriate conditions the Strong Szegő Limit Theorem (see, e.g., [4, 14]) states that

$$\det (a_{j-k})_{j,k=0,\dots,N-1} \sim (G[a])^N E[a] \tag{1}$$

as  $N \rightarrow \infty$ , where

$$G[a] = \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log a(e^{i\theta}) d\theta \right), \tag{2}$$

$$E[a] = \exp \left( \sum_{k=1}^{\infty} k s_k s_{-k} \right) \tag{3}$$

with  $s_k$  denoting the  $k$ th Fourier coefficient of  $\log a$ . The reader can check that in the case of linear statistics, where the function  $a$  is of the form  $e^{i\lambda f}$ , this implies that the probability distributions for linear statistics are asymptotically Gaussian ( $N \rightarrow \infty$ ).

It is also known that different types of random matrix ensembles lead to different classes of determinants. If one considers, for instance, averages for  $O^+(2N)$ , the set of orthogonal matrices with determinant equal to one (or, equivalently, the density function for certain linear statistics), then the corresponding determinant is that of a finite Toeplitz plus Hankel matrix. More specifically, it is of the form

$$\det (a_{j-k} + a_{j+k})_{j,k=0,\dots,N-1}$$

where the function  $a$  is assumed to be even. Because of this reason we are interested in the determinants of a sum of a finite Toeplitz plus a “certain type” of Hankel matrix. For other ensembles further kinds of determinants arise, namely

$$\det (a_{j-k} + a_{j+k+1})_{j,k=0,\dots,N-1}$$

and

$$\det (a_{j-k} - a_{j+k+2})_{j,k=0,\dots,N-1} .$$

We refer the interested reader to [1, 7] for derivations of the averages in the above cases and applications of the results in random matrix theory.

Our goal is to extend as much as possible the Strong Szegő Limit Theorem to these various types of determinants for both smooth and singular symbols. In this paper we address the case of smooth symbols.

An outline of the paper is as follows. In the next section we present some Banach algebra preliminaries and compute some operator determinants. Then we give some explicit examples of our general theory which correspond to the ones discussed earlier. We then return to the general setting and derive a Borodin-Okounkov-Geronimo-Case identity for the various classes of operators and establish the analogue of the Strong Szegő Limit Theorem. This will allow us to calculate the asymptotics of determinants of the form

$$(a_{i-j} \pm a_{i+j+1-k})_{i,j=0..N-1}$$

where  $k \in \mathbb{Z}$  is fixed. Finally, in the last section, we present some additional results about our classes of operators.

## 2. Compatible pairs and computation of operator determinants

We denote by  $\ell^2$  the space of all complex-valued square-summable sequences  $\{x_n\}_{n=0}^\infty$ . The set  $\mathcal{L}(\ell^2)$  is the set of all bounded linear operators on  $\ell^2$ . By  $\mathcal{C}_1(\ell^2)$  we denote the class of trace class operators on  $\ell^2$ . We refer to [11] for more information about trace class operators and the related notions of operator traces and determinants.

For  $a \in L^\infty(\mathbb{T})$  the Toeplitz operator  $T(a)$  and Hankel operator  $H(a)$  with symbol  $a$  are the bounded linear operator defined on  $\ell^2$  with matrix representations

$$T(a) = (a_{j-k}), \quad 0 \leq j, k < \infty,$$

and

$$H(a) = (a_{j+k+1}), \quad 0 \leq j, k < \infty.$$

It is well-known that Toeplitz and Hankel operators satisfy the fundamental identities

$$T(ab) = T(a)T(b) + H(a)H(\tilde{b}) \tag{4}$$

$$H(ab) = T(a)H(b) + H(a)T(\tilde{b}). \tag{5}$$

In the last two identities  $\tilde{b}(e^{i\theta}) = b(e^{-i\theta})$ . It is worthwhile to point out that these identities imply that

$$T(abc) = T(a)T(b)T(c), \quad H(ab\tilde{c}) = T(a)H(b)T(c) \tag{6}$$

for  $a, b, c \in L^\infty(\mathbb{T})$  if  $a_n = c_{-n} = 0$  for all  $n > 0$ .

The Riesz projection acting on  $L^p(\mathbb{T})$  ( $1 < p < \infty$ ) is defined by

$$P : \sum_{k=-\infty}^\infty a_k e^{ik\theta} \rightarrow \sum_{k=0}^\infty a_k e^{ik\theta}.$$

Finally, we introduce the notion of a *compatible pair*, which underlies our general theory in this paper.

Let  $\mathcal{S}$  stand for a unital Banach algebra of functions on the unit circle which is continuously embedded into  $L^\infty(\mathbb{T})$  and which has the following properties:

- (i) the Riesz projection  $P : \mathcal{S} \rightarrow \mathcal{S}$  is well-defined and bounded on  $\mathcal{S}$ ,
- (ii) the symmetric flip  $a \in \mathcal{S} \mapsto \tilde{a} \in \mathcal{S}$  is well defined and bounded on  $\mathcal{S}$ .

Then we can define

$$\mathcal{S}_\pm = \left\{ a \in \mathcal{S} : a_n = 0 \text{ for all } (\pm n) < 0 \right\}, \tag{7}$$

$$\mathcal{S}_0 = \left\{ a \in \mathcal{S} : a = \tilde{a} \right\}. \tag{8}$$

Moreover, we can make the following basic observations. Each  $a \in \mathcal{S}$  can be decomposed into  $a = a_+ + a_-$  with  $a_\pm \in \mathcal{S}_\pm$ . The decomposition can be made unique by requiring that  $[a_-]_0 = 0$ . Then the mappings  $a \mapsto a_\pm$  are linear and bounded.

Furthermore, each  $a \in \mathcal{S}$  can be decomposed into  $a = a_0 + a_-$  with  $a_0 \in \mathcal{S}_0$  and  $a_- \in \mathcal{S}_-$ . Indeed, this decomposition can be derived from the previous one by writing

$$a = a_+ + a_- = (a_+ + \tilde{a}_+) + (a_- - \tilde{a}_+)$$

and taking  $a_0 = a_+ + \tilde{a}_+$  and  $a_- - \tilde{a}_+$  as the new  $a_-$ . Again, we can make the decomposition unique by requiring  $[a_-]_0 = 0$ . The corresponding projections are bounded.

A pair  $[M, \mathcal{S}]$  will be called a *compatible pair* if  $\mathcal{S}$  is a Banach algebra with the properties described above and if  $M : a \in \mathcal{S} \mapsto M(a) \in \mathcal{L}(\ell^2)$  is a linear and continuous map such that the following conditions are fulfilled:

- (a) If  $a \in \mathcal{S}$ , then  $M(a) - T(a) \in \mathcal{C}_1(\ell^2)$  and

$$\|M(a) - T(a)\|_{\mathcal{C}_1(\ell^2)} \leq C \|a\|_{\mathcal{S}}.$$

- (b) If  $a \in \mathcal{S}_-, b \in \mathcal{S}, c \in \mathcal{S}_0$ , then

$$M(abc) = T(a)M(b)M(c).$$

- (c)  $M(1) = I$ .

We will refer to  $a$  as the symbol of  $M(a)$ .

Let us remark that, assuming (c), condition (b) is equivalent to the conditions that

$$M(ab) = T(a)M(b), \quad M(bc) = M(b)M(c) \tag{9}$$

whenever  $a \in \mathcal{S}_-, b \in \mathcal{S}, c \in \mathcal{S}_0$ .

