

## FINITE INTERTWININGS AND SUBSCALARITY

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*Abstract.* Quasinilpotent equivalence does not preserve subscalarity. However, if we replace quasinilpotent equivalence by “finite intertwining by the identity operator”, then subscalarity is preserved (in one direction). We shall prove that if  $A$ ,  $B$  and  $N$  are Banach space operators such that  $\Delta_{AB}^n(I) = \Delta_{AB}(\Delta_{AB}^{n-1}(I)) = \sum_{i=0}^n (-1)^i \binom{n}{i} A^{n-i} B^i = 0$  for some positive integer  $n$ , and if  $N$  is an algebraic operator which commutes with  $B$ , then  $A$  is subscalar implies  $B+N$  is subscalar. Applications to classes of Hilbert space operators, and the elementary operators  $L_A - R_B$  and  $L_A R_B - 1$  for certain choices of subscalar operators  $A$  and  $B^*$ , are considered.

### 1. Introduction

A Banach space operator  $A \in B(\mathcal{X})$  is generalized scalar if there exists a continuous algebra homomorphism  $\Phi$  from the space  $C^\infty(\mathbb{C})$  of infinitely differentiable complex valued functions into  $B(\mathcal{X})$ ,  $\Phi : C^\infty(\mathbb{C}) \rightarrow B(\mathcal{X})$ , such that  $\Phi(1) = I$  and  $\Phi(z) = A$ ; a subscalar operator is the restriction of a generalized scalar operator to a closed invariant subspace of the operator. Recall from Eschmeier and Putinar [12, Corollary 4.6] that an operator is subscalar if and only if it satisfies (the Eschmeier–Putinar–Bishop) property  $(\beta)_\varepsilon$ , where  $A \in B(\mathcal{X})$  satisfies property  $(\beta)_\varepsilon$  if for each open subset  $\mathcal{U}$  of the complex plane  $\mathbb{C}$  the operator  $A_z : f \rightarrow (A - z)f$  from the Fréchet space  $\mathcal{E}(\mathcal{U}, \mathcal{X})$  of  $\mathcal{X}$ -valued  $C^\infty$  functions into itself is a topological monomorphism (equivalently,  $A_z$  is injective and has closed range). Operators  $A \in B(\mathcal{X})$  satisfying property  $(\beta)_\varepsilon$  satisfy (Bishop’s) property  $(\beta)$ . Here an operator  $A \in B(\mathcal{X})$  satisfies property  $(\beta)$  if for each open subset  $\mathcal{U}$  of  $\mathbb{C}$  the operator  $A_z : f \rightarrow (A - z)f$  from the Fréchet space  $\mathcal{O}(\mathcal{U}, \mathcal{X})$  of  $\mathcal{X}$ -valued analytic functions into itself is a topological monomorphism.

If  $A, B$  are operators in  $B(\mathcal{X})$ , and  $\Delta_{AB}(X) \in B(B(\mathcal{X}))$  is the generalized derivation  $\Delta_{AB}(X) = AX - XB$ , then  $B$  is said to be asymptotically intertwined by  $X \in B(\mathcal{X})$  to  $A$  if

$$\lim_{n \rightarrow \infty} \|\Delta_{AB}^n(X)\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|\Delta_{AB}(\Delta_{AB}^{n-1}(X))\|^{\frac{1}{n}} = 0.$$

Asymptotically intertwined operators intertwined by the identity operator  $I \in B(\mathcal{X})$  share a number of properties; see [15, Lemmas 3.4.6, 3.4.7 and Proposition 3.7.11].

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In particular, if  $A$  has property  $(\beta)$  and  $\lim_{n \rightarrow \infty} \|\Delta_{AB}^n(I)\|^{\frac{1}{n}} = 0$ , then  $B$  has property  $(\beta)$ .

Recall, [15, p 253], that the operators  $A, B \in B(\mathcal{X})$  are said to be quasinilpotent equivalent if  $\lim_{n \rightarrow \infty} \|\Delta_{AB}^n(I)\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|\Delta_{BA}^n(I)\|^{\frac{1}{n}} = 0$ . Unlike the situation for property  $(\beta)$ ,  $B$  asymptotically intertwined to  $A$  (even,  $B$  quasinilpotent equivalent to  $A$ ) does not inherit property  $(\beta)_\varepsilon$  from  $A$ . Consider, for example,  $A = 0$  and  $B = Q$ , where  $Q$  is a non nilpotent quasinilpotent operator; then  $B$  is asymptotically intertwined to  $A$  by the identity operator,  $A$  has property  $(\beta)_\varepsilon$  but  $B$  does not have property  $(\beta)_\varepsilon$ .

Let us say that the operator  $B$  is finitely intertwined to  $A \in B(\mathcal{X})$  by the identity operator if there exists an integer  $k \geq 1$  such that  $\Delta_{AB}^k(I) = 0$ . (Such an intertwining of  $A$  and  $B$  by the identity has been called a Helton class of order  $k$ , denoted  $B \in \text{Helton}_k(A)$ , by Kim, Ko and Lee [14].) Using essentially the (localized version of) property  $(\beta)_\varepsilon$  and a straightforward algebraic argument, we prove that if  $B \in B(\mathcal{X})$  is finitely intertwined to  $A \in B(\mathcal{X})$  by the identity operator and if  $A$  satisfies property  $(\beta)_\varepsilon$ , then  $B + N$  satisfies property  $(\beta)_\varepsilon$  for every algebraic operator  $N \in B(\mathcal{X})$  such that  $BN = NB$ . Here, the hypothesis that  $A$  satisfies property  $(\beta)_\varepsilon$  may be weakened by requiring that  $g(A)$  satisfies property  $(\beta)_\varepsilon$  for some function  $g$  bi-holomorphic on a neighbourhood of  $\sigma(A)$ . We prove also that if  $\Delta_{AB}^k(I) = 0$  for some integer  $k \geq 1$ , then: (i)  $A$  is nilpotent if and only if  $B$  is nilpotent; (ii) if  $A$  and  $B^*$  have the single valued extension property, then  $A$  and  $B$  have the same spectrum and the same (Fredholm) essential (and Browder, and Weyl) spectrum.

As an application of our main results to classes of Hilbert space operators, it is proved that if  $A$  is  $p$ -hyponormal ( $0 < p \leq 1$ ), or  $w$ -hyponormal, or  $M$ -hyponormal, or  $p$ -quasihyponormal with  $A^{-1}(0) \subseteq A^{*-1}(0)$  (or, if  $A$  is an  $n$ -th root of an operator belonging to one these classes such that  $\sigma(A)$  is contained in an angle  $L < \frac{2\pi}{n}$  with vertex in the origin), and  $\Delta_{AB}^k(I) = 0$  for some integer  $k \geq 1$ , then  $B$  is subscalar. This generalizes a result of Kim *et al* [14] on  $p$ -hyponormal operators.

Subscalar operators satisfy Weyl's theorem [1]: as a further application of our results, it is proved that if  $A \in B(\mathcal{X})$  has property  $(\beta)_\varepsilon$  and  $B \in B(\mathcal{X})$  is finitely intertwined to  $A$  by the identity operator, then  $B + N$  satisfies Weyl's theorem and  $B^* + N^*$  satisfies  $a$ -Weyl's theorem for every algebraic operator  $N \in B(\mathcal{X})$  such that  $N$  commutes with  $B$ . Furthermore, it is proved that if  $A, B^*$  are subscalar Hilbert space operators such that their eigenspaces corresponding to distinct eigenvalues are orthogonal, then the elementary operator  $d_{AB} = L_A - R_B$  or  $L_A R_B - 1$  satisfies Weyl's theorem and  $d_{AB}^*$  satisfies  $a$ -Weyl's theorem. Here  $L_A$  and  $R_A$  denote the operators of left multiplication and right multiplication by  $A$ , respectively.

