

NON METRIZABILITY OF SOME TOPOLOGIZABLE ALGEBRAS

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Abstract. We compare topologization of certain algebras as topological algebras as well as metrizable semitopological algebras. The main aim of the paper is to prove that the algebra of continuous functions on a non Lindelöf paracompact space is not topologizable as a metrizable semitopological algebra. This gives us further examples of locally convex topologizable algebras that are not topologizable as metrizable semitopological algebras. On the other hand we prove that the free algebra generated by \aleph variables has at last $\max\{2^{\aleph_0}, \aleph\}$ topologies of metrizable semitopological locally convex algebras. We provide also, for all $p \in (0, 1]$, an example of locally bounded topological algebra which is not topologizable as a metrizable semitopological locally p -convex algebra. This is an investigation in the direction of some results of J. Esterle, V. Müller, W. Żelazko and the author in previous papers.

1. Introduction and preliminaries

In this paper, we study for certain algebras the existence of topologies of semitopological metrizable algebra and topologies of topological algebra. We begin by listing a series of results related to the context of the present paper. Firstly, we state two results due to J. Esterle [8]:

THEOREM 1. [8, Corollary 2.2, Theorem 3.3]. *Let E be a barreled locally convex space which is non normable, then the algebra $L_{FR}(E)$ of continuous finite rank operators on E is neither topologizable as a topological algebra nor as a metrizable semitopological algebra.*

THEOREM 2. [8, Theorem 4.3]. *Let E be a complete metrizable topological vector space which is separated by its topological dual (i.e., for all nonzero element $x \in E$, there is a continuous linear functional f satisfying $f(x) \neq 0$). Denote by $L_{FR}(E)$ the algebra of continuous finite rank operators on E , then the following statements are equivalent:*

- (1) $L_{FR}(E)$ is topologizable as a metrizable semitopological algebra,
- (2) $L_{FR}(E)$ is topologizable as a topological algebra,
- (3) $L_{FR}(E)$ is topologizable as a normable topological algebra.

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In the commutative case, V. Müller gave a counter example of non topologizable algebras (see [16, Theorem 1]). Other counter examples are provided by R. Frankiewicz and D. Plebanek [9, Theorem 3], where the considered algebra is of dimension \aleph_1 . This cardinal property is optimal and it is motivated by the following result due to W. Żelazko:

THEOREM 3. [25, Theorem 2]. *Let \mathcal{A} be a countably or finitely generated algebra. Then \mathcal{A} endowed with its maximal locally convex topology is a locally convex topological algebra.*

We will prove in Section 2 that every infinitely countably generated algebra possesses an infinitely many different topologies making it a metrizable locally convex topological algebra. We prove also that the free algebra generated by $(X_i)_{i \in I}$ variables has $\max\{2^{\aleph}, \aleph_I\}$ different metrizable locally convex topologies making it a topological algebra. Let mention here that such an algebra admits several topologies making it a complete locally convex topological algebra. This is proved in [19] and generalized in [3]. Namely,

THEOREM 4. [19, Proposition 2, 3 and 4]. *Let \mathcal{A} be a free algebra or an algebra of polynomials (resp. the algebra of finite support functions on a set of cardinal \aleph). Then, there exists 2^{\aleph_0} different complete locally convex (resp. m -convex) topologies on \mathcal{A} making it a topological algebra.*

THEOREM 5. [3, Théorème 3]. *Let \mathcal{A} be a free algebra of I non commuting variables. Then, there exists $\max\{2^{\aleph_0}, \aleph_I\}$ different complete locally convex (resp. m -convex) topologies on \mathcal{A} making it a topological algebra.*

Also in [3, Page 135] we fined the result (with some indication for the proof): Let \mathcal{A} be an algebra of polynomials of \aleph commuting variables (resp. the algebra of finitely supported functions on a set of cardinal \aleph). Then there exists $\max\{2^{\aleph_0}, \aleph_I\}$ different complete locally convex (resp. m -convex) topologies on \mathcal{A} making it a topological algebra. In the same direction we set

THEOREM 6. [4, Proposition 2]. *Let \mathcal{A} be an algebra with a Hamel basis \mathcal{E} satisfying:*

- (1) for all $(f, g) \in \mathcal{E}^2$, $f \cdot g \in \mathbb{K} \cdot \mathcal{E}$,
- (2) for all $e \in \mathcal{E}$, the set $\{(f, g) \in \mathcal{E}^2 \mid f \cdot g \in (\mathbb{K} \setminus \{0\}) \cdot e\}$ countable or finite.

Then, \mathcal{A} can be topologized as a locally convex complete topological algebra.

The Section 3, is devoted to the construction of counterexamples concerning respectively the existence of an algebra which is neither topologizable as a topological algebra nor as a metrizable semitopological algebra, the existence of topologizable algebra which is not topologizable as a metrizable semitopological algebra and for each

$p \in (0, 1]$ a locally bounded topologizable algebra which is not topologizable as a locally p -convex metrizable semitopological. The main result of this paper concerns the non metrizability of some function algebras. See Section 4 for precise statements. Below, we collect and fix some needed notations and basic definitions.

Let \mathbb{K} be the field \mathbb{R} (or \mathbb{C}) of real (or complex) numbers, N the set of positive integers and $\mathbb{N} = N \cup \{0\}$. Denote by \mathcal{A} an associative algebra (algebra for short) over the field \mathbb{K} and let τ be a topology defined on \mathcal{A} such that (\mathcal{A}, τ) becomes a Hausdorff topological vector space. Throughout this paper all topologies will be Hausdorff. Following W. Żelazko (see e.g. [19], [25], [21], [24]) we say that an algebra \mathcal{A} is topologizable as a topological (resp. semitopological) locally bounded, metrizable, locally pseudoconvex, locally convex, complete ... algebra if there is a topology on \mathcal{A} of topological (resp. semitopological) algebra such that the underlying topological vector space is locally bounded, metrizable, locally pseudoconvex, locally convex, complete ... Now we introduce some notations and recall standard facts needed for this paper (for more informations about these notions see [12], [19]-[21], for basic properties of topological vector spaces and algebras see [6], [7], [13], [14], [18], [26] and for further informations concerning topological spaces and continuous functions algebras see [10], [11], [15], [17]):

N° 1.1. Let \mathcal{A} be an algebra endowed with a topology τ of topological vector space.

- (1) (\mathcal{A}, τ) is a topological algebra if the multiplication on \mathcal{A} is jointly continuous i.e, the mapping $(\mathcal{A}, \tau) \times (\mathcal{A}, \tau) \rightarrow (\mathcal{A}, \tau), (x, y) \mapsto xy$ is continuous.
- (2) (\mathcal{A}, τ) is a semitopological algebra if the multiplication on \mathcal{A} is separately continuous i.e, for all $x \in \mathcal{A}$, the mappings defined on \mathcal{A} by $y \mapsto xy$ and $y \mapsto yx$ are continuous.
- (3) (\mathcal{A}, τ) is metrizable if the underlying topological vector space (\mathcal{A}, τ) is metrizable.

N° 1.2. Let E be a \mathbb{K} -vector space.