PROPOSITION 2.1. *Let  $[M, \mathcal{S}]$  be a compatible pair. Then*

- (i)  $H(a)H(b) \in \mathcal{C}_1(\ell^2)$  for each  $a, b \in \mathcal{S}$ , and there is a constant  $C$  such that

$$\|H(a)H(b)\|_{\mathcal{C}_1(\ell^2)} \leq C \|a\|_{\mathcal{S}} \|b\|_{\mathcal{S}} \quad \text{for each } a, b \in \mathcal{S},$$

- (ii) if  $a$  is invertible in  $\mathcal{S}$ , then  $T(a^{-1})M(a) - I$  and  $M(a)T(a^{-1}) - I$  are both in  $\mathcal{C}_1(\ell^2)$ .

*Proof.* (i): We first assume that  $b = \tilde{b}$ . By assumption (a) each of the operators

$$(M(a) - T(a))M(b), \quad T(a)(M(b) - T(b)), \quad T(ab) - M(ab)$$

is trace class. If we add these three operators together and use that  $M(ab) = M(a)M(b)$ , which follows from property (b), we obtain

$$T(ab) - T(a)T(b) = H(a)H(b),$$

which is trace class. With a more careful inspection we can derive the norm estimate

$$\|H(a)H(b)\|_{\mathcal{C}_1(\ell^2)} \leq C\|a\|_{\mathcal{S}}\|b\|_{\mathcal{S}}.$$

In general we write  $b = b_0 + b_-$  with  $b_0 \in \mathcal{S}_0$  and  $b_- \in \mathcal{S}_-$ . Then  $H(b) = H(b_0)$  and the result follows. The norm estimate also holds because, in particular, the map  $b \mapsto b_0$  is bounded.

(ii): Assume  $a$  is invertible. Then

$$T(a^{-1})(M(a) - T(a))$$

is trace class, and by the first part

$$T(a^{-1})T(a) - I = -H(a^{-1})H(\tilde{a})$$

is also trace class. Hence the sum  $T(a^{-1})M(a) - I$  is trace class. The proof for  $M(a)T(a^{-1}) - I$  is similar.  $\square$

Statement (i) of the previous proposition implies that if  $[M, \mathcal{S}]$  is a compatible pair, then  $\mathcal{S}$  is a suitable Banach algebra in the sense of [6]. It has been shown there that in such a setting the Strong Szegő Limit Theorem, i.e., the asymptotics (1), holds for symbols  $a = e^b$  with  $b \in \mathcal{S}$ .

Let  $B$  be a Banach algebra. In what follows we employ the notion of an analytic  $B$ -valued function. The definition of differentiability involves the appropriate norm. Despite this, we have the fact that an analytic  $\mathcal{L}(H)$ -valued function whose values are trace class operators is an analytic  $\mathcal{C}_1(H)$ -valued function. We refer to [10] for more details.

**PROPOSITION 2.2.** *Let  $[M, \mathcal{S}]$  be a compatible pair. Then for each  $a \in \mathcal{S}$*

$$F_1(\lambda) = T(e^{-\lambda a})M(e^{\lambda a}) - I \quad \text{and} \quad F_2(\lambda) = T^{-1}(e^{\lambda a})M(e^{\lambda a}) - I$$

*are analytic  $\mathcal{C}_1(\ell^2)$ -valued functions.*

*Proof.* The first function is obviously an analytic  $\mathcal{L}(\ell^2)$ -valued function. It is trace class valued because of (ii) of the previous proposition.

In order to consider the second function, we decompose  $a = a_+ + a_-$  with  $a_{\pm} \in \mathcal{S}_{\pm}$ . From this we derive (using (6)) that

$$T(e^{\lambda a}) = T(e^{\lambda a_-})T(e^{\lambda a_+}).$$

Hence the inverse exists and is given by

$$T(e^{\lambda a})^{-1} = T(e^{-\lambda a_+})T(e^{-\lambda a_-}). \tag{10}$$

This shows that also the second function is well defined and  $\mathcal{L}(\ell^2)$ -valued analytic.

From assumption (a) of the definition of a compatible pair it is easy to conclude that it is trace class valued.  $\square$

In the following two propositions we are going to compute some operator determinants. They will appear later as constants in our asymptotic relations. We use the following well known facts. If  $F(\lambda)$  is an analytic function of the form identity plus trace class, then its determinant  $\det F(\lambda)$  is well defined and an analytic function. Moreover,

$$(\log \det F(\lambda))' = \frac{(\det F(\lambda))'}{\det F(\lambda)} = \text{trace } F'(\lambda)F^{-1}(\lambda) = \text{trace } F^{-1}(\lambda)F'(\lambda).$$

The proof of the following propositions is similar to the proof of, for instance, [2, Thm. 2.5] and [6, Thm. 7.4], where more details are given.

PROPOSITION 2.3. *Let  $[M, \mathcal{S}]$  be a compatible pair. Then for  $a \in \mathcal{S}_0$ ,*

$$\det T(e^{-a})M(e^a) = \exp \left( \text{trace}(M(a) - T(a)) + \frac{1}{2}\text{trace } H(a)^2 \right). \quad (11)$$

*Proof.* Define the entire function

$$f(\lambda) := \det T(e^{-\lambda a})M(e^{\lambda a}).$$

Now consider the logarithmic derivative of  $f(\lambda)$ ,

$$\begin{aligned} \frac{f'(\lambda)}{f(\lambda)} &= \text{trace} \left( M(e^{-\lambda a})T^{-1}(e^{-\lambda a}) \right) \left( T(e^{-\lambda a})M(ae^{\lambda a}) - T(ae^{-\lambda a})M(e^{\lambda a}) \right) \\ &= \text{trace} \left( M(a) - T^{-1}(e^{-\lambda a})T(ae^{-\lambda a}) \right). \end{aligned}$$

Differentiating again yields

$$\begin{aligned} \left( \frac{f'(\lambda)}{f(\lambda)} \right)' &= \text{trace} \left( -T^{-1}(e^{-\lambda a})T(ae^{-\lambda a})T^{-1}(e^{-\lambda a})T(ae^{-\lambda a}) + T^{-1}(e^{-\lambda a})T(a^2e^{-\lambda a}) \right) \\ &= \text{trace} \left( -T(a)T(a) + T(a^2) \right) = \text{trace } H(a)^2. \end{aligned}$$

The last equality holds by writing  $a = a_- + a_+$  with  $a_{\pm} \in \mathcal{S}_{\pm}$  and considering the inverse of  $T(e^{-\lambda a})$  as in (10). Integration and fixing the constants by putting  $\lambda = 0$  yields

$$f(\lambda) = \exp \left( \lambda \text{trace}(M(a) - T(a)) + \frac{\lambda^2}{2}\text{trace } H(a)^2 \right).$$

This finishes the proof.  $\square$

PROPOSITION 2.4. *Let  $[M, \mathcal{S}]$  be a compatible pair. Then for  $a \in \mathcal{S}$ ,*

$$\det T^{-1}(e^a)M(e^a) = \exp\left(\text{trace}(M(a) - T(a)) - \frac{1}{2}\text{trace}H(a)^2\right). \quad (12)$$

*Proof.* We can decompose  $a = a_0 + a_-$  with  $a_- \in \mathcal{S}_-$ ,  $a_0 \in \mathcal{S}_0$ . Using (6) it is easy to see that

$$f(\lambda) := \det T^{-1}(e^{\lambda a})M(e^{\lambda a}) = \det T^{-1}(e^{\lambda a_0})M(e^{\lambda a_0}).$$