In the following, we define our notation and terminology progressively, on an as and when required basis.

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### 2. Results

For a Banach space  $\mathcal{X}$  and open subset  $\mathcal{U}$  of  $\mathbb{C}$ , let  $\mathcal{E}(\mathcal{U}, \mathcal{X})$  (resp.,  $\mathcal{O}(\mathcal{U}, \mathcal{X})$ ) denote the Fréchet space of all infinitely differentiable  $\mathcal{X}$ -valued functions on  $\mathcal{U}$  endowed with the topology of uniform convergence of all derivatives on compact subsets of  $\mathcal{U}$  (resp., of all analytic  $\mathcal{X}$ -valued functions on  $\mathcal{U}$  endowed with the topology of uniform convergence on compact subsets of  $\mathcal{U}$ ). Localising properties  $(\beta)_\varepsilon$  and  $(\beta)$ , we say that  $T \in B(\mathcal{X})$  satisfies:

- property  $(\beta)_\varepsilon$  at  $\lambda \in \mathbb{C}$  if there exists a neighbourhood  $\mathcal{N}$  of  $\lambda$  such that, for each open subset  $\mathcal{U}$  of  $\mathcal{N}$  and sequence  $\{f_n\}$  of  $\mathcal{X}$ -valued functions in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$ ,

$$(T - z)f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}) \implies f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X});$$

- property  $(\beta)$  at  $\lambda \in \mathbb{C}$  if there exists an  $r > 0$  such that, for every open subset  $\mathcal{U}$  of the open disc  $\mathbf{D}(\lambda; r)$  of radius  $r$  centered at  $\lambda$  and sequence  $\{f_n\}$  of  $\mathcal{X}$ -valued functions in  $\mathcal{O}(\mathcal{U}, \mathcal{X})$ ,

$$(T - z)f_n(z) \longrightarrow 0 \text{ in } \mathcal{O}(\mathcal{U}, \mathcal{X}) \implies f_n(z) \longrightarrow 0 \text{ in } \mathcal{O}(\mathcal{U}, \mathcal{X}).$$

Recall that an operator  $A \in B(\mathcal{X})$  has the single-valued extension property at a point  $\lambda_0 \in \mathbb{C}$ , SVEP at  $\lambda_0$ , if for every open disc  $\mathcal{D}$  centered at  $\lambda_0$  the only analytic function  $f : \mathcal{D} \rightarrow \mathcal{X}$  satisfying  $(A - \lambda)f(\lambda) = 0$  is the function  $f \equiv 0$ ;  $A$  has SVEP if it has SVEP at every  $\lambda \in \mathbb{C}$ . Evidently, property  $(\beta)_\varepsilon$  implies property  $(\beta)$ . It is well known that property  $(\beta)$  implies SVEP, and that if  $A$  satisfies property  $(\beta)$  then  $A^*$  satisfies property  $(\delta)$ . (We shall have no more than a passing interest in the (decomposition) property  $(\delta)$ , also Dunford’s condition  $(C)$ : the interested reader is invited to consult [15], in particular Definitions 1.2.18 and 1.2.28.)

Property  $(\beta)_\varepsilon$  is not preserved by quasi-affinities (see Remark 2.7 below). However, if  $A \in B(\mathcal{X})$  has property  $(\beta)_\varepsilon$  and  $AX = XB$  for some  $B \in B(\mathcal{X})$  and bounded below operator  $X \in B(\mathcal{X})$ , then  $B$  has property  $(\beta)_\varepsilon$ : this follows from the following argument. Let  $\{f_n(z)\}$  be a sequence in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$  such that  $(B - z)f_n(z) \rightarrow 0$  in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$ . Then  $(A - \lambda)Xf_n(z) \rightarrow 0$  in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$ . Since  $A$  has property  $(\beta)_\varepsilon$ , this implies that  $Xf_n(z) \rightarrow 0$  in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$ . But then, since  $X$  is injective and has closed range,  $f_n(z) \rightarrow 0$  in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$ . Using a similar argument we prove next that for finitely intertwined pair of operators  $(A, B) \in B(\mathcal{X})$  intertwined by the identity,  $B$  inherits property  $(\beta)_\varepsilon$  from  $A$ . The following terminology and technical lemmas will be required.

The quasi-nilpotent part  $H_0(T)$  of an operator  $T \in B(\mathcal{X})$  is the set

$$H_0(T) = \{x \in \mathcal{X} : \lim_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

$H_0(T)$  is, in general, a non-closed hyper-invariant subspace of  $T$  such that  $T^{-m}(0) \subseteq H_0(T)$  for all  $m = 0, 1, 2, \dots$ . (Observe that  $H_0(T)$  is closed if  $0 \in \text{iso}\sigma(T)$ .) Recall

that an operator  $N \in B(\mathcal{X})$  is said to be an algebraic operator if there exists a non-constant polynomial  $q(\cdot)$  such that  $q(N) = 0$ . Nilpotent operators, more generally operators  $F \in B(\mathcal{X})$  such that  $F^n$  is finite dimensional for some natural number  $n$ , are algebraic operators. If  $N \in B(\mathcal{X})$  is an algebraic operator ( $q(N) = 0$ ), then  $\sigma(N) = \{\mu_1, \mu_2, \dots, \mu_n\}$  for some integer  $n \geq 1$ , and  $H_0(N - \mu_i) = (N - \mu_i)^{-m_i}(0)$  for some integer  $m_i \geq 1$  (and all  $1 \leq i \leq n$ ). Let  $N_i = N|_{H_0(N - \mu_i)}$ . Then  $\mathcal{X} = \bigoplus_{i=1}^n H_0(N - \mu_i)$  and  $N = \bigoplus_{i=1}^n N_i$ . Apparently,  $\sigma(N_i) = \{\mu_i\}$ ,  $q(\mu_i) = q(\sigma(N_i)) = \sigma(q(N_i)) = \{0\}$  and  $0 = q(N_i) = q(N_i) - q(\mu_i) = (N_i - \mu_i)^{m_i} p(N_i)$  for some positive integer  $m_i$  and invertible operator  $p(N_i)$ . Consequently,  $N_i - \mu_i = N|_{H_0(N - \mu_i)} - \mu_i I|_{H_0(N - \mu_i)}$  is nilpotent.

LEMMA 2.1. *Let  $N \in B(\mathcal{X})$  be an algebraic operator, with  $\sigma(N) = \{\mu_1, \mu_2, \dots, \mu_n\}$ , which commutes with an operator  $B \in B(\mathcal{X})$ . Then  $B = \bigoplus_{i=1}^n B_i$  and  $B + N = \bigoplus_{i=1}^n B_i + N_i$ , where  $B_i = B|_{H_0(N - \mu_i)}$ ,  $N_i = N|_{H_0(N - \mu_i)}$  and  $N_i$  commutes with  $B_i$  for all  $1 \leq i \leq n$ .*

*Proof.* The (closed) subspace  $H_0(N - \mu_i)$ ,  $1 \leq i \leq n$ , coincides with the range of the spectral projection of  $N$  associated with  $\mu_i$  [1, Theorem 3.74]. Hence the hypothesis  $N$  commutes with  $B$  implies that  $N_i$  commutes with  $B_i$  for all  $1 \leq i \leq n$ .  $\square$

The following lemma is easily proved: we leave the proof to the reader.

