

STRONG COMMUTATIVITY PRESERVING MAPS ON TRIANGULAR RINGS

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Abstract. Let $\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular ring. It is shown, under some mild assumption, that every surjective strong commutativity preserving map $\Phi: \mathcal{U} \rightarrow \mathcal{U}$ (i.e. $[\Phi(T), \Phi(S)] = [T, S]$ for all $T, S \in \mathcal{U}$) is of the form $\Phi(T) = ZT + f(T)$, where Z is in $\mathcal{Z}(\mathcal{U})$, the center of \mathcal{U} , $Z^2 = I$ and f is a map from \mathcal{U} into $\mathcal{Z}(\mathcal{U})$. As an application, a characterization of general surjective maps that preserve the strong commutativity on the nest algebras of Banach space operators is given.

1. Introduction

Let \mathcal{R} be a ring. Then \mathcal{R} is a Lie ring under the Lie product $[A, B] = AB - BA$. Recall that a map Φ from \mathcal{R} into itself preserves commutativity if, for any $A, B \in \mathcal{R}$, $[\Phi(A), \Phi(B)] = 0$ whenever $[A, B] = 0$. The problem of characterizing linear (or additive) maps preserving commutativity had been studied intensively on various rings and algebras (ref. for eg., [3, 5, 7] and the references therein).

In [1], Bell and Daif gave the conception of strong commutativity preserving maps. A map Φ is strong commutativity preserving on \mathcal{R} if $[\Phi(T), \Phi(S)] = [T, S]$ for all $T, S \in \mathcal{R}$. Note that a strong commutativity preserving map must be commutativity preserving, but the inverse is not true generally. Bell and Daif [1] proved that \mathcal{R} must be commutative if \mathcal{R} is a prime ring and \mathcal{R} admits a derivation or a non-identity endomorphism which is strong commutativity preserving on a right ideal of \mathcal{R} . Brešar and Miers in [4] proved that every additive map that is strong commutativity preserving on a semiprime ring \mathcal{R} is of the form $A \mapsto \lambda A + \mu(A)$, where $\lambda \in \mathcal{C}$, the extended centroid of \mathcal{R} , $\lambda^2 = 1$ and $\mu: \mathcal{R} \rightarrow \mathcal{C}$ is an additive map. Recently, Lin and Liu in [9] obtained a similar result on a noncentral Lie ideal of a prime ring.

In recent years, more and more mathematicians are interested in discussing the general preserver problems (see [8, 10, 11, 14, 15]). Šemrl in [15] gave a characterization of general commutativity preserving maps on matrix algebras $M_n(\mathbb{C})$. Let X be a Banach space over \mathbb{F} with dimension at least 2, where \mathbb{F} is \mathbb{C} or \mathbb{R} in the infinite dimensional case and \mathbb{F} is \mathbb{C} in the finite dimensional case. Let $\Phi: \mathcal{B}(X) \rightarrow \mathcal{B}(X)$

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be a surjective map. Dolinar, Du, Hou and Legiša in [8] proved that, if Φ satisfies $\text{Lat}[\Phi(A), \Phi(B)] = \text{Lat}[A, B]$ ($\text{Lat}A$ denotes the lattice of all invariant subspaces under A), then there exist two maps $\phi, \psi : \mathcal{B}(X) \rightarrow \mathbb{F}$ such that $\Phi(A) = \phi(A)A + \psi(A)I$ for every $A \in \mathcal{B}(X)$, where $\phi(A) \neq 0$ if A is not a scalar operator. It is obvious that strong commutativity preserving maps must preserve the lattice of Lie product. Hence, by the result in [8] above one can obtain easily a characterization of general surjective strong commutativity preserving maps on $\mathcal{B}(X)$. More generally, Qi and Hou in [12] proved that every surjective map on a prime ring \mathcal{R} with a nontrivial idempotent that preserves the strong commutativity has the form $\Phi(A) = \alpha A + f(A)$ for all $A \in \mathcal{R}$, where $\alpha \in \{1, -1\}$ and f is a map from \mathcal{R} into its center.

The purpose of this paper is to continue studying this topic and to give a characterization of general surjective strong commutativity preserving maps on triangular rings.