Now consider the logarithmic derivative of  $f(\lambda)$ ,

$$\begin{aligned} \frac{f'(\lambda)}{f(\lambda)} &= \text{trace}\left(M(e^{-\lambda a_0})T(e^{\lambda a_0})\right) \\ &\quad \times \left(T^{-1}(e^{\lambda a_0})M(a_0 e^{\lambda a_0}) - T^{-1}(e^{\lambda a_0})T(a_0 e^{\lambda a_0})T^{-1}(e^{\lambda a_0})M(e^{\lambda a_0})\right) \\ &= \text{trace}\left(M(a_0) - T(a_0 e^{\lambda a_0})T^{-1}(e^{\lambda a_0})\right). \end{aligned}$$

Differentiating again yields

$$\begin{aligned} \left(\frac{f'(\lambda)}{f(\lambda)}\right)' &= \text{trace}\left(T(a_0 e^{\lambda a_0})T^{-1}(e^{\lambda a_0})T(a_0 e^{\lambda a_0})T^{-1}(e^{\lambda a_0}) - T(a_0^2 e^{\lambda a_0})T^{-1}(e^{\lambda a_0})\right) \\ &= \text{trace}\left(T(a_0)T(a_0) - T(a_0^2)\right) = -\text{trace}H(a_0)^2. \end{aligned}$$

Integration and fixing the constants by putting  $\lambda = 0$  yields

$$f(\lambda) = \exp\left(\lambda \text{trace}(M(a_0) - T(a_0)) - \frac{\lambda^2}{2}\text{trace}H(a_0)^2\right).$$

This implies the desired assertion by noting that  $H(a_-) = 0$  and  $M(a_-) = T(a_-)$  by parts (b) and (c) of the definition of a compatible pair.  $\square$

### 3. Concrete realizations of compatible pairs

While the above formulas are nice, it remains to show that there are some interesting classes of operators that satisfy the Banach algebra conditions as well as the algebraic conditions of the previous section. That is, we need to show that there are some compatible pairs. We would also like to have operators that correspond to the random matrix examples that were stated in the introduction. The purpose of this section is to introduce these examples, i.e., concrete realizations. We need to specify the Banach algebra  $\mathcal{S}$  and to identify the operators  $M(a)$ .

For our compatible pairs, it is convenient to take as Banach algebra the Besov class  $B_1^1$ . This is the algebra of all functions  $a$  defined on the unit circle for which

$$\|a\|_{B_1^1} := \int_{-\pi}^{\pi} \frac{1}{y^2} \int_{-\pi}^{\pi} |a(e^{ix+iy}) + a(e^{ix-iy}) - 2a(e^{ix})| dx dy < \infty.$$

A function  $a$  is in  $B_1^1$  if and only if the Hankel operators  $H(a)$  and  $H(\tilde{a})$  are both trace class. Moreover, the Riesz projection is bounded on  $B_1^1$ , and an equivalent norm is given by

$$|a_0| + \|H(a)\|_{\mathcal{S}_1} + \|H(\tilde{a})\|_{\mathcal{S}_1}.$$

A proof of these facts can be found in [8, 9]. Clearly, the symmetric flip  $a \mapsto \tilde{a}$  is bounded on  $B_1^1$ . Hence  $\mathcal{S} = B_1^1$  satisfies the Banach algebra conditions (i) and (ii) in the definition of a compatible pair.

In the following propositions we define four concrete realizations for  $M(a)$ . We only need to check that the algebraic conditions (a)–(c) are fulfilled. Introduce the projections

$$P_1 = \text{diag}(1, 0, 0, \dots), \quad Q_1 = I - P_1$$

acting on  $\ell^2$ .

PROPOSITION 3.1. *The following realizations for the operator  $M$  with symbols in the Besov class  $B_1^1$  define compatible pairs  $[M, B_1^1]$ :*

- (I)  $M(a) = T(a) + H(a)$
- (II)  $M(a) = T(a) - H(a)$
- (III)  $M(a) = T(a) - H(t^{-1}a)$
- (IV)  $M(a) = T(a) + H(ta)Q_1$

*Proof.* It is easy to see that conditions (a) and (c) are satisfied. Hence we focus on (b). Taking into account the remark made in connection with (9) we have to show that

$$M(ab) = M(a)M(b)$$

under the condition  $b = \tilde{b}$ . In order to verify the cases (I) and (II) use (4) and (5) to obtain

$$\begin{aligned} T(ab) \pm H(ab) &= T(a)T(b) + H(a)H(b) \pm T(a)H(b) \pm H(a)T(a) \\ &= \left(T(a) \pm H(a)\right) \left(T(b) \pm H(b)\right) \end{aligned}$$

as desired. In case (III), use in addition (6) to obtain

$$\begin{aligned} M(a)M(b) &= \left(T(a) - H(t^{-1}a)\right) \left(T(b) - H(t^{-1}b)\right) \\ &= T(a)T(b) + H(t^{-1}a)H(t^{-1}b) - H(t^{-1}a)T(b) - T(a)H(t^{-1}b) \\ &= T(ab) - H(a)P_1H(b) - H(t^{-1}ab) + T(t^{-1}a)H(b) - T(a)T(t^{-1})H(b) \\ &= T(ab) - H(t^{-1}ab) - H(a)P_1H(b) + H(a)H(t)H(b) \\ &= T(ab) - H(t^{-1}ab). \end{aligned}$$

Here  $P_1 = H(t) = H(t)^2 = I - Q_1$  and  $Q_1 = T(t)T(t^{-1})$ . In case (IV) we have

$$\begin{aligned}
 M(a)M(b) &= \left(T(a) + H(ta)Q_1\right)\left(T(b) + H(tb)Q_1\right) \\
 &= T(a)T(b) + H(ta)Q_1H(tb)Q_1 + H(ta)Q_1T(b) + T(a)H(tb)Q_1 \\
 &= T(a)T(b) + H(a)H(b)Q_1 + H(a)T(t^{-1}b) + T(a)H(tb)Q_1 \\
 &= T(ab) - H(a)H(b)P_1 + H(a)T(t^{-1}b) + H(tab)Q_1 - H(a)T(t^{-1}b)Q_1 \\
 &= T(ab) + H(a)T(b)T(t^{-1}) + H(tab)Q_1 - H(a)T(t^{-1}b)Q_1 \\
 &= T(ab) + H(tab)Q_1.
 \end{aligned}$$

This settles the proof. □

Let us remark that the operators (I)-(III) are precisely the infinite matrix versions of the finite Toeplitz plus Hankel matrices mentioned in the introduction. It is also easily seen that if we multiply the operator (IV) from the right with  $\text{diag}(2, 1, 1, \dots)$ , then we obtain  $T(a) + H(ta)$ . Finally notice the simple fact that the operators (I) and (II) are related with one another by multiplying from the left and right with  $\text{diag}(1, -1, 1, -1, \dots)$  and replacing the symbol  $a(t)$  by  $a(-t)$ .