LEMMA 2.2. *If  $T_i \in B(\mathcal{X}_i)$ ,  $1 \leq i \leq n$ , and  $T = \bigoplus_{i=1}^n T_i \in B(\bigoplus_{i=1}^n \mathcal{X}_i)$ , then  $T$  satisfies property  $(\beta)_\varepsilon$  if and only if  $T_i$  satisfies property  $(\beta)_\varepsilon$  for all  $1 \leq i \leq n$ .*

LEMMA 2.3. *If  $T \in B(\mathcal{X})$  satisfies property  $(\beta)_\varepsilon$ , then  $T + \mu$  satisfies property  $(\beta)_\varepsilon$  for every complex scalar  $\mu$ .*

*Proof.* Let  $\{f_n(z)\}$  be a sequence in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$  such that  $((T + \mu) - z)f_n(z) \rightarrow 0$  in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$ . Define the sequence  $\{g_n\}$  by  $g_n(z) = f_n(z + \mu)$ . Then  $g_n \in \mathcal{E}(\mathcal{U}, \mathcal{X})$  for all  $n = 1, 2, \dots$ , and  $(T - (z - \mu))g_n(z - \mu) \rightarrow 0$  in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$ . Since  $T$  satisfies property  $(\beta)_\varepsilon$ ,  $f_n(z) = g_n(z - \mu) \rightarrow 0$  in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$ . Thus  $T + \mu$  satisfies property  $(\beta)_\varepsilon$ .  $\square$

THEOREM 2.4. (i) *If  $B \in B(\mathcal{X})$  is finitely intertwined to a subsclar operator  $A \in B(\mathcal{X})$  by the identity operator, then  $B$  is subsclar.*

(ii) *If  $N \in B(\mathcal{X})$  is an algebraic operator, in particular a nilpotent operator, which commutes with a subsclar operator  $B \in B(\mathcal{X})$ , then  $B + N$  is subsclar.*

*Proof.* Since a Banach space operator  $T$  is subsclar if and only if it satisfies property  $(\beta)_\varepsilon$  [12], we prove that  $B$  and  $B + N$  satisfy property  $(\beta)_\varepsilon$ .

(i). Assume that  $\Delta_{AB}^k(I) = \sum_{i=0}^k (-1)^i \binom{k}{i} A^{k-i} B^i = 0$ , and let  $\{f_n(z)\}$  be a sequence in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$  such that

$$(B - z)f_n(z) \rightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}).$$

Then

$$(-1)^j \binom{k}{j} \{A^{k-j}B^j - zA^{k-j}B^{j-1}\}f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X})$$

for all  $j = 1, 2, \dots, k$ . Since

$$\begin{aligned} & \sum_{i=1}^k (-1)^i \binom{k}{i} \{A^{k-i}B^i - zA^{k-i}B^{i-1}\} \\ &= \left( \sum_{i=0}^k (-1)^i \binom{k}{i} A^{k-i}B^i \right) - (A^k + z \sum_{i=1}^k (-1)^i \binom{k}{i} A^{k-i}B^{i-1}), \\ & \{A^k + z \sum_{i=1}^k (-1)^i \binom{k}{i} A^{k-i}B^{i-1}\}f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}). \end{aligned} \quad (1)$$

Again, since

$$(-1)^j \binom{k}{j} \{A^{k-j}B^{j-1} - zA^{k-j}B^{j-2}\}f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X})$$

for all  $j = 2, \dots, k$ , the following implications hold:

$$\begin{aligned} & \sum_{i=2}^k (-1)^i \binom{k}{i} \{A^{k-i}B^{i-1} - zA^{k-i}B^{i-2}\}f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}) \\ \implies & \left\{ - \binom{k}{1} A^{k-1}f_n(z) + z \sum_{i=2}^k (-1)^i \binom{k}{i} A^{k-i}B^{i-2}f_n(z) \right\} \\ & - \left\{ \sum_{i=1}^k (-1)^i \binom{k}{i} A^{k-i}B^{i-1}f_n(z) \right\} \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}) \\ \implies & \left\{ \{A^k + (-1) \binom{k}{1}\} zA^{k-1} + z^2 \sum_{i=2}^k (-1)^i \binom{k}{i} A^{k-i}B^{i-2} \right\}f_n(z) \\ & - \{A^k + z \sum_{i=1}^k (-1)^i \binom{k}{i} A^{k-i}B^{i-1}\}f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}). \end{aligned}$$

This, by (1) above, implies that

$$\{A^k + (-1) \binom{k}{1}\} zA^{k-1} + z^2 \sum_{i=2}^k (-1)^i \binom{k}{i} A^{k-i}B^{i-2} \}f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}).$$

Repeating the above argument a finite number of times, it follows that

$$\sum_{i=0}^k (-1)^i \binom{k}{i} z^i A^{k-i}f_n(z) = (A - z)^k f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}).$$

But then, since  $A$  satisfies property  $(\beta)_\varepsilon$ ,

$$f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}),$$

i.e.,  $B$  satisfies property  $(\beta)_\varepsilon$ .

(ii). We start by assuming that  $N$  is a nilpotent operator which commutes with  $B$ , and prove that  $B + N$  is then a subscalar operator. This will then be extended to the case of a polynomial operator.

Let  $\{f_n(z)\}$  be a sequence in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$  such that

$$(B + N - z)f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}).$$

Then, since  $N^m = 0$ ,

$$N^{m-1}(B + N - z)f_n(z) = (B - z)N^{m-1}f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X})$$

implies that

$$N^{m-1}f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}).$$

This, since

$$N^{m-2}(B + N - z)f_n(z) = (B - z)N^{m-2}f_n(z) + N^{m-1}f_n(z),$$

implies that

$$(B - z)N^{m-2}f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}),$$

and hence that

$$N^{m-2}f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}).$$

Repeating this argument we eventually have that  $Nf_n(z) \longrightarrow 0$  in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$ . Hence

$$(B - z)f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}).$$

Thus

$$f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}),$$

which implies that  $B + N$  satisfies property  $(\beta)_\varepsilon$ .

Now let  $N \in \mathcal{B}(\mathcal{X})$  be such that  $q(N) = 0$  for some non-constant polynomial  $q(\cdot)$ . Let  $\sigma(N) = \{\mu_1, \mu_2, \dots, \mu_n\}$  for some natural number  $n$ . Then  $\mathcal{X} = \bigoplus_{i=1}^n \mathcal{X}_i$ , where  $\mathcal{X}_i = H_0(N - \mu_i)$  for all  $1 \leq i \leq n$ . Define the operators  $N_i$  and  $B_i \in \mathcal{B}(\mathcal{X}_i)$ ,  $1 \leq i \leq n$ , as in Lemma 2.1. The operators  $N_i - \mu_i$  are nilpotent and commute with  $B_i$  for all  $1 \leq i \leq n$ . Since  $B_i$  satisfies property  $(\beta)_\varepsilon$  for all  $1 \leq i \leq n$ , see Lemma 2.2,  $B_i + N_i - \mu_i$  satisfies property  $(\beta)_\varepsilon$  for all  $1 \leq i \leq n$ . This, by Lemma 2.3, implies that  $B_i + N_i$  satisfies property  $(\beta)_\varepsilon$  for all  $1 \leq i \leq n$ . Applying Lemma 2.2 it follows that  $B + N = \bigoplus_{i=1}^n (B_i + N_i)$  satisfies property  $(\beta)_\varepsilon$ .  $\square$

An obvious modification of the argument of the first part of the proof of Theorem 2.4 proves that if  $\Delta_{AB}^k(I) = 0$  and  $A$  has SVEP, or satisfies property  $(\beta)$ , then  $B$  has SVEP, or satisfies property  $(\beta)$ . If an operator  $A$  satisfies property  $(\beta)$  and  $\sigma(A)$  is

thick, then  $A$  has a non-trivial closed invariant subspace [15, Theorem 2.6.12]; hence, if  $A$  satisfies property  $(\beta)$ ,  $\Delta_{AB}^k(I) = 0$ , and  $\sigma(B)$  is thick, then  $B$  has a non-trivial closed invariant subspace. The following corollaries list a few further consequences of Theorem 2.4. We start by considering the case in which  $\Delta_{AB}^k(I) = 0$  for some integer  $k \geq 1$ , and either  $A$  or  $B$  is nilpotent.