- (1) A pseudoseminorm on E is a p -seminorm f , where $p \in (0, 1]$, i.e, $f: E \rightarrow [0, \infty)$ satisfying $f(x+y) \leq f(x) + f(y)$ and $f(\lambda x) = |\lambda|^p f(x)$, where $(\lambda, x, y) \in \mathbb{K} \times E^2$.
- (2) A pseudonorm (or a p -norm) is a pseudoseminorm (or a p -seminorm) f satisfying $f(x) = 0 \implies x = 0$. A norm (resp. seminorm) is a 1-norm (resp. 1-seminorm).
- (3) (E, τ) is locally bounded if there exists a bounded neighborhood V of zero (satisfying $\lambda V \subseteq V$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$). By [13, chap. 1, §15-10] (E, τ) is locally bounded if and only if the topology τ is defined by a p -norm for some $p \in (0, 1]$.
- (4) For $p \in (0, 1]$, (E, τ) is locally p -convex if the topology τ is given by mean of p -seminorms. (\mathcal{A}, τ) is locally convex if it is locally 1-convex.

- (5) (E, τ) is locally pseudoconvex if the topology τ is given by means of p -seminorms, $p \in (0, 1]$.

N° 1.3. (\mathcal{A}, τ) is a metrizable semitopological algebra if and only if there exists a fundamental system $(V_n)_{n \in \mathbb{N}}$ of neighborhood of zero in \mathcal{A} satisfying:

- (m_1) $\bigcap_{n \in \mathbb{N}} V_n = \{0\}$,
- (m_2) for all $n \in \mathbb{N}$, $\lambda V_n \subseteq V_n$ ($\lambda \in \mathbb{K}$, $|\lambda| \leq 1$),
- (m_3) for all $n \in \mathbb{N}$, $\bigcup_{m \in \mathbb{N}} m \cdot V_n = A$,
- (m_4) for all $n \in \mathbb{N}$, $V_{n+1} + V_{n+1} \subseteq V_n$,
- (m_5) for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$, there exists an integer $m \geq n$ such that $a \cdot V_m \subseteq V_n$ and $V_m \cdot a \subseteq V_n$.

Now, giving $p \in (0, 1]$, (\mathcal{A}, τ) is a locally p -convex metrizable semitopological algebra if and only if there exists a fundamental system $(V_n)_{n \in \mathbb{N}}$ of neighborhood of zero in \mathcal{A} satisfying (m_1) , (m_2) , (m_3) , (m_4) , (m_5) and a complementary property

- (m_6) for $(n, m) \in \mathbb{N}^2$ and $(\lambda_1, \dots, \lambda_m) \in [0, 1]^m$ with $\sum_{k=1}^m \lambda_k = 1$,

$$\lambda_1^{\frac{1}{p}} \cdot V_n + \dots + \lambda_m^{\frac{1}{p}} \cdot V_n \subseteq V_n$$

N° 1.4. We denote by \aleph_0 (resp. \aleph_1) the infinite countable cardinal (resp. the smallest uncountable cardinal), ω_0 (resp. ω_1) the first ordinal of cardinal \aleph_0 (resp. \aleph_1).

Now, for a nonempty set I , we denote by $\text{card}(I)$ the cardinal of I , $\text{Fin}(I)$ the set of nonempty finite subsets of I , \aleph_I the cardinal successor of $\text{card}(I)$ and ω_I the first ordinal of cardinal \aleph_I .

N° 1.5. For each nonempty set I , we denote $I^\infty = \bigcup_{n \in \mathbb{N}} I^n$, where $I^0 = (\circ)$ here \circ is a symbol such that $\circ \notin I$, $I^1 = I$ and for $n \in \mathbb{N}$, $I^{n+1} = I^n \times I$. We define a structure of ordered semigroup on I^∞ by

- (1) For $f \in I^n$ we put $c(f) = n$ ($n \in \mathbb{N}$).
- (2) For all $f = (f_1, \dots, f_m) \in I^n$ and $g = (g_1, \dots, g_n) \in I^m$, where $(n, m) \in \mathbb{N}^2$ we put $fg = (f_1, \dots, f_m, g_1, \dots, g_n)$ and for all $f \in I^\infty$, $(\circ)f = f(\circ) = f$.
- (3) If $(f, g) \in I^\infty \times I^\infty$, $f \leq g$ if $f = (\circ)$ or if $0 < c(f) \leq c(g)$ and if there exists $1 \leq i_1 < \dots < i_{c(f)} \leq c(g)$, such that $f_k = g_{i_k}$ ($k \in \{1, \dots, m\}$).

N° 1.6. Let I be a nonempty set I .

- (1) Denoting $S(I) = \{R \in S'(I) \mid R + R \subseteq R\}$, where $S'(I)$ is the set of all nonempty subsets R of \mathbb{N}^I such that $\bigcap_{r \in R} \{i \in I \mid r_i = 0\} = \emptyset$.

- (2) If E is a \mathbb{K} -vector space with a Hamel basis $(e_i)_{i \in I}$ and $(e_i^*)_{i \in I}$ is the associated dual basis. Then for each $x \in E$, we have $x = \sum_{i \in I} e_i^*(x)e_i$, where $\text{supp}(x) = \{i \in I \mid e_i^*(x) \neq 0\}$ is finite.
- (3) For every $p \in (0, \infty)$ and $r \in \mathbb{N}^I$, set $\bar{p} = \min\{1, \frac{1}{p}\}$, and consider the following pseudoseminorm:

$${}_p\| \|_r : E \rightarrow \mathbb{R}^+, \quad {}_p\|x\|_r = \left(\sum_{i \in I} |e_i^*(x)|^p r_i\right)^{\bar{p}}$$

Clearly if $p \in (0, 1]$, ${}_p\| \|_r$ is a p -seminorm and if $p \in [1, \infty)$, ${}_p\| \|_r$ is a seminorm (also in this case we will call it a p -seminorm).

- (4) For $R \in S'(I)$ and $P = (P_r)_{r \in R} \subseteq (0, \infty)^R$, let us consider the topology ${}_P\tau_R$ defined on E by the family of seminorms ${}_p\| \|_r$ ($r \in R$ and $p \in P_r$). Thus $(E, {}_P\tau_R)$ is a topological vector space.
- (5) For all p -seminorms $\| \|$, $p \in (0, \infty)$, we denote by $\| \|*$ the mapping $\| \| \xrightarrow{\max\{\frac{1}{p}, 1\}}$. Note that a p -seminorm $\| \| (E, {}_P\tau_R) \rightarrow \mathbb{R}^+$ is continuous if and only if there is a finite subset R' of R , a finite subset P'_r of P_r ($r \in R'$) and a positive real M such that

$$\|x\|^* \leq M \sum_{r \in R'} \sum_{q \in P'_r} q \|x\|_r^*$$

- (6) For each $J \subseteq I$, we denote by E_J the linear subspace of E spanned by $(e_j)_{j \in J}$ and $\Pi_J : E \rightarrow E$, $x \mapsto \sum_{j \in J} e_j^*(x)e_j$, thus $\Pi_{I \setminus J} = \Pi_I - \Pi_J$. Clearly for all $(p, r) \in (0, \infty) \times \mathbb{N}^I$ and $x \in E$,

$${}_p\|x\|_r^{\frac{1}{\bar{p}}} = {}_p\|\Pi_{I \setminus J}(x)\|_r^{\frac{1}{\bar{p}}} + {}_p\|\Pi_J(x)\|_r^{\frac{1}{\bar{p}}}$$

A characterization of the completeness of the topologies of type ${}_P\tau_R$ was given in [3, Proposition 1]. We include here such a proposition with its proof (also in the order to fix some mistakes in the notations).

THEOREM 7. *Let be $R \in S'(I)$ and $P = (P_r)_{r \in R} \in (0, \infty)^R$. The following statements are equivalent.*

- (1) $(E, {}_P\tau_R)$ is complete.
- (2) For all $J \subseteq I$ with $\text{card}(J) = \aleph_0$ and for all $(u_i)_{i \in J} \in \mathbb{N}^J$, there exists $r \in R$ satisfying

$$\sum_{j \in J} \frac{r_j}{u_j} = \infty$$

- (3) For all $J \subseteq I$ with $\text{card}(J) = \aleph_0$ and for all $(u_i)_{i \in J} \in \mathbb{N}^J$, there exists $J' \subseteq J$ and $r \in R$, such that $\text{card}(J') = \aleph_0$ and $r_j > u_j$ ($j \in J'$).