PROPOSITION 3.2. *Let  $a \in B_1^1$  and denote*

$$F[a] = \det T^{-1}(a)M(a) \tag{13}$$

where we assume that there exists a logarithm  $\log a \in B_1^1$ . Then in the above cases (I)–(IV) the corresponding constants evaluate as follows:

$$\begin{aligned}
 F_I[a] &= \exp\left(\sum_{n=0}^{\infty} [\log a]_{2n+1} - \frac{1}{2} \sum_{n=1}^{\infty} n [\log a]_n^2\right) \\
 F_{II}[a] &= \exp\left(-\sum_{n=0}^{\infty} [\log a]_{2n+1} - \frac{1}{2} \sum_{n=1}^{\infty} n [\log a]_n^2\right) \\
 F_{III}[a] &= \exp\left(-\sum_{n=1}^{\infty} [\log a]_{2n} - \frac{1}{2} \sum_{n=1}^{\infty} n [\log a]_n^2\right) \\
 F_{IV}[a] &= \exp\left(\sum_{n=1}^{\infty} [\log a]_{2n} - \frac{1}{2} \sum_{n=1}^{\infty} n [\log a]_n^2\right)
 \end{aligned}$$

*Proof.* We only need to note that  $\text{trace } H(\log a)^2$  is  $\sum_{n=1}^{\infty} n [\log a]_n^2$  and check that, for example,  $\text{trace } H(\log a) = \sum_{n=0}^{\infty} [\log a]_{2n+1}$ . □

The proof of the following proposition is almost the same as above.

PROPOSITION 3.3. *Let  $a \in B_1^1$  and denote*

$$\hat{F}[a] = \det T(a^{-1})M(a) \tag{14}$$

where we assume that there exists a logarithm  $\log a \in B_1^1$  and  $a = \tilde{a}$ . Then in the above cases (I)–(IV) the corresponding constants evaluate as follows:

$$\begin{aligned} \hat{F}_I[a] &= \exp\left(\sum_{n=0}^{\infty} [\log a]_{2n+1} + \frac{1}{2} \sum_{n=1}^{\infty} n [\log a]_n^2\right) \\ \hat{F}_{II}[a] &= \exp\left(-\sum_{n=0}^{\infty} [\log a]_{2n+1} + \frac{1}{2} \sum_{n=1}^{\infty} n [\log a]_n^2\right) \\ \hat{F}_{III}[a] &= \exp\left(-\sum_{n=1}^{\infty} [\log a]_{2n} + \frac{1}{2} \sum_{n=1}^{\infty} n [\log a]_n^2\right) \\ \hat{F}_{IV}[a] &= \exp\left(\sum_{n=1}^{\infty} [\log a]_{2n} + \frac{1}{2} \sum_{n=1}^{\infty} n [\log a]_n^2\right) \end{aligned}$$

**4. Exact identities for some determinants**

In this section we establish some exact identities for the finite sections of the operators considered in the previous section. These are of the Borodin/Okounkov/Geronimo /Case type and with these the asymptotics of the determinants will easily follow. For the Toeplitz analogue of this theorem see [3]. We define the projection  $P_N$  by

$$P_N : \{x_n\}_{n=0}^{\infty} \in \ell^2 \mapsto \{y_n\}_{n=0}^{\infty} \in \ell^2, \quad y_n = \begin{cases} x_n & \text{if } n < N \\ 0 & \text{if } n \geq N \end{cases}$$

and put  $Q_N = I - P_N$ . We are interested in the determinants (where the matrices or operators are always thought of as acting on the image of the projection of the appropriate space) of

$$P_N M(a) P_N.$$

We first take the case of even  $a$ . Recall the definition of the constant  $G[a]$  given in (2).

**PROPOSITION 4.1.** *Let  $[M, \mathcal{S}]$  be a compatible pair, and let  $b_+ \in \mathcal{S}_+$ . Put  $a = a_+ \tilde{a}_+ = \exp(b)$  with  $a_+ = \exp(b_+)$ ,  $b = b_+ + \tilde{b}_+$ . Then*

$$\det P_N M(a) P_N = G[a]^N \hat{F}[a] \det(I + Q_N K Q_N),$$

where

$$\hat{F}[a] = \det T(a^{-1})M(a) = \exp\left(\text{trace}(M(b) - T(b)) + \frac{1}{2} \text{trace } H(b)^2\right),$$

and  $K = M(a_+^{-1})T(a_+) - I$ .

*Proof.* We can write

$$\begin{aligned} P_N M(a) P_N &= P_N M(a) P_N \\ &= P_N T(a_+) T(a_+^{-1}) M(a) T(\tilde{a}_+) T(\tilde{a}_+^{-1}) P_N \\ &= P_N T(a_+) P_N T(a_+^{-1}) M(a) T(\tilde{a}_+^{-1}) P_N T(\tilde{a}_+) P_N. \end{aligned}$$

The last fact follows since for Toeplitz operators

$$P_N T(a_+) = P_N T(a_+) P_N, \quad T(\tilde{a}_+) P_N = P_N T(\tilde{a}_+) P_N.$$

At this point we have that

$$\begin{aligned} \det P_N M(a) P_N &= \det(P_N T(a_+) P_N T(a_+^{-1}) M(a) T(\tilde{a}_+^{-1}) P_N T(\tilde{a}_+) P_N) \\ &= \det(P_N T(a_+) P_N) \cdot \det(P_N T(\tilde{a}_+) P_N) \cdot \det P_N T(a_+^{-1}) M(a) T(\tilde{a}_+^{-1}) P_N \\ &= [a_+]_0^N \cdot [\tilde{a}_+]_0^N \cdot \det P_N T(a_+^{-1}) M(a) T(\tilde{a}_+^{-1}) P_N. \end{aligned}$$

First it is not hard to check that  $[a_+]_0 \cdot [\tilde{a}_+]_0 = G[a]$ . Now Jacobi's identity for invertible operators on Hilbert space which are of the form identity plus trace class operators states that for projections  $P$  and  $Q = I - P$  we have

$$\det PAP = (\det A) \cdot (\det QA^{-1}Q).$$

We apply this to the above with  $P = P_N$ ,  $Q = I - P_N$ , and  $A = T(a_+^{-1})M(a)T(\tilde{a}_+^{-1})$  to find that

$$\begin{aligned} \det P_N T(a_+^{-1}) M(a) T(\tilde{a}_+^{-1}) P_N &= \det T(a_+^{-1}) M(a) T(\tilde{a}_+^{-1}) \\ &\quad \times \det Q_N (T(a_+^{-1}) M(a) T(\tilde{a}_+^{-1}))^{-1} Q_N. \end{aligned}$$

To simplify the last two determinants we note that

$$\det T(a_+^{-1}) M(a) T(\tilde{a}_+^{-1}) = \det T(\tilde{a}_+^{-1}) T(a_+^{-1}) M(a) = \det T(a^{-1}) M(a) = \hat{F}[a]$$

and use Proposition 2.3. Moreover,

$$(T(a_+^{-1}) M(a) T(\tilde{a}_+^{-1}))^{-1} = T(\tilde{a}_+) M(a^{-1}) T(a_+) = M(a_+^{-1}) T(a_+)$$

which is also of the form  $I$  plus a trace class operator. We now put all these together,

$$\det P_N M(a) P_N = G[a]^N \hat{F}[a] \det Q_N M(a_+^{-1}) T(a_+) Q_N,$$

and make the observation that this last determinant is the same as

$$\det(P_N + Q_N M(a_+^{-1}) T(a_+) Q_N) = \det(I + Q_N K Q_N).$$

This proves the formula. □

Let us remark that the operator  $K$  appearing in the previous proposition becomes particularly simple in the cases of the four concrete realizations of operators  $M$  considered in the previous section. The precise expressions are as follows:

- (I)  $K = H(a_+^{-1} \tilde{a}_+)$
- (II)  $K = -H(a_+^{-1} \tilde{a}_+)$
- (III)  $K = -H(t^{-1} a_+^{-1} \tilde{a}_+)$
- (IV)  $K = H(t a_+^{-1} \tilde{a}_+) - T(a_+^{-1}) H(t \tilde{a}_+)$

As for case (IV) notice that the term  $T(a_+^{-1})H(\tilde{t}\tilde{a}_+)$  will be annihilated by multiplication with  $Q_N$  from the right ( $N \geq 1$ ).