**COROLLARY 2.5.** *If  $\Delta_{AB}^k(I) = 0$  for some operators  $A, B \in B(\mathcal{X})$  and integer  $k \geq 1$ , then  $B$  is nilpotent if and only if  $A$  is nilpotent.*

*Proof.* Let  $L_A, R_B \in B(B(\mathcal{X}))$  denote respectively the operators of left multiplication by  $A$  and right multiplication by  $B$ . Then

$$\Delta_{AB}^k(I) = 0 \implies (L_A - R_B)^k(I) = 0.$$

Since  $L_A$  and  $R_B$  commute,  $L_A$  and  $R_B$  are quasinilpotent equivalent, which [15, Proposition 3.4.11] implies that  $\sigma(A) = \sigma(L_A) = \sigma(R_B) = \sigma(B)$ . Thus, if either of  $A$  or  $B$  is nilpotent, then  $\sigma(A) = \sigma(B) = \{0\}$ . (In particular, if  $A$  (resp.,  $B$ ) is nilpotent, then  $B$  (resp.,  $A$ ) is quasinilpotent.) Observe that if  $A$  is nilpotent, then  $B$  satisfies property  $(\beta)_\varepsilon$  (see Theorem 2.4); again, since  $\Delta_{AB}^k(I) = 0 \implies \Delta_{B^*A^*}^k(I^*) = 0$ , if  $B$  is nilpotent (so that  $B^*$  is nilpotent and satisfies property  $(\beta)_\varepsilon$ ), then  $A^*$  satisfies property  $(\beta)_\varepsilon$ . Since a quasinilpotent operator satisfies property  $(\beta)_\varepsilon$  if and only if it is nilpotent, it follows that  $A$  (resp.,  $B$ ) nilpotent implies  $B$  (resp.,  $A$ ) nilpotent.  $\square$

**REMARK 2.6.** The following simple (algebraic) argument provides an alternative proof of Corollary 2.5. If  $\Delta_{AB}^k(I) = 0$  and  $A$  is  $n$ -nilpotent for some positive integer  $n$ , then let  $m = \max\{k, n\}$ . Evidently,  $\Delta_{AB}^m(I) = 0$  and  $A$  is  $m$ -nilpotent. Since  $A^m = 0$ ,  $A^{m-j}B^{m+j-1} = 0$  for  $j = 0$ . Assume that  $A^{m-j}B^{m+j-1} = 0$  for  $j = 0, 1, 2, \dots, r (< m)$ . Then

$$0 = A^{m-r-1} \{ \Delta_{AB}^m(I) \} B^r = \sum_{i=0}^m (-1)^i \binom{m}{i} A^{2m-i-r-1} B^{i+r} = (-1)^m A^{m-r-1} B^{m+r}.$$

Hence,  $A^{m-j}B^{m+j-1} = 0$  for all  $j = 0, 1, \dots, m$ . Choosing  $j = m$ ,  $B^{2m-1} = 0$ . Observe that  $\Delta_{AB}^k(I) = 0 \implies \Delta_{B^*A^*}^k(I^*) = 0$ . Hence it follows from the argument above that if  $B$  is nilpotent, then  $A$  is nilpotent. Thus, if  $\Delta_{AB}^k(I) = 0$ , then  $A$  is nilpotent if and only if  $B$  is nilpotent.

**REMARK 2.7.** For commuting  $A, B \in B(\mathcal{X})$  such that  $\Delta_{AB}^k(I) = 0$  for some integer  $k > 1$ ,  $A$  satisfies property  $(\beta)_\varepsilon$  if and only if  $B$  satisfies property  $(\beta)_\varepsilon$ . It is not clear if the hypothesis  $B$  satisfies property  $(\beta)_\varepsilon$  is sufficient for  $A$  to satisfy property  $(\beta)_\varepsilon$  for non commuting  $A, B \in B(\mathcal{X})$  such that  $\Delta_{AB}^k(I) = 0$  for some integer  $k > 1$ . Observe however that if  $\Delta_{AB}(X) = 0$  for some quasi-affinity  $X$ , then  $B$  satisfies property  $(\beta)_\varepsilon$  does not imply  $A$  satisfies property  $(\beta)_\varepsilon$ . Thus, let  $B \in \ell^2(\mathbf{N})$  be the forward unilateral shift,  $A \in \ell^2(\mathbf{N})$  the weighted forward unilateral shift with the weight sequence  $\{\frac{1}{n+1}\}$ , and  $X \in \ell^2(\mathbf{N})$  the multiplication operator defined by  $Xx = \{\frac{x_n}{n!}\}_{n \in \mathbf{N}}$  for all  $x = \{x_n\}_{n \in \mathbf{N}} \in \ell^2(\mathbf{N})$ . Then  $B$  satisfies property  $(\beta)_\varepsilon$ ,  $A$  is quasinilpotent and  $X$  is a quasi-affinity such that  $AX = XB$ . Evidently,  $A$  does not have property  $(\beta)_\varepsilon$ .

COROLLARY 2.8. *Let  $\{B_n\}$  be a sequence of operators in  $B(\mathcal{X})$  such that  $B_n$  converges to  $B$  in the operator norm topology. If  $A$  is subscalar and  $\Delta_{AB_n}^k(I) = 0$ , then  $B$  is subscalar.*

*Proof.* Since

$$\begin{aligned} \|\Delta_{AB}^k(I)\| &\leq \lim_{n \rightarrow \infty} \|\Delta_{AB_n}^k(I) - \Delta_{AB}^k(I)\| \\ &\leq \lim_{n \rightarrow \infty} \sum_{i=1}^k \binom{k}{i} \|A\|^{k-i} \|B_n^i - B^i\| = 0, \end{aligned}$$

$\Delta_{AB}^k(I) = 0$ . Hence  $B$  satisfies property  $(\beta)_\varepsilon$ .  $\square$

The following theorem is proved in [4, Theorem 2.2] for Hilbert space operators.

THEOREM 2.9. *If a function  $g$  is bi-holomorphic on a neighbourhood of  $\sigma(A)$  for some  $A \in B(\mathcal{X})$ , then  $A$  has property  $(\beta)_\varepsilon$  if and only if  $g(A)$  has property  $(\beta)_\varepsilon$ .*

*Proof.* The proof of the theorem is similar to that of [4, Theorem 2.2]: we include it here for completeness.