Proof. (1) \Rightarrow (2). Suppose that there exists $J = \{j_k : k \in N\} \subseteq I$ and $(u_i)_{i \in J} \in N^J$ such that for all $r \in R$, $\sum_{j \in J} \frac{r_j}{u_j} < \infty$. For all $n \in N$, we put $S_n = \sum_{k=1}^n \frac{1}{(u_{j_k})^k} e_{j_k}$. For arbitrary $r \in R$, $p \in P_r$ and $(m, n) \in N^2$ with $m > n$, we have ${}_p\|S_m - S_n\|_r^{\frac{1}{p}} \leq \sum_{k=n+1}^m \frac{r_{j_k}}{(u_{j_k})^{pk}}$. Then $(S_n)_{n \in N}$ is a Cauchy sequence in the complete topological space E . Set $S = \lim_{n \rightarrow \infty} S_n$, $L = \text{supp}(S) = \{i \in I \mid e_i^*(S) \neq 0\}$ and $m = \min\{n \in N \text{ for all } k \geq n, j_k \notin L\}$. The contradiction is a consequence of the fact that for all $n \geq m$,

$$\begin{aligned} {}_p\|S - S_n\|_r^{\frac{1}{p}} &= {}_p\|S - \Pi_L(S_n)\|_r^{\frac{1}{p}} + {}_p\|\Pi_{I \setminus L}(S_n)\|_r^{\frac{1}{p}} \\ &\geq {}_p\|\Pi_{I \setminus L}(S_n)\|_r^{\frac{1}{p}} \geq \frac{r_{j_m}}{(u_{j_m})^{pm}} > 0 \end{aligned}$$

(2) \Rightarrow (3). There exists $r \in R$ such that

$$\sum_{k=1}^{\infty} \frac{r_{j_k}}{k^2 u_{j_k}} = \infty, \text{ then the set } J' = \{j \in J \mid \frac{r_{j_k}}{u_{j_k}} > 1\} \text{ is infinite.}$$

(3) \Rightarrow (1). Let $(S_\lambda)_\lambda$ be a Cauchy net, for all $i \in I$, $(e_i^*(S_\lambda))_\lambda$ is also a Cauchy net, then it converges to a limit a_i .

We claim that the set $L = \{i \in I \mid a_i \neq 0\}$ is finite. Indeed, suppose the contrary, choose an infinite countable subset $\{i_n \mid n \in N\}$ of L . There is $r \in R$ such that $J = \{n \in N \mid r_{i_n} \cdot \min(|a_{i_n}|^n, |a_{i_n}|^{1/n}) > n\}$ is infinite. Choose $p \in P_r$ and denote $M_r = \lim_p \|S_\lambda\|_r$.

Thus,

$$n < |a_{i_n}|^p \cdot r_{i_n} = \lim_\lambda |e_{i_n}^*(S_\lambda)|^p \cdot r_{i_n} \leq \lim_\lambda ({}_p\|S_\lambda\|_r)^{\frac{1}{p}} \leq (M_r)^{\frac{1}{p}}$$

for all $n \in J$ satisfying $n \geq \max\{p, 1/p\}$, which is a contradiction. Hence $S = \sum_{i \in I} a_i e_i \in E$, and we can suppose that $\lim_\lambda e_i^*(S_\lambda) = 0 \ (i \in I)$. Choose $r \in R$, $p \in P_r$ and put $M_r = \lim_p \|S_\lambda\|_r$. If $M_r \neq 0$, then there exists λ_0 such that for all $\lambda \geq \lambda_0$,

$${}_p\|S_\lambda - S_{\lambda_0}\|_r \leq \frac{M_r}{2}.$$

Put $J = \text{supp}(S_{\lambda_0})$, J is finite, then $\lim_p \|\Pi_J(S_\lambda)\|_r = 0$. The property

$${}_p\|S_\lambda - S_{\lambda_0}\|_r^{\frac{1}{p}} = {}_p\|\Pi_J(S_\lambda) - S_{\lambda_0}\|_r^{\frac{1}{p}} + {}_p\|\Pi_{I \setminus J}(S_\lambda)\|_r^{\frac{1}{p}}$$

is in contradiction with the hypothesis $M_r \neq 0$. Indeed

$$M_r = \lim_p \|S_\lambda\|_r = \lim_p \|\Pi_{I \setminus J}(S_\lambda)\|_r \leq \frac{M_r}{2} \quad \square$$

REMARK 1.7. As a consequence we have:

- (1) if ${}_p\tau_{R'} \geq {}_p\tau_R$, where $(R', R) \in S'(I)^2$ and $(E, {}_p\tau_R)$ is complete, then $(E, {}_p\tau_{R'})$ is also complete. This consequence is obvious (see e.g, [3, Corollary 1]).

- (2) Let R be in $S'(I)$ and $P = (P_r)_{r \in R} \in (0, \infty)^R$. Then $(E, {}_P\tau_R)$ can not be metrizable and complete simultaneously. Indeed, Assume that ${}_P\tau_R$ is metrizable, then we can suppose that $R = \{r(n) \mid n \in N\}$ is countable. Let $u_n = n^2 \sum_{1 \leq i, j \leq n} r(i)_j$. By (1) of Theorem 7, $(E, {}_P\tau_R)$ is not complete.

2. Non uniqueness of the topologies of topological metrizable algebras for some algebras

In this section we prove the metrizability of some topologizable algebras. This gives further examples in the direction of Theorem 1 and Theorem 2.

2.1. Non uniqueness of the topologies of topological metrizable algebras for countably generated algebras

PROPOSITION 2.1. *Every algebra of countable dimension has \aleph_1 different topologies of metrizable locally convex topological algebras.*

Proof. Let \mathcal{A} be an algebra with a countable Hamel basis $(e_k)_{k \in N}$. By Theorem 3, the algebra \mathcal{A} endowed with the maximal locally convex topology is a topological algebra. Thus, for every seminorm $\| \cdot \|$ in \mathcal{A} there exists a seminorm $\| \cdot \|'$ in \mathcal{A} satisfying for all $(x, y) \in \mathcal{A}^2$,

$$\|x\| \leq \|x\|' \text{ and } \|xy\| \leq \|x\|' \|y\|' \tag{2.1}$$

Let $(\| \cdot \|_{\mu, m})_{(\mu, m) \in [0, \omega_1) \times N}$ be a family of norms in \mathcal{A} defined by [5, E.III.18, §2, n° 2, C.59] as follows:

Step 1 $\|x\|_{0,1} = \sum_{n \in N} |e_n^*(x)|$, and $\| \cdot \|_{0, m+1} = \| \cdot \|'_{0, m}$.

Step 2 For given $\mu \in (0, \omega_1)$, let

$$\sigma : N \rightarrow Fin([0, \mu) \times N)$$

be a onto mapping and define a norm in \mathcal{A} by

$$\|x\|_{\mu, 1} = \sum_{n \in N} |e_n^*(x)| \|e_n\|_{\mu, 1}$$

where

$$\|e_n\|_{\mu, 1} = n \cdot \sum_{i \in \sigma(n)} \|e_n\|_i$$

Step 3 $\| \cdot \|_{\mu, m+1} = \| \cdot \|'_{\mu, m}$, is the seminorm associated to $\| \cdot \|_{\mu, m}$ by the formula 2.1.