Similar to the definition of triangular algebras (see [6]), one can introduce a conception of triangular rings. Let \mathcal{A} and \mathcal{B} be unital rings, and \mathcal{M} be a $(\mathcal{A}, \mathcal{B})$ -bimodule, which is faithful as a left \mathcal{A} -module and also as a right \mathcal{B} -module, that is, for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $A\mathcal{M} = \mathcal{M}B = \{0\}$ imply $A = 0$ and $B = 0$, respectively. The ring

$$\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} : A \in \mathcal{A}, M \in \mathcal{M}, B \in \mathcal{B} \right\}$$

under the usual matrix operations will be called a triangular ring. It is easily checked that the center $\mathcal{Z}(\mathcal{U})$ of \mathcal{U} coincides with the set

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{U} : A \in \mathcal{Z}(\mathcal{A}), B \in \mathcal{Z}(\mathcal{B}) \text{ and } AM = MB \text{ for all } M \in \mathcal{M} \right\}.$$

We call the idempotent element $P = \begin{pmatrix} I_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}$ the standard idempotent in the triangular ring \mathcal{U} . Let $Q = I - P = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathcal{B}} \end{pmatrix}$, where $I, I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ are units of \mathcal{U}, \mathcal{A} and \mathcal{B} , respectively. In addition, \mathcal{M} is said to be loyal if, for any $A \in \mathcal{A}, B \in \mathcal{B}, A\mathcal{M}B = 0$ implies $A = 0$ or $B = 0$ (see [2]). Obviously, each loyal (A, B) -bimodule \mathcal{M} is faithful as a left A -module and also as a right B -module.

Let \mathcal{A} and \mathcal{B} be unital rings with \mathcal{A} or \mathcal{B} noncommutative, and \mathcal{M} a loyal $(\mathcal{A}, \mathcal{B})$ -bimodule. Let $\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular ring and P the standard idempotent of it. Assume that $\Phi : \mathcal{U} \rightarrow \mathcal{U}$ is a surjective map. In this paper, we prove that, if \mathcal{U} satisfies $P\mathcal{Z}(\mathcal{U})P = \mathcal{Z}(P\mathcal{U}P)$ and $Q\mathcal{Z}(\mathcal{U})Q = \mathcal{Z}(Q\mathcal{U}Q)$, then Φ is strong commutativity preserving if and only if there exist some nonzero element $Z \in \mathcal{Z}(\mathcal{U})$ with $Z^2 = I$ and a map $f : \mathcal{U} \rightarrow \mathcal{Z}(\mathcal{U})$ such that $\Phi(T) = ZT + f(T)$ for all $T \in \mathcal{U}$ (Theorem 2.1). By using of this result, we give a characterization of nonlinear surjective strong commutativity preserving maps on nest algebras of Banach space operators. Let \mathcal{N} be a nontrivial nest on a real or complex Banach space X of dimension ≥ 3 . Assume that \mathcal{N} has a nontrivial complemented element. Let $\Phi : \text{Alg } \mathcal{N} \rightarrow \text{Alg } \mathcal{N}$ be a nonlinear surjective map. We show that Φ preserves the strong

commutativity if and only if there exists a nonlinear functional f of $\text{Alg } \mathcal{N}$ such that $\Phi(T) = T + f(T)I$ for all $T \in \text{Alg } \mathcal{N}$ or $\Phi(T) = -T + f(T)I$ for all $T \in \text{Alg } \mathcal{N}$ (Theorems 2.2–2.3).

2. Main results and proofs

In this section, we consider the question of characterizing general strong commutativity preserving maps on triangular rings. The following is our main result.

THEOREM 2.1. *Let \mathcal{A} and \mathcal{B} be unital rings with \mathcal{A} or \mathcal{B} noncommutative, and \mathcal{M} a loyal $(\mathcal{A}, \mathcal{B})$ -bimodule. Let $\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular ring and P the standard idempotent of it. Assume that \mathcal{U} satisfies $P\mathcal{Z}(\mathcal{U})P = \mathcal{Z}(P\mathcal{U}P)$ and $Q\mathcal{Z}(\mathcal{U})Q = \mathcal{Z}(Q\mathcal{U}Q)$. Then a surjective map $\Phi : \mathcal{U} \rightarrow \mathcal{U}$ is strong commutativity preserving if and only if there exist a map $f : \mathcal{U} \rightarrow \mathcal{Z}(\mathcal{U})$ and a nonzero element $Z \in \mathcal{Z}(\mathcal{U})$ with $Z^2 = I$ such that $\Phi(T) = ZT + f(T)$ for all $T \in \mathcal{U}$.*

As an application of Theorem 2.1, we get a characterization of nonlinear surjective maps that preserve the strong commutativity on nest algebras.