By symmetry, it suffices to verify that if  $g(A)$  satisfies property  $(\beta)_\varepsilon$  for some bi-holomorphic  $g$  on a neighbourhood  $\mathcal{V}$  of  $\sigma(A)$ , then so does  $A$ . (Indeed, consider  $g^{-1}$  for the other direction.) Thus, assume that  $g$  is bi-holomorphic on a neighbourhood  $\mathcal{V}$  of  $\sigma(A)$ . Then, for  $z$  and  $\lambda$  in  $\mathcal{V}$ ,

$$g(z) - g(\lambda) = (z - \lambda)p_\lambda(z),$$

where  $p_\lambda$  is a uniformly bounded (on  $\lambda$ ) holomorphic function. Assume that  $g(A)$  satisfies property  $(\beta)_\varepsilon$ . Let  $\{f_n(z)\}$  be a sequence in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$  such that  $(A - z)f_n(z) \rightarrow 0$  in  $\mathcal{E}(\mathcal{U}, \mathcal{X})$ . Then

$$(g(A) - g(z))f_n(z) = p_z(A)(A - z)f_n(z) \rightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}).$$

Let  $g(z) = w$ . Then

$$(g(A) - w)f_n(g^{-1}w) \rightarrow 0 \text{ in } \mathcal{E}(g(\mathcal{U}), \mathcal{X}),$$

or

$$(g(A) - w)(f_n \circ g^{-1})(w) \rightarrow 0 \text{ in } \mathcal{E}(g(\mathcal{U}), \mathcal{X}).$$

Since  $\{f_n \circ g^{-1}\}$  is a sequence in  $\mathcal{E}(g(\mathcal{U}), \mathcal{X})$  and since  $g(A)$  has property  $(\beta)_\varepsilon$ , it follows that

$$f_n \circ g^{-1}(w) \rightarrow 0 \text{ in } \mathcal{E}(g(\mathcal{U}), \mathcal{X}).$$

Equivalently,

$$f_n(z) \rightarrow 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{X}).$$

Hence  $A$  satisfies property  $(\beta)_\varepsilon$ .  $\square$

Combining Theorems 2.4 and 2.9, we have: “If the function  $g$  is bi-holomorphic on a neighbourhood of  $\sigma(A)$  for some  $A \in B(\mathcal{X})$ ,  $g(A)$  is subscalar and  $B \in B(\mathcal{X})$  is finitely intertwined to  $A$  by the identity operator, then  $B + N$  is subscalar for every algebraic operator  $N \in B(\mathcal{X})$  such that  $B$  and  $N$  commute”.

### Hilbert space operators

A Hilbert space operator  $T \in B(\mathcal{H})$  is:  $p$ -hyponormal if  $|T^*|^{2p} \leq |T|^{2p}$  for some  $0 < p \leq 1$  (a 1-hyponormal operator is hyponormal);  $M$ -hyponormal if there exists a scalar  $M > 0$  such that  $\|(T - \lambda)^*x\|^2 \leq M\|(T - \lambda)x\|^2$  for all  $\lambda \in \mathbb{C}$  and  $x \in \mathcal{H}$ ;  $w$ -hyponormal if  $|\tilde{T}^*| \leq |T| \leq |\tilde{T}|$ , where  $\tilde{T}$  is the Aluthge transform  $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$  of  $T$  and  $U$  is as in the polar decomposition  $T = U|T|$  of  $T$  [3];  $p$ -quasihyponormal for some  $0 < p \leq 1$  if  $T^*(|T|^{2p} - |T^*|^{2p})T \geq 0$ . It is known that hyponormal,  $p$ -hyponormal,  $M$ -hyponormal and  $w$ -hyponormal operators are subscalar [12, 5]. However,  $p$ -quasihyponormal operators  $T$  do not satisfy property (satisfied by all subscalar operators [1, Page 175] that)  $H_0(T - \lambda) = (T - \lambda)^{-m}(0)$  for some non-negative integer  $m = m(\lambda)$  and all  $\lambda \in \sigma(T)$  [2], hence are not subscalar. For brevity, let  $p - Q_*$  denote the class of  $p$ -quasihyponormal operators  $T$  such that  $T^{-1}(0) \subseteq T^{*-1}(0)$ . Then 0, whenever it is in the point spectrum of  $T$ , is a normal eigenvalue of  $T$  (i.e.,  $T^{-1}(0)$  is reducing), and  $T = T_1 \oplus T_2$  with respect to some decomposition  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  of  $\mathcal{H}$  such that  $T_1 = T|_{\mathcal{H}_1}$  is normal and  $T_2 = T|_{\mathcal{H}_2}$  is an injective, completely non-normal  $p$ -quasihyponormal operator. Let  $T_i$ ,  $i = 1, 2$ , have the polar decomposition  $T_i = U_i|T_i|$ . Then the partial isometry  $U_2$  is an isometry, and we may choose the partial isometry  $U_1$  to be a unitary such that  $U_1$  commutes with  $|T_1|$ . Define  $S \in B(\mathcal{H})$  by  $S = S_1 \oplus S_2 = (|T_1| \oplus |T_2|)(U_1 \oplus U_2)$ . Then  $\sigma(S) = \sigma(T)$  [5]. Furthermore, since  $|T_2|$  is a quasi-affinity, it follows from the equivalence

$$T_2^*(|T_2|^{2p} - |T_2^*|^{2p})T_2 \geq 0 \iff U_2^*(|T_2|^{2p} - |T_2^*|^{2p})U_2 \geq 0$$

that

$$|S_2|^{2p} \leq |T_2|^{2p} = U_2^*|T_2^*|^{2p}U_2 \leq U_2^*|T_2|^{2p}U_2 \leq |S_2|^{2p},$$

where the last inequality is a consequence of the fact that  $U_2^*|T_2|^{2p}U_2 \leq (U_2^*|T_2|^2U_2)^p$  for all  $0 < p \leq 1$ . Evidently,  $S$  is  $p$ -hyponormal, hence subscalar. Recall from [5] that, for operators  $L, M \in B(\mathcal{H})$ ,  $LM$  satisfies property  $(\beta)_\varepsilon$  if and only if  $ML$  satisfies property  $(\beta)_\varepsilon$ . Hence,  $p - Q_*$  operators are subscalar.

Let  $\Xi$  denote the class of Hilbert space operators which are either hyponormal or  $M$ -hyponormal or  $p$ -hyponormal or  $w$ -hyponormal or  $p - Q_*$ , and  $\sqrt[n]{\Xi}$  denote the class of operators  $A$  such that  $A^n \in \Xi$  for some positive integer  $n \geq 2$ . It is then evident from Theorem 2.9, and the above, that if an  $A \in \sqrt[n]{\Xi}$  is such that  $\sigma(A)$  is contained in an angle  $L < \frac{2\pi}{n}$  with vertex in the origin, then  $A$  is subscalar.

A version of the following corollary has been proved by Kim *et al* [14] for  $p$ -hyponormal operators.

**COROLLARY 2.10.** *Suppose that  $A \in \Xi$ , or  $A \in \sqrt[n]{\Xi}$  with  $\sigma(A)$  contained in an angle  $L < \frac{2\pi}{n}$  with vertex in the origin. If  $\Delta_{AB}^k(I) = 0$  for some  $B \in B(\mathcal{H})$  and integer  $k \geq 1$ , and if  $N \in B(\mathcal{H})$  is an algebraic operator which commutes with  $B$ , then  $B + N$  is subscalar.*

*Proof.* Evident.  $\square$

### 3. Applications I: Browder, Weyl spectra

In this section we consider the relationship between certain distinguished parts of the spectrum of operators  $A$  and  $B$  satisfying  $\Delta_{AB}^k(I) = 0$  for some integer  $k$ .