For every $\mu \in [0, \omega_1)$ we denote by τ_μ the topology defined by $(\| \cdot \|_{\mu, m})_{m \in N}$. Whence (\mathcal{A}, τ_μ) is a metrizable topological algebra. Furthermore $\tau_\mu \neq \tau_\nu$ for $\nu \neq \mu$. Indeed, if $\nu < \mu$, then $\| \cdot \|_{\mu, 1}$ is not continuous for τ_ν . \square

2.2. Non uniqueness of the topology of topological metrizable algebras for the free algebras

We denote by $F(X)$ the free algebras of, non commuting, symbols $X = (X_i)_{i \in I}$, where I is a nonempty set. Thus, $(X^f)_{f \in I^\infty}$ is a Hamel basis of $F(X)$. Every element P of $F(X)$ can be represented by $P = \sum_{f \in I^\infty} X_f^*(P)X^f$ where $(X_f^*(P))_{f \in I^\infty} \in \mathbb{K}^{I^\infty}$ is a family of finite support. Thus, $(X_f^*(P))_{f \in I^\infty}$ is the dual basis of the Hamel basis $(X_f(P))_{f \in I^\infty}$. Recall that the product in $F(X)$ is given by

$$PQ = \sum_{f \in I^\infty} \left(\sum_{\substack{(g,h) \in I^\infty \times I^\infty \\ gh=f}} X_g^*(P)X_h^*(Q) \right) \cdot X^f$$

For every $p \in (0, \infty)$, $u = (u_n)_{n \in \mathbb{N}} \in N^{\mathbb{N}}$ and $w = (w_i)_{i \in I} \in N^I$, let $p\|\|_{u,w}$ the p -seminorm on $F(X)$ defined by

$$p\|P\|_{u,w} = \left(\sum_{f \in I^\infty} |X_f^*(P)|^p \cdot u_{|f|_w} \right)^{\overline{p}} \tag{2.2}$$

where $|f|_w = w_{f_1} + \dots + w_{f_n}$ for $f = (f_1, \dots, f_n) \in I^\infty$ and $|\circ|_w = 0$. Now we will give two lemmas needed for the sequel. Note that such lemmas can be used to understand the inequality given in [3, page 134] and the proof of [3, Lemme 4].

LEMMA 2.2. For all $p \in (0, \infty)$, $u = (u_n)_{n \in \mathbb{N}} \in N^{\mathbb{N}}$ and $w = (w_i)_{i \in I} \in N^{*I}$, there is $\hat{u} = (\hat{u}_n)_{n \in \mathbb{N}} \in N^{\mathbb{N}}$ such that

$$p\|PQ\|_{u,w} \leq p\|P\|_{\hat{u},w} \cdot p\|Q\|_{\hat{u},w}$$

Proof. Let $\hat{u} = (\hat{u}_n)_{n \in \mathbb{N}}$ be the sequence defined by

$$\hat{u}_n = (2n + 1)^{[p]} \max\{u_m \mid 0 \leq m \leq 2n\}$$

where $[p] = \max\{n \in \mathbb{N} \mid n \leq p\}$. Clearly $(n + m + 1)^{[p]} u_{n+m} \leq \hat{u}_n \hat{u}_m$ ($(n, m) \in \mathbb{N}^2$). Remark that for $f \in I^\infty$,

$$\text{card}(\{(g, h) \in I^\infty \times I^\infty \mid gh = f\}) = c(f) + 1.$$

Then, for all $(P, Q) \in F(X)^2$, we have

$$\begin{aligned} p\|PQ\|_{u,w} &= \left(\sum_{f \in I^\infty} \left| \sum_{gh=f} X_g^*(P) \cdot X_h^*(Q) \right|^p \cdot u_{|g|_w + |h|_w} \right)^{\overline{p}} \\ &= \left(\sum_{f \in I^\infty} \left| \sum_{gh=f} \frac{1}{c(f)+1} \cdot ((c(f) + 1) \cdot X_g^*(P) \cdot X_h^*(Q)) \right|^p \cdot u_{|g|_w + |h|_w} \right)^{\overline{p}} \\ &= \left(\sum_{f \in I^\infty} \sum_{gh=f} \frac{|((c(f)+1) \cdot X_g^*(P) \cdot X_h^*(Q))|^p}{c(g)+c(h)+1} \cdot u_{|g|_w + |h|_w} \right)^{\overline{p}} \\ &\leq \left(\sum_{f \in I^\infty} \sum_{gh=f} (|g|_w + |h|_w + 1)^{[p]} |X_g^*(P) \cdot X_h^*(Q)|^p \cdot u_{|g|_w + |h|_w} \right)^{\overline{p}} \\ &\leq \left(\sum_{f \in I^\infty} \sum_{gh=f} |X_g^*(P)|^p \cdot |X_h^*(Q)|^p \cdot \hat{u}_{|g|_w} \cdot \hat{u}_{|h|_w} \right)^{\overline{p}} \\ &= (p\|P\|_{\hat{u},w})(p\|Q\|_{\hat{u},w}) \quad \square \end{aligned}$$

LEMMA 2.3. Let $p \in (0, \infty)$ and $U \times W \subseteq N^{\mathbb{N}} \times N^{I^\infty}$ such that

$$\aleph_0 \leq \text{card}(U \times W) \leq \text{card}(I).$$

(1) There exists a p -seminorm $p \|\cdot\|_{\bar{u}, \bar{w}}$ which is not continuous on

$$(F(X), (p \|\cdot\|_{u,w})_{(u,w) \in U \times W}).$$

(2) If I is uncountable, there exists a p -seminorm $p \|\cdot\|_{\bar{u}, \bar{w}}$ which is not continuous on

$$(F(X), (q \|\cdot\|_{u,w})_{(q,u,w) \in Q})$$

where

$$Q = (p, \infty) \times N^{\mathbb{N}} \times N^{I^\infty} \cup \{p\} \times U \times W.$$

Proof. We have $\text{card}(U \times W) \leq \text{card}(I)$. Then the cardinal of the set $\text{Fin}(U \times W)$ of finite nonempty subsets of $U \times W$ is less than $\text{card}(I)$. Thus, we can write $\text{Fin}(U \times W) = \{I_i \mid i \in I\}$. Now, let $\bar{w} = (\bar{w}_i)_{i \in I}$ be the element of N^I and $\bar{u} = (\bar{u}_n)_{n \in \mathbb{N}}$ the element of $N^{\mathbb{N}}$ defined by:

$$\bar{w}_i = \sum_{(u,w) \in I_i} u_{|(i)|_w} \quad (i \in I), \quad \bar{u}_n = n^2 + 1 \quad (n \in \mathbb{N}).$$

Hence, $p \|\cdot\|_{\bar{u}, \bar{w}}$ is the suitable p -seminorm. Indeed,

(1) Let J be a nonempty subset of $U \times W$ and $M \in N$. Choose $i \in I$, such that $J \in I_i$ and $\text{card}(I_i) > M$. Thus,

$$p \|X_i\|_{\bar{u}, \bar{w}} = \left(\sum_{(u,w) \in I_i} u_{|(i)|_w} \right)^2 + 1 > M \cdot \left(\sum_{(u,w) \in J} u_{|(i)|_w} \right) \geq M \cdot \sum_{(u,w) \in J} p \|X_i\|_{u,w}.$$

Hence, $p \|\cdot\|_{\bar{u}, \bar{w}}$ is not continuous on $(F(X), (p \|\cdot\|_{u,w})_{(u,w) \in U \times W})$.