The above proposition needs to be slightly changed for non-even functions  $a$ . We include the result for completeness sake, although in our applications  $a$  is always even.

**PROPOSITION 4.2.** *Let  $[M, \mathcal{S}]$  be a compatible pair, and let  $b_{\pm} \in \mathcal{S}_{\pm}$ . Put  $a = a_+a_- = \exp(b)$  with  $a_{\pm} = \exp(b_{\pm})$ ,  $b = b_+ + b_-$ . Then*

$$\det P_N M(a) P_N = G[a]^N E[a] F[a] \det(I + Q_N K Q_N),$$

where

$$\begin{aligned} F[a] &= \det T^{-1}(a)M(a) = \exp\left(\text{trace}(M(b) - T(b)) - \frac{1}{2}\text{trace}H(b)^2\right) \\ E[a] &= \det T(a^{-1})T(a) = \exp\left(\text{trace}H(b)H(\tilde{b})\right), \end{aligned}$$

and  $K = M(a_- a_+^{-1} \tilde{a}_+^{-1})T(a_+ \tilde{a}_+ a_-^{-1}) - I$ .

*Proof.* The only real difference in the proof is that we must replace  $\tilde{a}_+$  by  $a_-$ . The subsequent computations must be modified as follows. Firstly,

$$\det T(a_+^{-1})M(a)T(a_-^{-1}) = \det T(a_-^{-1})T(a_+^{-1})M(a) = \det T(a^{-1})M(a).$$

This we can write as the product

$$\det T(a^{-1})T(a) \cdot \det T(a)^{-1}M(a)$$

and use Proposition 2.4 to identify the second factor. The first factor is well known from Toeplitz theory [4, 14]. Since in our setting the Banach algebras  $\mathcal{S}$  is not specified, one way to settle the issue is to use the same ideas as in Propositions 2.3 and 2.4. Another possibility would be to apply a formula due to Pincus (see, e.g., [5]).

Secondly, one computes that

$$\begin{aligned} (T(a_+^{-1})M(a)T(a_-^{-1}))^{-1} &= (T(a_+^{-1})T(a_- \tilde{a}_+^{-1})M(a_+ \tilde{a}_+)T(a_-^{-1}))^{-1} \\ &= T(a_-)M(a_+^{-1} \tilde{a}_+^{-1})T(a_-^{-1} \tilde{a}_+)T(a_+) \\ &= M(a_- a_+^{-1} \tilde{a}_+^{-1})T(a_-^{-1} \tilde{a}_+ a_+). \end{aligned}$$

This is again identity plus trace class by Proposition 2.1. □

The above proposition immediately establishes an asymptotic formula for the determinants since the operators  $Q_N$  tend to zero strongly as do their adjoints. Thus we arrive at the final results of this section.

**THEOREM 4.3.** *Let  $[M, \mathcal{S}]$  be a compatible pair, let  $b \in \mathcal{S}$  and  $a = \exp(b)$ . Then*

$$\det P_N M(a) P_N \sim G[a]^N \hat{E}[a] \quad \text{as } N \rightarrow \infty,$$

where

$$\hat{E}[a] = \exp\left(\text{trace}(M(b) - T(b)) - \frac{1}{2}\text{trace}H(b)^2 + \text{trace}H(b)H(\tilde{b})\right).$$

It is clear that the formula for  $E[a]$  in the previous two theorem simplifies if  $b$  is assumed to be even and then correspond to Proposition 4.1. Also, in the case of the concrete realizations the traces of  $M(a) - T(a)$  are explicit in Proposition 3.2.

We remark here that in the examples of our concrete realizations and with the symbol  $e^{i\lambda f}$  these theorems tell us that the distributions of the linear statistics are all Gaussian since as a function of  $\lambda$  the transforms are exponentials of quadratic functions.

We end this section with an application of the above asymptotics which yield an expansion for determinants of finite sections of operators of the form  $T(a) \pm H(ar^k)$ . These operators are (for general  $k$ ) not the ones that yield a compatible pair realization, but using Jacobi's identity

$$\det PAP = (\det A) \cdot (\det QA^{-1}Q), \tag{15}$$

we can still compute the determinants of their finite sections asymptotically. We prepare with a basic result still relating to compatible pairs.

PROPOSITION 4.4. *Let  $[M, \mathcal{S}]$  be a compatible pair. Suppose that  $a \in \mathcal{S}$  such that  $\log a \in \mathcal{S}$ . Then there exists a unique factorization of the form*

$$a(t) = a_-(t)a_0(t) \tag{16}$$

such that  $a_-, a_-^{-1} \in \mathcal{S}_-, a_0, a_0^{-1} \in \mathcal{S}_0$ , and  $[a_-]_0 = 1$ . Moreover,  $M(a)$  is invertible and

$$M(a)^{-1} = M(a_0^{-1})T(a_-^{-1}). \tag{17}$$

*Proof.* We can decompose  $\log a$  into  $b_- + b_0$  with  $b_- \in \mathcal{S}_-, b_0 \in \mathcal{S}_0$ , and  $[b_-]_0 = 0$ . Then we simply put  $a_- = \exp(b_-)$  and  $a_0 = \exp(b_0)$  in order to obtain the factorization. Notice that  $\mathcal{S}_-$  and  $\mathcal{S}_0$  are unital Banach subalgebras of  $\mathcal{S}$ . To obtain the uniqueness of the factorization write

$$a = a_-^{(1)}a_0^{(1)} = a_-^{(2)}a_0^{(2)}$$

whence  $(a_-^{(2)})^{-1}a_-^{(1)} = a_0^{(2)}(a_0^{(1)})^{-1}$ . Apply the fact that the intersection of  $\mathcal{S}_-$  and  $\mathcal{S}_0$  are the constant functions only and use the normalization condition to conclude that the last products are in fact equal to one.

Using the factorization and the basic properties of  $M(a)$  we have  $M(a) = T(a_-)M(a_0)$  and

$$T(a_-)T(a_-^{-1}) = T(a_-^{-1})T(a_-) = I, \quad M(a_0)M(a_0^{-1}) = M(a_0^{-1})M(a_0) = I.$$

Hence the invertibility of  $M(a)$  follows. □

THEOREM 4.5. *Let  $a \in B_1^1$  such that  $\log a \in B_1^1$ . Assume that  $a = a_-a_0$  is a factorization of the form (16).*

1. Suppose that  $k$  is a negative even integer ( $k = -2l, l \geq 1$ ). Then

$$\det P_N(T(a) \pm H(at^k))P_N \sim G[a]^{N+l}E_{1,\pm}[a] \det P_l(T(a_0^{-1}) \pm H(a_0^{-1}))P_l$$

as  $N \rightarrow \infty$ , where

$$E_{1,\pm}[a] = \exp \left( \pm \sum_{n=1}^{\infty} \log a_{2n+1} - \frac{1}{2} \sum_{n=1}^{\infty} n[\log a]_n^2 + \sum_{n=1}^{\infty} n[\log a]_{-n}[\log a]_n \right).$$