An operator  $A \in B(\mathcal{X})$  is upper semi-Fredholm (resp., lower semi-Fredholm) if  $A\mathcal{X}$  is closed and  $\alpha(A) = \dim A^{-1}(0) < \infty$  (resp.,  $A\mathcal{X}$  is closed and  $\beta(A) = \dim(\mathcal{X}/A\mathcal{X}) < \infty$ ).  $A$  is Fredholm if it is both upper and lower semi-Fredholm, and then the Fredholm index of  $A$  is the integer  $\text{ind}(A) = \alpha(A) - \beta(A)$ . The Browder spectrum (resp., the Weyl spectrum) of an  $A \in B(\mathcal{X})$  is the set  $\sigma_b(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Fredholm or one of } \text{asc}(A - \lambda) \text{ and } \text{dsc}(A - \lambda) \text{ is not finite}\}$  (resp.,  $\sigma_w(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Fredholm or } \text{ind}(A - \lambda) \neq 0\}$ ).

Subscalar operators  $A$  have the property that their quasinilpotent part satisfies

$$H_0(A - \lambda) = (A - \lambda)^{-m}(0)$$

for some integer  $m = m(\lambda) \geq 0$  and all  $\lambda \in \mathbb{C}$ . Let  $K(A - \lambda)$  denote the analytic core

$$K(A - \lambda) = \{x \in \mathcal{X} : \text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for which } x = x_0, (A - \lambda)(x_{n+1}) = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, \dots\}$$

of  $A - \lambda$ .  $K(A - \lambda)$  is (generally) a non-closed hyperinvariant subspace of  $(A - \lambda)$  such that  $(A - \lambda)K(A - \lambda) = K(A - \lambda)$  [16]. If  $A$  is subscalar and  $\lambda$  is an isolated point of  $\sigma(A)$ ,  $\lambda \in \text{iso}\sigma(A)$ , then

$$\begin{aligned} \mathcal{X} &= H_0(A - \lambda) \oplus K(A - \lambda) = (A - \lambda)^{-m}(0) \oplus K(A - \lambda) \\ \implies \mathcal{X} &= (A - \lambda)^{-m}(0) \oplus (A - \lambda)^m \mathcal{X}, \end{aligned}$$

so that isolated points of  $\sigma(A)$  are poles of the resolvent of  $A$  (in the terminology of [7], *subscalar operators are polaroid*).

Recall from [15, Corollary 3.4.5] that if  $\lim_{n \rightarrow \infty} \|\Delta_{AB}^n(I)\|^{\frac{1}{n}} = 0$ , then  $\sigma_A(x) \subseteq \sigma_B(x)$  for all  $x \in \mathcal{X}$ . Here  $\sigma_T(x)$  denotes the *local spectrum of  $T$  at  $x$*  (see [15, p 16] for definition). Since  $\bigcup\{\sigma_T(x) : x \in \mathcal{X}\}$  equals the surjectivity spectrum  $\sigma_s(T)$  of  $T$ , and since  $\sigma(T) = \sigma(T^*) = \sigma_s(T)$  whenever  $T$  has SVEP [15, Proposition 1.3.2], it follows that if  $\lim_{n \rightarrow \infty} \|\Delta_{AB}^n(I)\|^{\frac{1}{n}} = 0$  and  $A$  has SVEP then  $\sigma(A) \subseteq \sigma_s(B) \subseteq \sigma(B)$ . Furthermore, since  $\lim_{n \rightarrow \infty} \|\Delta_{AB}^n(I)\|^{\frac{1}{n}} = 0 \implies \lim_{n \rightarrow \infty} \|\Delta_{B^*A^*}^n(I^*)\|^{\frac{1}{n}} = 0 \implies \sigma_{B^*}(y) \subseteq \sigma_{A^*}(y)$  for all  $y \in \mathcal{X}^*$ , if also  $B^*$  has SVEP, then  $\sigma(B) = \sigma_s(B^*) \subseteq \sigma_s(A^*) \subseteq \sigma(A^*)$  ( $\implies \sigma(B) \subseteq \sigma(A)$ ). Hence, if  $\lim_{n \rightarrow \infty} \|\Delta_{AB}^n(I)\|^{\frac{1}{n}} = 0$  and both  $A, B^*$  have SVEP, then  $\sigma(A) = \sigma(B)$ . The following theorem proves more. Let  $\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Fredholm}\}$  denote the (Fredholm) essential spectrum of  $A$ .

**THEOREM 3.1.** *Let  $A, B \in B(\mathcal{X})$  be such that  $\Delta_{AB}^k(I) = 0$  for some integer  $k \geq 1$ . If  $A$  and  $B^*$  have SVEP, then  $\sigma(A) = \sigma(B)$  and  $\sigma_e(A) = \sigma_b(A) = \sigma_w(A) = \sigma_e(B) = \sigma_b(B) = \sigma_w(B)$ .*

*Proof.* It is easily seen that the hypothesis  $A$  and  $B^*$  have SVEP, coupled with the hypothesis  $\Delta_{AB}^k(I) = 0$ , implies that  $A, A^*, B$  and  $B^*$  all have SVEP. Hence,

[1, Theorem 3.52],  $\sigma_e(A) = \sigma_b(A) = \sigma_w(A)$  and  $\sigma_e(B) = \sigma_b(B) = \sigma_w(B)$ . Evidently,  $\lambda \in \sigma(A) \setminus \sigma_e(A) \iff \lambda \in \sigma(A) \setminus \sigma_b(A) \iff \lambda \in \sigma(A)$  is a finite rank pole of the resolvent of  $A$ . (In particular, since  $\sigma(A) = \sigma(B)$ , see above,  $\lambda \in \text{iso}\sigma(B)$ .) Recall now from [15, Corollary 3.4.5] that if  $\Delta_{AB}^k(I) = 0$ , then  $H_0(B - \lambda) \subseteq H_0(A - \lambda)$ . Hence,  $H_0(B - \lambda)$  is finite dimensional. Consequently,  $\lambda \notin \sigma_e(B)$  [15, Proposition 3.7.5] ( $\iff \lambda \in \sigma(B) \setminus \sigma_e(B)$ ). Conversely,  $\lambda \in \sigma(B) \setminus \sigma_e(B) \iff \lambda \in \sigma(B^*) \setminus \sigma_e(B^*) \implies \dim H_0(B^* - \lambda I^*) < \infty$  and  $\lambda \in \text{iso}\sigma(A^*)$ . Since  $\Delta_{B^*A^*}^k(I^*) = 0$  implies  $H_0(A^* - \lambda I^*) \subseteq H_0(B^* - \lambda I^*)$ ,  $\lambda \notin \sigma_e(A^*) \iff \lambda \in \sigma(A) \setminus \sigma_e(A)$ . This completes the proof.  $\square$

REMARK 3.2. The hypothesis  $\Delta_{AB}^k(I) = 0$  for some integer  $k \geq 1$  can not be replaced by the hypothesis that  $B$  is intertwined to  $A$  by a quasi-affinity (or, even quasi-similarity), as the following example from Stampfli [17, Example, page 11] shows. Let  $\{e_n\}_{n=1}^\infty$  be an orthonormal basis for the Hilbert space  $\mathcal{H}$ . Let, for each  $m = 1, 2, \dots$ ,  $A_m e_n = w_n^{(m)} e_{n+1}$ , where  $w_n^{(m)} = 1$  for  $n \leq 0$ ,  $w_n^{(m)} = 1 + \frac{n}{m}$  for  $1 \leq n \leq m$ ,  $w_n^{(m)} = 2$  for  $m+1 \leq n \leq 2m$ ,  $w_n^{(m)} = 1 + \frac{3m-n}{m}$  for  $2m \leq n \leq 3m$  and  $w_n^{(m)} = 1$  for  $3m+1 \leq n$ . Then the operator  $A = \bigoplus_1^\infty A_m$  (is essentially normal and) has SVEP. Let  $B = \bigoplus_1^\infty U$ , where  $U \in B(\mathcal{H})$  is the bilateral shift. Evidently,  $B$  and  $B^*$  have SVEP,  $AX = XB$  for some quasi-affinity  $X$  (indeed, since each  $A_m$  is similar to  $U$ ,  $A$  and  $B$  are quasisimilar),  $\sigma(A) = \{\lambda : 1 \leq |\lambda| \leq 2\}$  and  $\sigma(B)$  is the unit circle.