(2) Furthermore, if I is uncountable, let J be a nonempty subset of $U \times W$, $M \in N$ and choose an arbitrary $(q, u, w) \in (p, \infty) \times N^{\mathbb{N}} \times N^{I^\infty}$. Then, there is an infinite set $K = \{i_n \mid n \in N\} \subseteq I$ such that $\{q \|X_i\|_{u,w} \mid i \in K\}$ is bounded and for all $i \in K$, $J \in I_i$ and $\text{card}(I_i) > 2M$. Put $P_n = \sum_{k=1}^n k^{-\frac{1}{q}} X_{i_k}$. As in (1), we have

$$\frac{1}{2} \cdot (p \|P_n\|_{\bar{u}, \bar{w}}^*) > M \cdot \sum_{(u,w) \in J} p \|P_n\|_{u,w}^*$$

further $(q \|P_n\|_{u,w}^*)_{n \in N}$ is bounded but $\frac{1}{2} \cdot (p \|P_n\|_{\bar{u}, \bar{w}}^*)$ is unbounded. Thus, $p \|\cdot\|_{\bar{u}, \bar{w}}$ is noncontinuous on $(F(X), (q \|\cdot\|_{u,w})_{(q,u,w) \in Q})$. \square

PROPOSITION 2.4. For every set of non commuting symbols $X = (X_i)_{i \in I}$, the free algebra $F(X)$ has $\max\{2^{\aleph_0}, \aleph_I\}$ different topologies of topological metrizable algebras.

Proof. Set $U = \{u^n \mid n \in \mathbb{N}\}$, where $u^1 = (m+1)_{m \in \mathbb{N}}$ and let $u^{n+1} = \widehat{u}^n$ be the element given by Lemma 2.2. Let w^μ , $\mu \in [0, \omega_I)$ be the family defined by the transfinite induction [5, E.III.18, §2, n° 2, C.59] as follow:

Step 1 $w^0 = (1)_{i \in I}$,

Step 2 for $\mu \in (0, \omega_I)$, suppose that w^ν , $\nu \in [0, \mu)$, are defined.

Step 3 Denote $W^\mu = \{w^\nu \mid \nu \in [0, \mu)\}$. We have

$$\text{card}(U \times W^\mu) \leq \omega_0 \cdot \omega_I = \omega_I$$

Finally denote by w^μ the element \bar{w} associated to $U \times W^\mu$ by Lemma 2.3.

Now for every $\mu \in [0, \omega_I)$, let τ_μ be the topology on $F(X)$ defined by the seminorms $(\| \cdot \|_{u^n, w^\mu})_{n \in \mathbb{N}}$. Then $(F(X), \tau_\mu)$ is a metrizable topological algebra. And for $\nu \neq \mu$, we have $\tau_\nu \neq \tau_\mu$. Thus, we have at last \aleph_I different topologies of metrizable locally convex topological algebras.

By [19, Proposition 3], for all real $c \geq 1$, there is a complete locally convex topology τ_c making $F(X)$ a topological algebra and the restriction of τ_c to $P(X_1)$ is exactly the topology τ_c in the sense of [19, Proposition 1], where $P(X_1)$ is the subalgebra of $F(X)$ generated by X_1 . By [19, Proposition 3] and the proof of [19, Proposition 1, pages 114, 115], there exists a τ_c -continuous norm $\| \cdot \|_c$ which discontinuous in the topology $\tau_{c'}$, for every $c' > c$. For all $c \in [1, \infty)$, choose a sequence $(\| \cdot \|_{c,n})_{n \in \mathbb{N}}$ of τ_c -continuous norms satisfying $\|xy\|_{c,n} \leq \|x\|_{c,n+1} \|y\|_{c,n+1}$. Thus, we have at last 2^{\aleph_0} different topologies of metrizable locally convex topological algebras. \square

Similar proof to the one of Proposition 2.4 with the notation of [19, Proposition 2] yields the following result:

PROPOSITION 2.5. *For every set of commuting symbols $(t_i)_{i \in I}$, the algebra $P((t_i)_{i \in I})$ of all polynomials in these variables has $\max\{2^{\aleph_0}, \aleph_I\}$ different topologies of topological metrizable algebras.*

3. A metrizable topologized algebra which is not topologizable as a locally convex metrizable semitopological

The following lemma will be used in the sequel.

LEMMA 3.1. *Let \mathcal{A} be an algebra. Suppose that there exist nonempty subsets X and Y of $\mathcal{A} \setminus \{0\}$, $(z_y)_{y \in Y} \in (\mathcal{A} \setminus \{0\})^Y$ and $(\alpha_{x,y})_{(x,y) \in X \times Y} \in \mathbb{K}^{X \times Y}$ with the property:*

- (1) *For all $(x,y) \in X \times Y$, $xy = \alpha_{x,y} z_y$.*
- (2) *Y is uncountable and for all uncountable subset Y' of Y there exists $x \in X$ such that $\{\alpha_{x,y} \mid y \in Y'\}$ is unbounded in \mathbb{K} .*

Then \mathcal{A} is not topologizable as a metrizable semitopological algebra.

Proof. Suppose the contrary, i.e \mathcal{A} can be topologized as a metrizable semitopological algebra. Then it is a well known fact that there exists a fundamental system $(V_n)_{n \in \mathbb{N}}$ of neighborhood of zero in \mathcal{A} satisfying the properties (m_1) , (m_2) , (m_3) , (m_4) and (m_5) given in \mathbf{N}° 1.3.

By (m_1) we have $\bigcup_{n \in \mathbb{N}} \{y \in Y \mid z_y \notin V_n\} = Y$. Then, there exist an integer k_0 such that $Y' = \{y \in Y \mid z_y \notin V_{k_0}\}$ is uncountable. Thus $\{z_y \mid y \in Y'\} \cap V_{k_0} = \emptyset$. By (m_3) we can choose an integer $r_n > 0$ and an uncountable subset Y'' of Y' such that $Y'' \subseteq r_n V_n$ for every fixed $n \in \mathbb{N}$. Choose $x \in X$ such that $\{\alpha_{x,y} \mid y \in Y''\}$ is unbounded in \mathbb{K} . The property (m_5) , implies the existence of an integer $n_0 > k_0$ such that $x \cdot V_{n_0} \subseteq V_{k_0}$. Hence for each $y \in Y''$ with $\alpha_{x,y} \neq 0$,

$$z_y = \frac{1}{\alpha_{x,y}}xy \in \frac{r_{n_0}}{\alpha_{x,y}}xV_{n_0} \subseteq \frac{r_{n_0}}{\alpha_{x,y}}V_{k_0}$$

There exists $y \in Y''$ such that $|\alpha_{x,y}| \geq r_{n_0}$. Hence by (m_2) , we get $z_y \in V_{k_0}$, which is a contradiction. \square

The following functions will be used together with the above lemma in the sequel

\mathbf{N}° 3.2. For every ordinal $\mu \in [\omega_0, \omega_1)$, choose a mapping $f_\mu : [1, \omega_1) \rightarrow \mathbb{N}$ such that

- (1) $f_\mu^{-1}(\{0\}) = [\mu, \omega)$,
- (2) the restriction of f_μ to $[1, \mu)$, $f_{\mu|_{[1, \mu)}} : [1, \mu) \rightarrow \mathbb{N}$ is one to one.