Recall that a nest \mathcal{N} on a Banach space X is a collection of closed (under norm topology) subspaces of X containing $\{0\}$ and X , which is a chain under the inclusion relation, and is closed under the formation of arbitrary closed linear span (denote by \bigvee) and intersection (denote by \bigwedge). The nest algebra associated to the nest \mathcal{N} , denoted by $\text{Alg } \mathcal{N}$, is the weakly closed operator algebra consisting of all operators that leave \mathcal{N} invariant, i.e.,

$$\text{Alg } \mathcal{N} = \{T \in \mathcal{B}(X) : TN \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

When $\mathcal{N} \neq \{0, X\}$, we say that \mathcal{N} is non-trivial. If X is a Hilbert space, then every $N \in \mathcal{N}$ corresponds to a projection P_N satisfying $P_N = P_N^* = P_N^2$ and $N = P_N(X)$. However, it is not always the case for general Banach space nests there are idempotent operators $P_N \in \mathcal{B}(X)$ such that $P_N(X) = N$ because $N \in \mathcal{N}$ may be not complemented. Note that, $\text{Alg } \mathcal{N} = \mathcal{B}(X)$ if the nest \mathcal{N} is trivial. Since $\mathcal{B}(X)$ is prime and the general surjective maps that preserve the strong commutativity on prime rings were characterized in [12], we always assume that the nests are nontrivial in this paper.

THEOREM 2.2. *Let X be an infinite dimensional Banach space over the real or complex field \mathbb{F} . Let \mathcal{N} be a nest on X which contains a nontrivial element complemented in X . Let $\Phi : \text{Alg } \mathcal{N} \rightarrow \text{Alg } \mathcal{N}$ be a surjective map. Then Φ preserves the strong commutativity if and only if there exist a map $f : \text{Alg } \mathcal{N} \rightarrow \mathbb{F}$ and a nonzero scalar λ with $\lambda \in \{-1, 1\}$ such that $\Phi(T) = \lambda T + f(T)I$ for all $T \in \text{Alg } \mathcal{N}$.*

Proof. By the assumption on the nest, there is a non-trivial element $N_1 \in \mathcal{N}$ such that N_1 is complemented in X . Thus, there exists an idempotent operator E with $\text{ran}(E) = N_1 \in \mathcal{N}$. It is clear that $E \in \text{Alg } \mathcal{N}$. Now, by the proof of [13, Theorem 2.2], $\text{Alg } \mathcal{N}$ is a triangular algebra with standard idempotent E and meets all hypotheses of Theorem 2.1. Note that $\mathcal{Z}(\text{Alg } \mathcal{N}) = \mathbb{F}I$. Therefore, Φ preserves the strong commutativity if and only if it is of the form $\Phi(T) = \lambda T + f(T)I$ for all $T \in \text{Alg } \mathcal{N}$, where λ is a scalar with $\lambda^2 = 1$ and f is a general functional. \square

We remark that, if X is a Hilbert space, then the assumption in Theorem 2.2 that there exists a non-trivial complemented element in \mathcal{N} is superfluous.

For the case that X is finite dimensional, it is clear that every nest algebra on X is isomorphic to an upper triangular block matrix algebra. Let $\mathcal{M}_n(\mathbb{F})$ denote the algebra of all $n \times n$ matrices over \mathbb{F} . Recall that an upper triangular block matrix algebra $\mathcal{T} = \mathcal{T}(n_1, n_2, \dots, n_k)$ is a subalgebra of $\mathcal{M}_n(\mathbb{F})$ consisting of all $n \times n$ matrices of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix},$$

where $\{n_1, n_2, \dots, n_k\}$ is a finite sequence of positive integers satisfying $0 < n_i < n$ for each $i = 1, 2, \dots, k$, $n_1 + n_2 + \dots + n_k = n$ and $A_{ij} \in \mathcal{M}_{n_i \times n_j}(\mathbb{F})$, the space of all $n_i \times n_j$ matrices over \mathbb{F} .

For the upper triangular block matrix algebra, we have

THEOREM 2.3. *Let \mathbb{F} be the real or complex field and n be a positive integer greater than 2. Let $\mathcal{T} = \mathcal{T}(n_1, n_2, \dots, n_k) \subseteq M_n(\mathbb{F})$ be an upper triangular block matrix algebra and $\Phi : \mathcal{T} \rightarrow \mathcal{T}$ be a surjective map. Then Φ preserves the strong commutativity if and only if there exist a map $f : \mathcal{T} \rightarrow \mathbb{F}$ and a nonzero scalar λ with $\lambda \in \{-1, 1\}$ such that $\Phi(T) = \lambda T + f(T)I$ for all $T \in \mathcal{T}$.*

Now, let us start the proof of Theorem 2.1.

As every map of the form $T \rightarrow ZT + f(T)$ preserves the strong commutativity, where Z is a central element with $Z^2 = I$ and f is a map into the center, to prove Theorem 2.1, we need only check the ‘‘only if’’ part. We will do this by checking several lemmas.