Recall that an operator  $A$  satisfies *Weyl's theorem* if  $\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A)$ , where  $\pi_{00}(A) = \{\lambda \in \text{iso}\sigma(A) : \lambda \text{ is a finite multiplicity eigenvalue of } A\}$ . Observe that if  $A$  satisfies Weyl's theorem, then  $\pi_{00}(A) = p_{00}(A)$ , where  $p_{00}(A) = \{\lambda \in \text{iso}\sigma(A) : \lambda \text{ is a finite rank pole of the resolvent of } A\}$ . Let  $\sigma_a(A)$  denote the approximate point spectrum of  $A$ , and let  $\sigma_{aw}(A) = \{\lambda \in \sigma_a(A) : A - \lambda \text{ is not upper semi-Fredholm or the index of } A - \lambda \text{ is not less than or equal to } 0\}$  denote the essential approximate Weyl spectrum of  $A$ .  $A$  satisfies *a-Weyl's theorem* if the complement of  $\sigma_{aw}(A)$  in  $\sigma_a(A)$  is the set  $\pi_{00}^a(A) = \{\lambda \in \sigma_a(A) : 0 < \dim(A - \lambda)^{-1}(0) < \infty\}$ . (See [1, Chapter 3, Section 8] or [9, 7] for further information on Weyl and *a*-Weyl theorems.) Subscalar operators satisfy Weyl's theorem [1, Theorem 3.99].

THEOREM 3.3. *Let  $B \in B(\mathcal{X})$  be such that  $\Delta_{AB}^k(I) = 0$  for some subscalar operator  $A \in B(\mathcal{X})$  and some integer  $k \geq 1$ . If  $N \in B(\mathcal{X})$  is an algebraic operator which commutes with  $B$ , then  $B + N$  satisfies Weyl's theorem and  $B^* + N^*$  satisfies *a*-Weyl's theorem.*

*Proof.* The operator  $B + N$  is subscalar (by Theorem 2.4), and so satisfies Weyl's theorem. Apparently, see above,  $B + N$  has SVEP and is polaroid. SVEP implies that  $\sigma(B + N) = \sigma_a(B^* + N^*)$  [15, Proposition 1.3.2]; consequently,  $\pi_{00}(B^* + N^*) = \pi_{00}^a(B^* + N^*)$ . Again, since  $\lambda \notin \sigma_{aw}(B^* + N^*)$  if and only if  $B^* + N^* - \lambda$  is upper semi-Fredholm and  $\text{ind}(B^* + N^* - \lambda) \leq 0$ , and since  $B + N$  has SVEP and  $B^* + N^* - \lambda$  is upper semi-Fredholm implies that  $\text{ind}(B^* + N^* - \lambda) \geq 0$  [1, Corollary 3.19],

$$\lambda \notin \sigma_{aw}(B^* + N^*) \implies \lambda \notin \sigma_w(B^* + N^*).$$

This, since  $\sigma_{aw}(T) \subseteq \sigma_w(T)$  for every  $T \in B(\mathcal{X})$ , implies that  $\sigma_{aw}(B^* + N^*) = \sigma_w(B^* + N^*) = \sigma_w(B + N)$ . As stated above,  $B + N$  is polaroid. Hence  $B^* + N^*$  is polaroid, with  $p_{00}(B + N) = p_{00}(B^* + N^*)$ . This, since  $B + N$  satisfies Weyl's theorem, implies that  $\pi_{00}(B + N) = p_{00}(B + N) = p_{00}(B^* + N^*) \subseteq \pi_{00}(B^* + N^*) = \pi_{00}^a(B^* + N^*)$  ( $= \{\lambda \in \text{iso}\sigma_a(B^* + N^*) : B^* + N^* - \lambda$  is upper semi-Fredholm  $\}$ ). Again, since  $\lambda \in \pi_{00}^a(B^* + N^*)$  implies  $\lambda \in \text{iso}\sigma(B + N) \implies \lambda \in p_{00}(B + N)$ , we have that  $\pi_{00}(B + N) = \pi_{00}^a(B^* + N^*)$ . Putting it all together, it follows that

$$\sigma(B + N) \setminus \sigma_w(B + N) = \pi_{00}(B + N) \Rightarrow \sigma_a(B^* + N^*) \setminus \sigma_{aw}(B^* + N^*) = \pi_{00}^a(B^* + N^*),$$

i.e.,  $B^* + N^*$  satisfies  $a$ -Weyl's theorem.  $\square$

#### 4. Applications II: Elementary operator $d_{AB}$

In this section we restrict ourselves to Hilbert space operators  $A, B^* \in B(\mathcal{H})$ , and consider an application of the results of Section 2 to the elementary operator  $d_{AB} \in B(B(\mathcal{H}))$ ,  $d_{AB} = L_A - R_B$  or  $L_A R_B - 1$ . (Recall that  $L_A$  and  $R_B$  denote, respectively, the operators of left multiplication by  $A$  and right multiplication by  $B$ .) If the operators  $A$  and  $B^*$  are subscalar, then (both)  $L_A$  and  $R_B$  satisfy Dunford's condition (C) [15, Corollary 3.6.11]. This, since  $L_A$  and  $R_B$  commute, by [15, Theorem 3.6.3 and Note 3.6.19, Page 283] implies that  $L_A - R_B$  and  $L_A R_B$  have SVEP (everywhere). Hence:

PROPOSITION 4.1. *If  $A, B^* \in B(\mathcal{H})$  are subscalar, then  $d_{AB}$  has SVEP.*

We remark here that the conclusion  $d_{AB}$  has SVEP does not require the full force of the hypothesis on the subscalarity of  $A$  and  $B^*$ : the hypothesis that  $A$  and  $B^*$  satisfy property  $(\beta)$  would do just as well.

As seen above, isolated points of the spectrum of a subscalar operator are poles of the resolvent of the operator. We prove in the following that the operator  $d_{AB}$  satisfies a similar property in the case in which eigenvectors corresponding to distinct eigenvalues of the subscalar operators  $A$  and  $B^*$  are (mutually) orthogonal.