3.1. Example of an algebra which is neither topologizable as a topological algebra nor as a metrizable semitopological algebra

Using the algebra \mathcal{A} constructed in [9, Theorem 3], we obtain the following result

PROPOSITION 3.3. *The algebra \mathcal{A} is neither topologizable as a topological algebra nor as a metrizable semitopological algebra.*

Proof. Let \mathcal{A} be the \mathbb{K} -vector space generated by the symbols x_μ , $\mu \in [1, \omega_1)$, y_μ , $\mu \in [\omega_0, \omega_1)$ and c . A commutative multiplication is defined on \mathcal{A} by $c\mathcal{A} = \{0\}$, $x_\nu x_\mu = 0$, $y_\mu y_\nu = 0$ and $x_\nu y_\mu = f_\mu(\nu)c$ where $(f_\mu)_{\mu \in [1, \omega_1)}$ is the family defined in \mathbf{N}° 3.2. The first assertion of the proposition is exactly the [9, Theorem 3]. The second assertion is a consequence of Lemma 3.1 with the notations $X = \{a_\mu \mid \mu \in [1, \omega_1)\}$, $Y = \{b_\nu \mid \nu \in [1, \omega_1)\}$, $z_y = c$ for $y \in Y$, and $\alpha_{a_\mu, b_\nu} = f_\mu(\nu)$. \square

3.2. Example of a topologizable algebra which is not topologizable as a metrizable semitopological algebra

The following result (written in a quite similar form) was established by the author in [1] but not yet published so far.

PROPOSITION 3.4. *There exists a commutative algebra \mathcal{A} with dimension equal to \aleph_1 satisfying:*

- (1) \mathcal{A} is not topologizable as a metrizable semitopological algebra,
- (2) \mathcal{A} is topologizable as a complete m -convex algebra.

Proof. Let \mathcal{A} be the \mathbb{K} -vector space generated by the symbols $a_\mu, \mu \in [1, \omega_1]; b_\nu, \nu \in [\omega_0, \omega_1]$ and $c_\xi, \xi \in [\omega_0, \omega_1]$. Let $(f_\mu)_{\mu \in [\omega_0, \omega_1]}$ be the family given in $\mathbf{N}^\circ 3.2$. We define a commutative multiplication on \mathcal{A} by:

$$\begin{aligned} a_\mu a_\nu &= 0, & (\mu, \nu) &\in [1, \omega_1] \times [1, \omega_1] \\ c_\nu a &= \{0\}, & \nu &\in [\omega_0, \omega_1] \\ b_\mu b_\nu &= 0, & (\mu, \nu) &\in [\omega_0, \omega_1] \times [\omega_0, \omega_1] \\ a_\nu b_\mu &= f_\mu(\nu)c_\mu, & (\nu, \mu) &\in [1, \omega_1] \times [\omega_0, \omega_1] \end{aligned}$$

(1) With the notations $X = \{a_\mu \mid \mu \in [1, \omega_1]\}, Y = \{b_\nu \mid \nu \in [1, \omega_1]\}, z_y = c_\nu$ if $y = b_\nu$, and $\alpha_{a_\mu, b_\nu} = f_\mu(\nu)$, the first assertion is a consequence of Lemma 3.1.

(2) The algebra \mathcal{A} is satisfying the conditions of Theorem 6. Then there exists on \mathcal{A} a complete locally convex topology. Recall that the product of all three elements is equal to zero then such topology is m -convex. \square

REMARK 3.5. In fact in [1], the assertion (2) is given in the form: " \mathcal{A} is topologizable as a topological m -convex algebra. Furthermore, \mathcal{A} can be endowed with a m -convex topology defined by a family of submultiplicatives seminorms whose cardinal is the first uncountable cardinal". This can be handled as follows: Let $(f_\mu)_{\mu \in [1, \omega_1]}$ be the family defined in $\mathbf{N}^\circ 3.2$ and $F = \{f_{v_1} + \dots + f_{v_n} \mid (v_1, \dots, v_n) \in [\omega_0, \omega_1]^n, n \in \mathbf{N}\}$. For all $f \in F$ and $a = \sum_{\mu \in [1, \omega_1]} \alpha_\mu a_\mu + \sum_{\nu \in [\omega_0, \omega_1]} \beta_\nu b_\nu + \sum_{\xi \in [\omega_0, \omega_1]} \gamma_\xi c_\xi \in \mathcal{A}$ we put $\|a\|_f =$

$$\sum_{\mu \in [1, \omega_1]} f(\mu)|\alpha_\mu| + \sum_{\substack{\nu \in [\omega_0, \omega_1] \\ f_\nu \leq f}} |\beta_\nu| + \sum_{\substack{\xi \in [\omega_0, \omega_1] \\ \gamma_\xi \leq f}} |\gamma_\xi|. \text{ Thus, the seminorm } \|\cdot\|_f \text{ is sub-}$$

multiplicative. Indeed, let a be as above and $a' = \sum_{\mu' \in [1, \omega_1]} \alpha'_{\mu'} a_{\mu'} + \sum_{\nu' \in [\omega_0, \omega_1]} \beta'_{\nu'} b_{\nu'} +$

$\sum_{\xi' \in [\omega_0, \omega_1]} \gamma'_{\xi'} c_{\xi'} \in \mathcal{A}$. We have

$$\begin{aligned} \|aa'\|_f &= \left\| \sum_{(\mu, \mu') \in [1, \omega_1]^2} \sum_{\substack{\nu \in [\omega_0, \omega_1] \\ f_\nu \leq f}} (f_\nu(\mu)\alpha_\mu\beta'_\nu + f_\nu(\mu')\alpha'_{\mu'}\beta_\nu)c_\nu \right\|_f \\ &\leq \sum_{\mu \in [1, \omega_1]} \sum_{\substack{\nu' \in [\omega_0, \omega_1] \\ f_{\nu'} \leq f}} f(\mu)|\alpha_\mu| \cdot |\beta'_{\nu'}| \\ &\quad + \sum_{\mu' \in [1, \omega_1]} \sum_{\substack{\nu \in [\omega_0, \omega_1] \\ f_\nu \leq f}} f(\mu')|\alpha'_{\mu'}| \cdot |\beta_\nu| \leq \|a\|_f \|a'\|_f \end{aligned}$$

Now we have to prove that $(\mathcal{A}, (\|\cdot\|_f)_{f \in F})$ is Hausdorff. Let a be a nonzero element of \mathcal{A} . Using the canonical representation of a . If there exists $\alpha_\mu \neq 0$, then for all $f \in F, \|a\|_f \neq 0$. If not, there exists necessarily $\nu \in [\omega_0, \omega_1]$ such that $\beta_\nu \neq 0$ or $\gamma_\nu \neq 0$, thus for $\mu \in (\nu, \omega_1), \|a\|_{f_\mu} \neq 0$. Hence, $(\|\cdot\|_f)_{f \in F}$ is the required family. \square

3.3. Example of a locally bounded algebra which is not topologizable as a metrizable locally p -convex semitopological algebra

In fact we prove the following result:

PROPOSITION 3.6. *For all $p \in (0, 1]$, there exists a commutative locally bounded algebra \mathcal{A}_p (Further, such an algebra is q -normed for all q in $(0, \frac{p}{p+2})$), which can neither be topologizable as a locally p -convex metrizable semi-topological algebra nor as a locally p -convex topological algebra.*