2. Suppose that  $k$  is a negative odd integer less than  $-1$  ( $k = -1 - 2l, l \geq 1$ ). Then

$$\det P_N(T(a) - H(at^k))P_N \sim G[a]^{N+l}E_2[a] \det P_l(T(a_0^{-1}) - H(a_0^{-1}t^{-1}))P_l$$

as  $N \rightarrow \infty$ , where

$$E_2[a] = \exp \left( - \sum_{n=1}^{\infty} \log a_{2n} - \frac{1}{2} \sum_{n=1}^{\infty} n[\log a]_n^2 + \sum_{n=1}^{\infty} n[\log a]_{-n}[\log a]_n \right).$$

3. Suppose that  $k$  is a negative odd integer ( $k = 1 - 2l, l \geq 1$ ). Then

$$\det P_N(T(a) + H(at^k))P_N \sim G[a]^{N+l}E_3[a] \det P_l(T(a_0^{-1}) + H(a_0^{-1}t))P_l$$

as  $N \rightarrow \infty$ , where

$$E_3[a] = \exp \left( - \log 2 + \sum_{n=1}^{\infty} \log a_{2n} - \frac{1}{2} \sum_{n=1}^{\infty} n[\log a]_n^2 + \sum_{n=1}^{\infty} n[\log a]_{-n}[\log a]_n \right).$$

4. We have

$$\det P_N(T(a) + H(at^k))P_N = 0 \quad \text{if } N \geq k \geq 2,$$

$$\det P_N(T(a) - H(at^k))P_N = 0 \quad \text{if } N \geq k \geq 1.$$

*Proof. Case 1.* Consider the matrix  $P_N(T(a) \pm H(at^k))P_N$ . We observe that it is indeed the right bottom  $N \times N$  corner of the  $(N + l) \times (N + l)$  matrix

$$A_N = P_{N+l}(T(a) \pm H(a))P_{N+l}.$$

But this is the same as the matrix  $Q_l A_N Q_l$ . Using Jacobi's identity (15) with  $P = Q_l, Q = P_l$ , we obtain

$$\det(P_N(T(a) \pm H(at^k))P_N) = \det(P_l A_N^{-1} P_l) \cdot (\det A_N).$$

Each of these last two factors can be computed asymptotically. For the second we use Theorem 4.3 with  $M(a) = T(a) \pm H(a)$  and the results of Section 3 to conclude that  $\det A_N$  is asymptotically  $G[a]^{N+l}E_{1,\pm}[a]$ .

For the first we use the fact ([4], Theorem 7.20) that the inverses of the finite sections  $P_{N+l}(T(a) \pm H(a))P_{N+l}$  converge strongly to the inverse of  $M(a) = T(a) \pm H(a)$ .

Notice that  $M(a)$  is  $T(a)$  plus a compact operator and that  $M(a)$  is invertible. The inverse equals  $M(a)^{-1} = M(a_0^{-1})T(a_-^{-1})$ . Hence  $\det P_l A_N^{-1} P_l$  converges to

$$\det P_l M(a_0^{-1}) T(a_-^{-1}) P_l = \det P_l M(a_0^{-1}) P_l T(a_-^{-1}) P_l = \det P_l M(a_0^{-1}) P_l.$$

Here we use the basic fact that  $T(a_-^{-1}) P_l = P_l T(a_-^{-1}) P_l$  and the normalization  $[a_-]_0 = 1$ .

**Case 2.** The proof is nearly the same only that we now use

$$A_N = P_{N+l}(T(a) - H(at^{-1}))P_{N+l}$$

and  $M(a) = T(a) - H(at^{-1})$ .

**Case 3.** Here an additional modification must be made. We consider

$$A_N = P_{N+l}(T(a) + H(at))P_{N+l}$$

and using Jacobi's identity we can write

$$\det(P_N(T(a) - H(at^k))P_N) = \det(P_l A_N^{-1} P_l) \cdot (\det A_N).$$

Furthermore, we observe that

$$M(a) = T(a) + H(at)Q_1 = (T(a) + H(at))R$$

where

$$R = \text{diag}(1/2, 1, 1, 1, \dots).$$

This leads to

$$\det A_N = 2 \det(P_{N+l}M(a)P_{N+l}), \quad \det(P_l A_N^{-1} P_l) = \frac{1}{2} \det(P_l(P_{N+l}M(a)P_{N+l})^{-1} P_l).$$

The last determinant converges to

$$\det(P_l M^{-1}(a) P_l) = \det(P_l M(a_0^{-1}) P_l) = \frac{1}{2} \det(P_l(T(a_0^{-1}) + H(a_0^{-1}t))P_l).$$

For this reason we get an additional factor  $1/2$ .

**Case 4.** Observe that  $P_N(T(a) \pm H(at^k))P_N$  is given by the matrix

$$(a_{i-j} \pm a_{i+j-k+1})_{i,j=0 \dots N-1}.$$

Thus the first column ( $j = 0$ ) and the  $k$ th column ( $j = k - 1$ ) are given by  $a_i \pm a_{i-k+1}$  and  $a_{i-k+1} \pm a_i$ , respectively. Hence they are either equal or the negative of each other. This settles the statements in the case  $k \geq 2$ . The case  $k = 1$  with the "minus" is special. Then the first (=  $k$ th column) equals zero.  $\square$

### 5. The general form of $M(a)$

In this section we consider the question of how general the operator  $M(a)$  can be. Let us first state some implications of  $[M, \mathcal{S}]$  being a compatible pair. For simplicity, we will assume that  $\mathcal{S}$  contains the trigonometric polynomials as a dense subset.

Let us write

$$K(a) = M(a) - T(a).$$

Recall that the main property (b) for compatible pairs implies the two conditions stated in (9), i.e.,

$$M(ab) = T(a)M(b), \quad M(bc) = M(b)M(c)$$

whenever  $a \in \mathcal{S}_-, b \in \mathcal{S}, c \in \mathcal{S}_0$ . These conditions can be restated as

$$K(ab) = T(a)K(b) \tag{18}$$

and

$$K(bc) = K(b)K(c) + T(b)K(c) + K(b)T(c) - H(b)H(c) \tag{19}$$

for  $a \in \mathcal{S}_-, b \in \mathcal{S}, c \in \mathcal{S}_0$ . Condition (c) for compatible pairs implies  $K(1) = 0$ , and using (18) this shows that  $K(a) = 0$  whenever  $a \in \mathcal{S}_-$ . In particular,  $K(t^n) = 0$  for  $n \leq 0$ .

From (18) it also follows that  $T(t^{-1})K(t^{n+1}) = K(t^n)$ . Since  $T(t^{-1})$  is the backward shift operator and  $K(1) = 0$ , it follows that (with respect to the usual matrix representation)  $K(t^n)$  can have nonzero entries only in the first  $n$  rows. Furthermore, all but the first row of  $K(t^{n+1})$  can be obtained from the rows of  $K(t^n)$ . More specifically, we can conclude that there exist  $x_1, x_2, \dots \in \ell^2$  such that

$$K(t^n) = (x_n, x_{n-1}, \dots, x_1, 0, 0, \dots)^T,$$

where we think of the  $x_k$ 's and the 0's as infinite column vectors. Equivalently,

$$K(t^n) = e_0 x_n^T + e_1 x_{n-1}^T + \dots + e_{n-1} x_1^T, \tag{20}$$

where  $e_i = \{\delta_{i,k}\}_{k=0}^\infty$  ( $i \geq 0$ ) is the standard basis in  $\ell^2$ . Here  $yx^T$  stands for the rank one operator

$$z \in \ell^2 \mapsto \langle z, x \rangle \cdot y \in \ell^2.$$

The vectors  $x_k \in \ell^2$  can be obtained recursively from  $x_1$ . To see this notice that (for  $n \geq 1$ )

$$K(t^{n+1}) = K(t(t^n + t^{-n})),$$

which using (19) implies

$$K(t^{n+1}) = K(t)K(t^n) + T(t)K(t^n) + K(t)T(t^n + t^{-n}) - H(t)H(t^n).$$

Now multiply from the left with  $e_0^T$  to obtain the relation

$$x_{n+1}^T = x_1^T K(t^n) + x_1^T T(t^n + t^{-n}) - e_{n-1}^T,$$

and finally

$$x_{n+1} = \sum_{k=1}^n [x_1]_{n-k} \cdot x_k + T(t^n + t^{-n})x_1 - e_{n-1}, \quad [x_1]_{n-k} = \langle x_1, e_{n-k} \rangle. \quad (21)$$