In the sequel, we always assume that $\Phi : \mathcal{U} \rightarrow \mathcal{U}$ is a surjective map that preserves the strong commutativity and \mathcal{A} is noncommutative (for the case that \mathcal{B} is noncommutative, the proof is similar).

LEMMA 2.4. $\Phi(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{U})$.

Proof. Let Z be an arbitrary element in $\mathcal{Z}(\mathcal{U})$. Then for any $T \in \mathcal{U}$, we have $[\Phi(Z), \Phi(T)] = [Z, T] = 0$, which implies that $\Phi(Z) \in \mathcal{Z}(\mathcal{U})$ since Φ is surjective. Hence $\Phi(\mathcal{Z}(\mathcal{U})) \subseteq \mathcal{Z}(\mathcal{U})$. On the other hand, there is some $T \in \mathcal{U}$ such that $\Phi(T) = Z$ as Φ is surjective. Then we have $[T, S] = [Z, \Phi(S)] = 0$ for all $S \in \mathcal{U}$, which entails that $T \in \mathcal{Z}(\mathcal{U})$. Hence $\mathcal{Z}(\mathcal{U}) \subseteq \Phi(\mathcal{Z}(\mathcal{U}))$, completing the proof. \square

LEMMA 2.5. *For any $T, S \in \mathcal{U}$, there exists an element $Z_{T,S} \in \mathcal{Z}(\mathcal{U})$ depending on T, S such that $\Phi(T + S) = \Phi(T) + \Phi(S) + Z_{T,S}$.*

Proof. For any $T, S, R \in \mathcal{U}$, we have

$$\begin{aligned} & [\Phi(T + S) - \Phi(T) - \Phi(S), \Phi(R)] \\ &= [\Phi(T + S), \Phi(R)] - [\Phi(T), \Phi(R)] - [\Phi(S), \Phi(R)] \\ &= [T + S, R] - [T, R] - [S, R] = 0. \end{aligned}$$

By the surjectivity of Φ and the above equation, we obtain that $Z_{T,S} = \Phi(T + S) - \Phi(T) - \Phi(S) \in \mathcal{Z}(\mathcal{U})$. \square

LEMMA 2.6. *There exist a nonzero element $Z \in \mathcal{Z}(\mathcal{U})$ and an element $Z' \in \mathcal{Z}(\mathcal{U})$ such that $\Phi(P) = ZP + Z'$.*

Proof. For any $T \in \mathcal{U}$, it is easy to check that $[P, [P, [P, T]]] = [P, T]$. So we have $[P, [P, [\Phi(P), \Phi(T)]]] = [\Phi(P), \Phi(T)]$. It follows from the surjectivity of Φ that

$$[P, [P, [\Phi(P), T]]] = [\Phi(P), T] \quad \text{for all } T \in \mathcal{U}.$$

Thus we obtain that,

$$P\Phi(P)T - PT\Phi(P) - 2P\Phi(P)TP + 2PT\Phi(P)P + \Phi(P)TP - T\Phi(P)P = \Phi(P)T - T\Phi(P) \tag{1}$$

holds for all $T \in \mathcal{U}$. Taking $T = QSQ$ in Eq. (1), one gets $P\Phi(P)QSQ = \Phi(P)QSQ - QSQ\Phi(P)$, that is, $Q\Phi(P)QSQ = QSQ\Phi(P)Q$ holds for all $S \in \mathcal{U}$. This implies that

$$Q\Phi(P)Q = \begin{pmatrix} 0 & 0 \\ 0 & Z_B \end{pmatrix} \quad \text{for some } Z_B \in \mathcal{Z}(\mathcal{B}). \tag{2}$$

Similarly, letting $T = PSP$ in Eq. (1), we get

$$P\Phi(P)P = \begin{pmatrix} Z_A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for some } Z_A \in \mathcal{Z}(\mathcal{A}). \tag{3}$$

For $Z_B \in \mathcal{Z}(\mathcal{B})$, by the assumption on $\mathcal{Z}(\mathcal{U})$ that $Q\mathcal{Z}(\mathcal{U})Q = \mathcal{Z}(Q\mathcal{U}Q)$, there exists $Z_1 = \begin{pmatrix} Z'_A & 0 \\ 0 & Z'_B \end{pmatrix} \in \mathcal{Z}(\mathcal{U})$ such that $QZ_1Q = \begin{pmatrix} 0 & 0 \\ 0 & Z_B \end{pmatrix}$. It follows that $Z'_B = Z_B$.