Proof. Let p be in $(0, 1]$ and K an uncountable set and $D = N^{N \times K}$. For every $d = (d_{ij})_{(i,j) \in N \times K} \in D$ and $n \in N$ there exists $d_n \in N$ and a subset K_{dn} of K such that $d_n = \text{card}(K_{dn}) = d_{nk}$ for all $k \in K_{dn}$. Let \mathcal{A} be the \mathbb{K} -vector space spanned by the symbols $c, x_{nk}, (n, k) \in N \times K, \mathcal{A}_d, d \in D$ and $y_{dnk}, (d, n) \in D \times N, k \in K_{dn} \subseteq K$. Define a commutative multiplication on \mathcal{A} by $cA = y_{dnk}A = \{0\}, \mathcal{A}_d a_{d'} = 0, x_{nk} \cdot x_{n'k'} = 0$ and for $(d, n) \in D \times N,$

$$x_{nk} \cdot a_d = \begin{cases} (d_n)^{\frac{1}{p}} y_{dnk} & \text{if } k \in K_{dn} \\ 0 & \text{if } k \in K \setminus K_{dn} \end{cases}$$

Let M be the linear subspace of \mathcal{A} spanned by the elements $c - d_n^{-\frac{1}{p}} \sum_{k \in K_{dn}} y_{dnk}, (d, n) \in D \times N$. By [2] \mathcal{A} is an algebra and M is an ideal of \mathcal{A} . Let \mathcal{A}_p be the quotient algebra \mathcal{A}/M and $\pi : \mathcal{A} \rightarrow \mathcal{A}_p$ the canonical projection. By [2, Corollary 1] the algebra \mathcal{A}_p is a locally bounded algebra. Further, \mathcal{A}_p is q -normable algebra for all $q \in (0, \frac{p}{p+2})$. Now, suppose that the algebra \mathcal{A}_p is topologizable as a metrizable locally p -convex semi-topological algebra. Then, there exists a fundamental system $(V_n)_{n \in N}$ of neighborhoods of zero in \mathcal{A} satisfying the property $(m_1), (m_2), (m_3), (m_4), (m_5)$ and (m_6) given in \mathbf{N}° 1.3.

Choose $u \in N$ such that $\pi(c) \notin V_u$. Then, there is $(s_{nk})_{(n,k) \in N \times K} \in N^{N \times K}$ such that $x_{nk} \in s_{nk} \cdot V_n, (n, k) \in N \times K$. Choose $\hat{d} \in D$ such that $\hat{d}_{nk} > (s_{nk})^p$. By (m_5) there is an integer $v \geq u$ satisfying $\mathcal{A}_{\hat{d}} V_v \subseteq V_u$. We have for all $k \in K_{\hat{d}_v}$

$$\pi(y_{\hat{d}_v k}) = \left(\frac{1}{\hat{d}_v}\right)^{\frac{1}{p}} \pi(a_{\hat{d}}) \pi(x_{vk}) \in \frac{s_{vk}}{(\hat{d}_v)^{\frac{1}{p}}} \pi(a_{\hat{d}}) V_v \subseteq \frac{s_{vk}}{(\hat{d}_v)^{\frac{1}{p}}} V_u \subseteq V_u$$

Therefore, using $(m_6),$

$$\pi(c) = \sum_{k \in K_{\hat{d}_v}} \left(\frac{1}{\hat{d}_v}\right)^{\frac{1}{p}} \pi(y_{\hat{d}_v k}) \in \sum_{k \in K_{\hat{d}_v}} \left(\frac{1}{\hat{d}_v}\right)^{\frac{1}{p}} V_u \subseteq V_u$$

Thus, \mathcal{A}_p is not topologizable as a locally p -convex metrizable semitopological algebra. \square

REMARK 3.7. Using the notations of the proofs of [2, Corollaire 1, Théorème] and the techniques of Proposition 3.6 we obtain the following results:

- (1) there exists a commutative locally pseudoconvex metrizable algebra which is neither be topologizable as a locally p -convex metrizable semitopological algebra nor as a locally p -convex topological algebra for any $p \in (0, 1]$.
- (2) there exists a commutative metrizable topological algebra which is neither be topologizable as a locally pseudoconvex metrizable semitopological algebra nor as a locally pseudoconvex topological algebra.

4. Metrizability of some function algebras

In the sequel we will use the following (For more information, see [6], [7], [11] and [17]):

- E a Hausdorff topological space and $C(E, \mathbb{K})$ the algebra of continuous functions $f : E \rightarrow \mathbb{K}$.
- $\|f\|_X = \sup_{x \in X} |f(x)|$, for every non empty subset X of E and $f \in C(E, \mathbb{K})$.

Clearly the algebra $(C(E, \mathbb{K}), (\| \cdot \|_X)_{X \in Fin(E)})$, where $Fin(E)$ is the set of nonempty finite subsets of E , is a m -convex topological algebras.

- A family $(X_i)_{i \in I}$ is locally finite if for each $x \in E$ there is a neighborhood V of x such that $\{i \in I \mid V \cap X_i \neq \emptyset\}$ is finite or empty.
- A topological space E is a Lindelöf space if and only if each open cover of E has a countable subcover.
- A topological space E is a paracompact space if and only if each open cover of E has a open locally finite refinement.
- A family $(X_i)_{i \in I}$ of subsets of E is discrete if $(X_i)_{i \in I}$ is locally finite and $X_i \cap X_j = \emptyset$ whenever $i \neq j$.
- for $f \in C(E, \mathbb{K})$, the support of f ($supp(f)$) is the closure of the set $\{x \in E \mid f(x) \neq 0\}$.
- Let $(f_i)_{i \in I}$ be a family of elements in $C(E, \mathbb{K})$ such that $(supp(f_i))_{i \in I}$ is locally finite. It is easy to see that the sum of the family $(f_i)_{i \in I}$ given by

$$\begin{aligned} \sum_{i \in I} f_i : E &\longrightarrow \mathbb{K} \\ x &\longmapsto \sum_{i \in I} f_i(x) \end{aligned}$$

is a well defined mapping and it is continuous [7, TG.IX. 46].

- A pseudocompact (or weierstrassien) space (see [7, IX.88, exercice 20] or [10, I.4]) is a topological space on which each real valued continuous function is bounded. For example every compact space, countably compact, sequentially compact space is a pseudocompact space. Then we have

PROPOSITION 4.1. *Let E be a topological space*

- (1) *Suppose that there exists a sequence $(E_n)_{n \in \mathbb{N}}$ of pseudocompact topological subspaces of E such that $\bigcup_{n \in \mathbb{N}} E_n$ is a dense subspace of E . Then $(C(E, \mathbb{K}), \| \cdot \|_{E_n})_{n \in \mathbb{N}}$ is a metrizable m -convex topological algebra.*

(2) If E is a completely regular non pseudocompact space, then $C(E, \mathbb{K})$ cannot be topologized as a locally bounded semitopological algebra.

Proof. (1) Is obvious.

(2) Choose a non bounded function f' in $C(E, \mathbb{K})$. Then there is $(x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$ with $0 < |f'(x_n)| < |f'(x_{n+1})|$ and $\lim_{n \rightarrow \infty} |f'(x_n)| = +\infty$. For all $n \in \mathbb{N}$ set

$$V_n = \left\{ x \in E \mid \frac{|f'(x_{n-1})| + |f'(x_n)|}{2} < |f'(x)| < \frac{|f'(x_n)| + |f'(x_{n+1})|}{2} \right\}$$

there exists a function $f_n \in C(E, \mathbb{K})$ such that $\text{supp}(f_n) \subseteq V_n$ and $f_n(x_n) = 1$. Hence, $(V_n)_{n \in \mathbb{N}}$ is a discrete family of open nonempty subsets of E . For each $n \in \mathbb{N}$, there exists $(u_n, v_n) \in N^2$ such that $f_n \in u_n B$ and $v_n f_n^2 \notin B$. Thus $(\text{supp}(nu_n v_n f_n))_{n \in \mathbb{N}}$ is a discrete family, therefore $f = \sum_{n \in \mathbb{N}} nu_n v_n f_n$ is a well defined continuous mapping. Hence $f \in C(E, \mathbb{K})$ and there is $r \in N$ such that $f \cdot B \subseteq rB$. Then

$$v_n f_n^2 = \frac{1}{nu_n} f f_n \in \frac{u_n}{nu_n} f B \subseteq \frac{ru_n}{nu_n} B = \frac{r}{n} B$$

Finally we get $v_n f_n^2 \in B$, for $n > r$, a contradiction. \square

REMARK 4.2. [11, Problem E, page 162] If $W(\omega_1) = [0, \omega_1)$ endowed with its natural topology of order. Then the space $W(\omega_1)$ is pseudocompact, locally compact, satisfying the first axiom, and on the other hand

- (1) $W(\omega_1)$ is a pseudocompact and locally compact space satisfying the first axiom of countability.
- (2) $W(\omega_1)$ is not σ -compact and not Lindelöf space. In particular it is non separable.