In other words, the operators  $K(a)$  are determined for all trigonometric polynomials once we know the operator  $K(t) = e_0x_1^T$ , i.e.,  $x_1 \in \ell^2$ . We can also see this last fact in another way by observing that

$$M((t + t^{-1})^n) = (T(t + t^{-1}) + e_0x_1^T)^n \quad (22)$$

for  $n \geq 1$ . This simply follows from the multiplicative property of  $M(a)$  for even symbols. Recall that  $M(t^{-n}) = T(t^{-n})$  for  $n \geq 0$ .

Now the question is whether each  $x_1 \in \ell^2$  gives rise by, for instance, the definition (22) to operators  $M(a)$  which are well-defined for all trigonometric polynomials and satisfy the properties (a)–(c) for compatible pairs. If this is the case and assuming the density of the trigonometric polynomials in  $\mathcal{S}$ , then the operator  $M$  can be extended by continuity to all of  $\mathcal{S}$ . In fact, this question has an algebraic and an analytic part. The answer to the algebraic part is positive as the following theorem shows. As we will see later the analytic part can also be answered affirmative provided that the Banach algebra  $\mathcal{S}$  is chosen properly.

In what follows let  $\mathcal{T}$  stand for the algebra of all trigonometric polynomials. We define  $\mathcal{T}_-$  and  $\mathcal{T}_0$  as in (7) and (8).

**THEOREM 5.1.** *Let  $x_1 \in \ell^2$ . Then the definitions*

$$M(t^{-n}) = T(t^{-n}), \quad M((t + t^{-1})^n) = (T(t + t^{-1}) + e_0x_1^T)^n, \quad n \geq 0,$$

*determine uniquely a well defined linear operator  $M : \mathcal{T} \rightarrow \mathcal{L}(\ell^2)$ , which satisfies the conditions (b) and (c) of the definition of compatible pairs.*

*Proof.* It is obvious that the definition determines (by linearity) a well defined linear operator  $M$  on  $\mathcal{T}$  in a unique way. It is also clear that (c) is satisfied. It remains to show that condition (b) holds, i.e., we have to show that

$$M(abc) = T(a)M(b)M(c)$$

holds for  $a \in \mathcal{T}_-$ ,  $b \in \mathcal{T}$ , and  $c \in \mathcal{T}_0$ . Since an arbitrary function in  $\mathcal{T}$  can be represented as a linear combination of  $t^{-n}$  and  $(t + t^{-1})^n$ ,  $n \geq 0$ , it is not hard to see (see also (9)) that the only problem is to prove that  $M(ab) = T(a)M(b)$  for  $a \in \mathcal{T}_-$ ,  $b \in \mathcal{T}_0$ . We will consider  $a = t^{-n}$  and  $b = t^m + t^{-m}$  and use induction on  $n + m$ . There is nothing to prove when  $n = 0$  or  $m = 0$ . The case  $n = m = 1$  follows from

$$T(t^{-1})M(t + t^{-1}) = T(t^{-1})(T(t + t^{-1}) + e_0x_1^T) = T(1 + t^{-2}) = M(1 + t^{-2}).$$

Now let  $n, m \geq 1$  and  $n + m > 2$ . Then

$$T(t^{-n})M(t^m + t^{-m}) = T(t^{-n+1})T(t^{-1})M(t^m + t^{-m})$$

which by the induction arguments equals

$$\begin{aligned} T(t^{-n+1})M(t^{m-1} + t^{-m-1}) &= T(t^{-n+1})M(t^{m-1} + t^{-m+1}) + T(t^{-n+1})T(t^{-m-1} - t^{-m+1}) \\ &= M(t^{m-n} + t^{-m-n}) + T(t^{-m-n} - t^{-n-m+2}) \\ &= M(t^{m-n} + t^{-n-m+2}) \end{aligned}$$

as desired. □

The remaining analytic problem is the following. Given  $x_1 \in \ell^2$ , does there exist a Banach algebra  $\mathcal{S}$  (satisfying the basic assumptions (i) and (ii) of Section 2) such that the trigonometric polynomials  $\mathcal{T}$  are a dense subset in  $\mathcal{S}$  and such that

$$\|M(a) - T(a)\|_{\mathcal{E}_1(\ell^2)} \leq C \cdot \|a\|_{\mathcal{S}}$$

holds for all  $a \in \mathcal{T}$ , where  $M$  is defined on  $\mathcal{T}$  as in the previous theorem? If this is the case, then  $M$  can be extended by continuity to all of  $\mathcal{S}$ , and  $[M, \mathcal{S}]$  is a compatible pair.

We will see below that such Banach algebras  $\mathcal{S}$  always exist. They depend, in general, on the given  $x_1 \in \ell^2$ . Of course, one would like to have  $\mathcal{S}$  as large as possible. Our choices of  $\mathcal{S}$  will probably be more restrictive than necessary. We will not pursue the question of what might be the “optimal” Banach algebra  $\mathcal{S}$  any further.

The Banach algebras that we are going to consider are the following ones. For  $\alpha \geq 0$  and  $\beta \geq 1$  let  $F\ell^1_{\alpha,\beta}$  stand for the set of all functions  $a$  defined on the unit circle such that its Fourier coefficients satisfy

$$\|a\|_{F\ell^1_{\alpha,\beta}} := \sum_{n \in \mathbb{Z}} (1 + |n|)^\alpha \beta^{|n|} |a_n| < \infty.$$

It is easy to see that  $F\ell^1_{\alpha,\beta}$  are Banach algebras which satisfy the basic assumptions (i) and (ii) of Section 2 and that  $\mathcal{T}$  is a dense subset of  $\mathcal{S}$ .

Let us start with a basic observation. If it is the case that the operators  $M(t^n)$  are uniformly bounded in the operator norm for all  $n > 0$ , i.e.,

$$\sup_{n>0} \|M(t^n)\|_{\mathcal{L}(\ell^2)} < \infty,$$

then  $[M, \mathcal{S}]$  is a compatible pair with  $\mathcal{S} = F\ell^1_{1,1}$ . Indeed, if the operators  $M(t^n)$  are uniformly bounded in the operator norm, then so are the operators  $K(t^n)$ . Because  $K(t^n) = P_n K(t^n)$  for  $n > 0$ , we have that  $\|K(t^n)\|_{\mathcal{E}_1(\ell^2)} = O(n)$ . Recall that  $K(t^n) = 0$  for  $n \leq 0$ . Hence

$$\|K(a)\|_{\mathcal{E}_1(\ell^2)} \leq \sum_{n>0} |a_n| \cdot \|K(t^n)\|_{\mathcal{E}_1(\ell^2)} \leq C \sum_{n>0} |a_n| \cdot n \leq C \cdot \|a\|_{F\ell^1_{1,1}}$$

for all  $a \in \mathcal{S}$  as desired.