THEOREM 4.2. *If  $A, B^* \in B(\mathcal{H})$  are polaroid operators (i.e., isolated points of the spectrum of  $A$ , similarly  $B^*$ , are poles of the resolvent of  $A$ , respectively  $B^*$ ), and if eigen-spaces corresponding to distinct eigen-values of  $A$  (similarly,  $B^*$ ) are orthogonal, then  $d_{AB}$  is polaroid.*

*Proof.* It is known, [11, Theorem 3.2], that  $\sigma(L_A - R_B) = \bigcup \{ \alpha - \beta : \alpha \in \sigma(A), \beta \in \sigma(B) \}$  and  $\sigma(L_A R_B - 1) = \{ \alpha\beta - 1, \alpha \in \sigma(A), \beta \in \sigma(B) \}$ . If  $\lambda \in \text{iso}\sigma(d_{AB})$ , then there exist finite sets  $\{ \alpha_1, \dots, \alpha_n \} \subset \text{iso}\sigma(A)$  and  $\{ \beta_1, \dots, \beta_n \} \subset \text{iso}\sigma(B)$  such that  $\alpha_i - \beta_i = \lambda$  if  $d_{AB} = L_A - R_B$ , and  $\alpha_i\beta_i - 1 = \lambda$  if  $d_{AB} = L_A R_B - 1$ , for all  $1 \leq i \leq n$ . Assuming  $H_0(A - \lambda) = (A - \lambda)^{-p}(0)$  and  $H_0(B^* - \bar{\lambda}) = (B^* - \bar{\lambda})^{-q}(0)$  for some non-negative integers  $p, q$  and all complex numbers  $\lambda$ , let

$$H'_1 = \bigvee_{i=1}^n (A - \alpha_i)^{-p}(0), H_1 = \bigvee_{i=1}^n (B^* - \bar{\beta}_i)^{-q}(0), H'_2 = \mathcal{H} \ominus H'_1 \text{ and } H_2 = \mathcal{H} \ominus H_1.$$

The subspaces  $H'_1$  and  $H_1$  being invariant for  $A$  and  $B^*$ , respectively,  $A$  and  $B$  have the triangular representations:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} H'_1 \\ H'_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}.$$

Then  $0 \notin \sigma(d_{A_{ii}B_{jj}} - \lambda)$  for all  $i, j = 1, 2$  except for  $i = j = 1$ . Letting  $X \in H_0(d_{AB} - \lambda)$ ,  $X : H'_1 \oplus H'_2 \rightarrow H_1 \oplus H_2$ , have the matrix representation  $X = [X_{ij}]_{i,j=1}^2$ , it follows that

$$(d_{AB} - \lambda)^m = \begin{pmatrix} * & * \\ * & (d_{A_{22}B_{22}} - \lambda)^m X_{22} \end{pmatrix}$$

for every natural number  $m$  (and for some as yet to be determined entries “\*”). The hypothesis  $X \in H_0(d_{AB} - \lambda)$  implies that

$$\lim_{m \rightarrow \infty} \|(d_{A_{22}B_{22}} - \lambda)^m X_{22}\|^{\frac{1}{m}} = 0.$$

Since  $d_{A_{22}B_{22}} - \lambda$  is invertible, it follows that  $X_{22} = 0$ , and hence that

$$(d_{AB} - \lambda)^m = \begin{pmatrix} * & (d_{A_{11}B_{22}} - \lambda)^m X_{12} \\ (d_{A_{22}B_{11}} - \lambda)^m X_{21} & 0 \end{pmatrix}$$

for every natural number  $m$  (and for some as yet to be determined entry “\*”). Again, the hypothesis  $X \in H_0(d_{AB} - \lambda)$  implies that

$$\lim_{m \rightarrow \infty} \|(d_{A_{11}B_{22}} - \lambda)^m X_{12}\|^{\frac{1}{m}} = \lim_{m \rightarrow \infty} \|(d_{A_{22}B_{11}} - \lambda)^m X_{21}\|^{\frac{1}{m}} = 0;$$

hence, since  $d_{A_{11}B_{22}} - \lambda$  and  $d_{A_{22}B_{11}} - \lambda$  are invertible,  $X_{12} = X_{21} = 0$ . Consequently,

$$(d_{AB} - \lambda)^m = \begin{pmatrix} (d_{A_{11}B_{11}} - \lambda)^m & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\lim_{m \rightarrow \infty} \|(d_{A_{11}B_{11}} - \lambda)^m X_{11}\|^{\frac{1}{m}} = 0.$$

Suppose now that the eigenvectors corresponding to distinct eigenvalues of  $A$ , similarly  $B^*$ , are orthogonal. Then  $A_{11}$  and  $B_{11}^*$  are normal, hence generalized scalar operators. Consequently,  $d_{A_{11}B_{11}} - \lambda$  is a generalized scalar operator [6, 4.3.3 Theorem]; hence

$$H_0(d_{A_{11}B_{11}} - \lambda) = (d_{A_{11}B_{11}} - \lambda)^{-t}(0)$$

for some positive integer  $t$  [6, Theorem 4.4.5]. Evidently,  $H_0(d_{AB} - \lambda) = H_0(d_{A_{11}B_{11}} - \lambda)$ ; This, as earlier seen, implies that  $d_{AB}$  is polaroid at  $\lambda$ .  $\square$

Paranormal operators (i.e., operators  $T$  such that  $\|Tx\|^2 \leq \|T^2x\|^2$  for every unit vector  $x \in \mathcal{H}$ ) satisfy the hypotheses of Theorem 4.2. A condition guaranteeing orthogonality of the eigen-spaces corresponding to distinct eigen-values of the operator  $A_{11}$  (similarly,  $B_{11}^*$ ) of the proof of Theorem 4.2 above is that every part of the operator  $A$  is both polaroid and normaloid. (A part of an operator is its restriction to an invariant subspace, and an operator  $T$  is normaloid if its norm equals its spectral radius  $r(T)$ .)

LEMMA 4.3. *If  $A \in B(\mathcal{H})$  is such that every part of  $A$  is both polaroid and normaloid, then the eigen-spaces corresponding to distinct eigen-values of  $A$  are orthogonal.*

*Proof.* Let  $\lambda_1$  and  $\lambda_2$  be two distinct eigenvalues of  $A$ , and let  $M$  denote the subspace generated by the eigenvectors corresponding to these eigenvalues. Let  $A_1 = A|_M$ . Then  $\sigma(A_1) = \{\lambda_1, \lambda_2\}$ , and  $A_1$  is a normaloid operator such that all its spectral points are poles of the resolvent. Letting  $|\lambda_1| \leq |\lambda_2|$ , an application of [13, Proposition 54.4] implies that  $\|x\| \leq \|x + \mu y\|$  for all  $\mu \in \mathbb{C}$ , every  $x \in \text{null}(T_1 - \lambda_2)$  and  $y \in \text{null}(T_1 - \lambda_1)$ . This, in turn, implies that  $(x, y) = 0$ .  $\square$

The following theorem applies (in particular) to operators  $A, B^* \in \Xi \cap B(\mathcal{H})$ . Recall that  $T$  satisfies Weyl's theorem (resp.,  $a$ -Weyl's theorem) if  $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$  (resp.,  $\sigma_a(T) \setminus \sigma_{wa}(T) = \pi_{00}^a(T)$ ).

THEOREM 4.4. *If  $A$  and  $B^* \in B(\mathcal{H})$  are subscalar operators such that the eigenvectors corresponding to distinct eigen-values of  $A$ , similarly  $B^*$ , are orthogonal, then  $d_{AB}$  satisfies Weyl's theorem and (the dual operator)  $d_{AB}^*$  satisfies  $a$ -Weyl's theorem.*

*Proof.* Since  $d_{AB}$  has SVEP by Proposition 4.1 and  $d_{AB}$  is polaroid by Theorem 4.2,  $d_{AB}$  satisfies Weyl's theorem [1, Theorem 3.85]. Now argue as in the proof of Theorem 3.3 to prove that  $d_{AB}^*$  satisfies  $a$ -Weyl's theorem.  $\square$

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