LEMMA 4.3. Let E be a completely regular topological space. If E admits an uncountable discrete family of nonempty open subsets, then there is a subalgebra \mathcal{A} of $C(E, \mathbb{K})$ satisfying:

- (1) the dimension of the \mathbb{K} -vector space \mathcal{A} is \aleph_1 ,
- (2) the algebra \mathcal{A} is not topologizable as a semitopological metrizable algebra,
- (3) the algebra \mathcal{A} is topologizable as an m -convex topological algebra. Furthermore, such a topology can be defined by \aleph_1 submultiplicative seminorms.

Proof. Let $(f_\mu)_{\mu \in [\omega_0, \omega_1)}$ be the family giving in $\mathbf{N}^\circ 3.2$. Choose an uncountable discrete family $(\Omega_\mu)_{\mu \in [1, \omega_1)}$ of nonempty open subsets of E . For every μ in $[1, \omega_1)$ let x_μ be an arbitrary element of Ω_μ . Then there exists a continuous function $u_\mu : E \rightarrow [0, 1]$ such that $u_\mu(x_\mu) = 1$ and $\text{supp}(u_\mu) \subseteq \Omega_\mu$. The family $(\text{supp}(u_\mu))_{\mu \in [1, \omega_1)}$ is discrete,

thus $\phi_\mu = \sum_{\nu \in [1, \omega_1]} f_\nu(\mu) u_\nu$ is a well defined and continuous mapping. Denote by \mathcal{A} the subalgebra of $C(E, \mathbb{K})$ generated by the set $\{\phi_\mu, u_\nu : (\mu, \nu) \in [1, \omega_1] \times [\omega_0, \omega_1]\}$. Clearly $\nu \cdot \phi_\mu = f_\mu(\nu) u_\nu^2$.

(1) The dimension of the \mathbb{K} -vector space \mathcal{A} is \aleph_1 . Indeed, \mathcal{A} is linearly generated by the set of finite products of elements of $\{u_\mu, \phi_\nu : (\mu, \nu) \in [1, \omega_1] \times [\omega_0, \omega_1]\}$ and the family $(u_\mu)_{\mu \in [1, \omega_1]}$ is linearly independent in \mathcal{A} . The cardinal of both sets is \aleph_1 .

(2) With the notations $X = \{\phi_\mu : \mu \in [1, \omega_1]\}$, $Y = \{u_\nu : \nu \in [1, \omega_1]\}$, $z_y = u_\nu^2$ if $y = u_\nu$, and $\alpha_{\phi_\mu, u_\nu} = f_\mu(\nu)$. The second assertion is a consequence of Lemma 3.1.

(3) It suffices to consider the topology defined by the family of submultiplicative seminorms $(\|\Omega_\mu\|)_{\mu \in [1, \omega_1]}$. \square

As a consequence of Lemma 4.3 we obtain

COROLLARY 4.4. *Let E be a completely regular topological space having an uncountable discrete family of nonempty open subsets. Then, the algebra $C(E, \mathbb{K})$ can not be topologized as a metrizable semitopological algebra.*

Clearly a regular Lindelöf space does not satisfy the conditions of Lemma 4.3. It is well known that a regular Lindelöf space is paracompact [7, IX.76, Proposition 2]. The main result of this section is the following

THEOREM 4.5. *For every paracompact non Lindelöf space E , the algebra $C(E, \mathbb{K})$ is not topologizable as a metrizable semitopological algebra.*

Proof. Since E is not a Lindelöf space, then there is an open cover $(U_i)_{i \in I}$ of E without countable subcover. By ([11, V, Theorem 28] or [7, TG.IX.107, exercise 27]) $(U_i)_{i \in I}$ has a σ -discrete refinement. Thus, there exists $(V_n)_{n \in \mathbb{N}}$ such that $\bigcup_{n \in \mathbb{N}} V_n = E$ and $V_n = \bigcup_{j \in J_n} V_{n,j}$ where $(V_{n,j})_{j \in J_n}$ is a discrete family of open subsets of E satisfying for all $n \in \mathbb{N}$ and $j \in J_n$, $V_{n,j} \subseteq U_i$ for some $i \in I$. Hence, there is a positive integer n such that $(V_{n,j})_{j \in J_n}$ is uncountable. Then by Corollary 4.4, $C(E, \mathbb{K})$ is not topologizable as a metrizable semitopological algebra. \square

COROLLARY 4.6. *Let E be a metrizable space. Then the following statements are equivalent:*

- (1) E is separable.
- (2) E is Lindelöf.
- (3) The algebra $C(E, \mathbb{K})$ is topologizable as a metrizable semitopological algebra.

Proof. By [7, XI.51, Théorème 4], the space E is paracompact.

(1) \iff (2) By [11, Theorems 1.15, 4.17 and 4.18] E is Lindelöf if and only if E is separable.

(3) \implies (2) Follows by applying Theorem 4.5. Clearly, (1) \implies (3) is obvious. \square

COROLLARY 4.7. *Let E be a nonempty set. The product algebra \mathbb{K}^E is topologizable as a semitopological metrizable algebra if and only if E is countable.*

Proof. Endowed with the discrete topology E is a metrizable space, in this case $\mathbb{K}^E = C(E, \mathbb{K})$. Clearly E is separable if and only if E is countable. Thus we may apply Corollary 4.6 to get the result. \square

REMARK 4.8. (1) Comparing Propositions 4.1, 4.6 and Theorem 2 on the other hand, we can see that the rapports between the notions of normability, metrizability and topologizability in the commutative case (i.e. algebras of continuous functions) is not analogous to the noncommutative case (i.e. algebras of continuous finite rank operators).

- (2) Let E be a regular topological space, and assume that $C(E, \mathbb{K})$ is topologizable as a semitopological metrizable algebra. By Theorem 4.5 and [7, IX.76, Proposition 2]: E is paracompact if and only if E is Lindelöf.
- (3) As a result, the space $W(\omega_1)$ of Remark 4.2 is not paracompact. This result is well known [17, VI.5, example 1].
- (4) If E is a topological uncountable sum of topological spaces $(E_i)_{i \in I}$ [6, TG.I.15, n° 4, exemple 3], then the algebra $C(E, \mathbb{K})$ is non topologizable as a semitopological metrizable algebra. Indeed, there is a one to one algebra homomorphism $\mathbb{K}^I \rightarrow C(E, \mathbb{K})$, $\phi \mapsto \Phi$, where $\Phi(x) = \phi_i$ for all $x \in E_i$. The result is a consequence of the fact that \mathbb{K}^I is not topologizable as a semitopological metrizable algebra.
- (5) We were unable to answer the following question: Does there exist a metrizable semitopological algebra which is not topologizable as a topological algebra?

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