The hypothesis of the previous assertion is fulfilled in some cases, e.g., for the four concrete examples considered in Section 3. These examples correspond to the choices  $\pm e_0$  or  $e_1$  or the zero vector for  $x_1$ . The hypothesis is not fulfilled in general as one can see from straightforward computations based on the recurrence relation (21).

In order to give some information about the general case, and a somewhat less general, but still important case, we conclude with the following result.

**THEOREM 5.2.** *Let  $x_1 \in \ell^2$ , and define  $M : \mathcal{T} \rightarrow \mathcal{L}(\ell^2)$  as in Theorem 5.1.*

- (i) *If  $x_1 \neq 0$ , then  $M$  extends by continuity to  $\mathcal{S} = F\ell_{0,1+\sigma}^1$  with  $\sigma = \|x_1\|_{\ell^\infty}$ , and  $[M, \mathcal{S}]$  is a compatible pair.*
- (ii) *If  $x_1 \in \ell^1$ , then  $M$  extends by continuity to  $\mathcal{S} = F\ell_{\alpha,\beta}^1$ , and  $[M, \mathcal{S}]$  is a compatible pair, where*

$$(\alpha, \beta) = \begin{cases} (1, 1) & \text{if } \|x_1\|_{\ell^1} < 1 \\ (2, 1) & \text{if } \|x_1\|_{\ell^1} = 1 \\ (0, \|x_1\|_{\ell^1}) & \text{if } \|x_1\|_{\ell^1} > 1. \end{cases}$$

*Proof.* We start with the general observation that if

$$\|K(t^n)\|_{\mathcal{C}_1(\ell^2)} \leq Cn^\alpha \beta^n$$

for all  $n \geq 1$ , then  $M$  extends by continuity to a linear bounded operator on  $\mathcal{S} = F\ell_{\alpha,\beta}^1$ , and the pair  $[M, \mathcal{S}]$  is compatible. Indeed, if this holds true, then similar as above it follows that

$$\|K(a)\|_{\mathcal{C}_1(\ell^2)} \leq C \cdot \|a\|_{F\ell_{\alpha,\beta}^1}$$

for all  $a \in \mathcal{T}$ . Hence the operator  $K$  can be extended by continuity to all of  $\mathcal{S}$ , and its values are trace class. The operator  $M(a) = T(a) + K(a)$  can be extended by continuity as well. It is not hard to see that conditions (a) and (c) of the definition of compatible pairs are fulfilled. Condition (b) is also fulfilled because  $\mathcal{T}_0$  and  $\mathcal{T}_-$  are dense in  $\mathcal{S}_0$  and  $\mathcal{S}_-$ , respectively.

Thus our goal is now to estimate the trace norm of  $K(t^n)$  for  $n \geq 1$ . Using (20) it follows that

$$\|K(t^n)\|_{\mathcal{C}_1(\ell^2)} \leq \sum_{k=1}^n \|x_k\|_{\ell^2} =: s_n.$$

Here we use the elementary fact that the trace norm of the rank one operator  $yx^T$  equals  $\|x\| \cdot \|y\|$ . Let us now consider the two cases separately in order to estimate  $s_n$ .

- (i): Put  $\sigma = \|x_1\|_{\ell^\infty} > 0$  and  $\gamma = 2\|x_1\|_{\ell^2} + 1$ . The recursion (21) implies

$$\|x_{n+1}\|_{\ell^2} \leq \sigma \sum_{k=1}^n \|x_k\|_{\ell^2} + \gamma,$$

whence

$$\sum_{k=1}^{n+1} \|x_k\|_{\ell^2} \leq (1 + \sigma) \sum_{k=1}^n \|x_k\|_{\ell^2} + \gamma,$$

i.e.,  $s_{n+1} \leq (1 + \sigma)s_n + \gamma$ . It is now easy to see that  $s_n = O((1 + \sigma)^n)$ , which implies the assertion.

- (ii): Put  $\varrho = \|x_1\|_{\ell^1}$  and  $\gamma = 2\|x_1\|_{\ell^2} + 1$ . We use again the recurrence relation (21) in order to conclude that

$$\|x_{n+1}\|_{\ell^2} \leq \varrho \max_{1 \leq k \leq n} \|x_k\|_{\ell^2} + \gamma.$$

In case  $\varrho < 1$ , it follows that  $\|x_n\| = O(1)$ , thus  $s_n = O(n)$ . In case  $\varrho = 1$ , it follows that  $\|x_n\| = O(n)$ , thus  $s_n = O(n^2)$ . Finally, in case  $\varrho > 1$ , it follows that  $\|x_n\| = O(\varrho^n)$ , thus  $s_n = O(\varrho^n)$ . This implies the statements.  $\square$

## REFERENCES

- [1] J. BAIK, E. M. RAINS, *Algebraic aspects of increasing subsequences*, Duke Math. J. **109**, no. 1 (2001), 1–65.
- [2] E. L. BASOR, T. EHRHARDT, *Asymptotic formulas for determinants of a sum of finite Toeplitz and Hankel matrices*, Math. Nachr. **228** (2001), 5–45.
- [3] A. BORODIN AND A. OKOUNKOV, *A Fredholm determinant formula for Toeplitz determinants*, Int. Eqns. Operator Th. **37** (2000) 386–396.
- [4] A. BÖTTCHER AND B. SILBERMANN, *Analysis of Toeplitz Operators*, 2nd edition, Springer, Berlin 2006.
- [5] T. EHRHARDT, *A generalization of Pincusi formula and Toeplitz operator determinants*, Arch. Math. (Basel), **80** (2003), 302–309.
- [6] T. EHRHARDT, *An new algebraic proof of the Szegő-Widom limit theorem*, Acta Math. Hungar. **99**, no. 3 (2003), 233–261.
- [7] P. J. FORRESTER, N. E. FRANKEL, *Applications and generalizations of Fisher-Hartwig asymptotics*, J. Math. Phys. **45**, no. 5 (2004), 2003–2028.
- [8] V. V. PELLER, *Hankel operators of the class  $\mathcal{S}_p$  and their applications*, Math. USSR Sbornik, **41**, (1980/82), 443–479.
- [9] V. V. PELLER, *Hankel operators and their applications*, Springer monographs in Mathematics, Springer, New York 2003.
- [10] W. RUDIN, *Functional Analysis*, McGraw-Hill, Inc., New York 1991.
- [11] I. C. GÖHBERG AND M. G. KREIN, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monographs, Vol. 18, Amer. Math. Soc., Providence, 1969.
- [12] M. L. MEHTA, *Random Matrices*, Academic Press, San Diego, 1991.
- [13] C. A. TRACY, H. WIDOM, *Introduction to random matrices*, in: Lecture Notes in Physics, Vol. 424, Springer, Berlin 1993, pp. 103–130.
- [14] H. WIDOM, *Block Toeplitz matrices and Determinants*, Adv. in Math. **13**, no. 3 (1974).

(Received August 5, 2008)

*Estelle L. Basor*  
*American Institute of Mathematics*  
*Palo Alto, California*  
*94306, USA*

e-mail: ebasor@aimath.org

*Torsten Ehrhardt*  
*Department of Mathematics*  
*POSTECH*

*Pohang 790-784, South Korea*  
 e-mail: ehrhardt@postech.ac.kr