Thus $Z_1 = \begin{pmatrix} Z'_A & 0 \\ 0 & Z_B \end{pmatrix} \in \mathcal{Z}(\mathcal{U})$ and

$$\Phi(P) = \begin{pmatrix} Z_A - Z'_A & 0 \\ 0 & 0 \end{pmatrix} + Z_1 + P\Phi(P)Q.$$

Note that, by Lemma 2.4 and Lemma 2.5, $\Phi(I) \in \mathcal{Z}(\mathcal{U})$ and $\Phi(I) - \Phi(P) - \Phi(Q) \in \mathcal{Z}(\mathcal{U})$. So there exists an element $Z_0 \in \mathcal{Z}(\mathcal{U})$ such that

$$\Phi(Q) = Z_0 - \begin{pmatrix} Z'_A - Z_A & 0 \\ 0 & 0 \end{pmatrix} - P\Phi(P)Q. \tag{4}$$

On the other hand, it is easily seen that $[Q, [Q, T]] = [Q, T]$ for all $T \in \mathcal{U}$. It follows that $[Q, [\Phi(Q), \Phi(T)]] = [\Phi(Q), \Phi(T)]$, and so

$$[Q, [\Phi(Q), S]] = [\Phi(Q), S] \quad \text{for all } S \in \mathcal{U}.$$

Replacing S by $PSP + QSQ$ in the above equation, we get

$$P\Phi(Q)PSP - PSP\Phi(Q)P + Q\Phi(Q)QSQ + 2P\Phi(Q)QSQ - 2PSP\Phi(Q)Q - QSQ\Phi(Q)Q = 0.$$

Combining the equation with Eq. (4), we obtain $PSP\Phi(P)Q = P\Phi(P)QSQ$. So

$$\begin{aligned} SP\Phi(P)Q &= (PSP + PSQ + QSQ)P\Phi(P)Q = PSP\Phi(P)Q \\ &= P\Phi(P)QSQ = P\Phi(P)Q(PSP + PSQ + QSQ) \\ &= P\Phi(P)QS \end{aligned}$$

holds for all $S \in \mathcal{U}$, which implies that $P\Phi(P)Q \in \mathcal{Z}(\mathcal{U})$. This forces that $P\Phi(P)Q = 0$. So

$$\Phi(P) = \begin{pmatrix} Z_A - Z'_A & 0 \\ 0 & 0 \end{pmatrix} + Z_1.$$

For $Z_A - Z'_A \in \mathcal{Z}(\mathcal{A})$, by the assumption on $\mathcal{Z}(\mathcal{U})$ that $P\mathcal{Z}(\mathcal{U})P = \mathcal{Z}(P\mathcal{U}P)$, there must exist $Z_2 = \begin{pmatrix} Z''_A & 0 \\ 0 & Z''_B \end{pmatrix} \in \mathcal{Z}(\mathcal{U})$ such that $PZ_2P = \begin{pmatrix} Z_A - Z'_A & 0 \\ 0 & 0 \end{pmatrix}$. So $Z_A - Z'_A = Z''_A$. Now let $Z = Z_2$ and $Z' = Z_1$. We obtain

$$\Phi(P) = \begin{pmatrix} Z_A - Z'_A & 0 \\ 0 & 0 \end{pmatrix} + Z_1 = \begin{pmatrix} Z''_A & 0 \\ 0 & Z''_B \end{pmatrix} P + Z' = ZP + Z'.$$

Finally, we still need to prove that $Z \neq 0$. If, on the contrary, $Z = 0$, then $\Phi(P) = Z'$. By Lemma 2.4, we get $P \in \mathcal{Z}(\mathcal{U})$, which is impossible. The proof is complete. \square

LEMMA 2.7. *Let Z be the element as in Lemma 2.6. The following statements are true.*

- (1) $Z^2 = I$.
- (2) For any $T \in \mathcal{U}$, there exists $Z_{PTQ} \in \mathcal{Z}(\mathcal{U})$ such that

$$\Phi(PTQ) = ZPTQ + Z_{PTQ}.$$

- (3) For any $T \in \mathcal{U}$, there exists $Z_{PTP} \in \mathcal{Z}(\mathcal{U})$ such that

$$\Phi(PTP) = ZPTP + Z_{PTP}.$$

- (4) For any $T \in \mathcal{U}$, there exists $Z_{QTQ} \in \mathcal{Z}(\mathcal{U})$ such that

$$\Phi(QTQ) = ZQTQ + Z_{QTQ}.$$

Proof. Take any $T \in \mathcal{U}$. We will prove the lemma by three steps.

Step 1. $Z^2 = I$ and there exists $Z_{PTP} \in \mathcal{Z}(\mathcal{U})$ such that $\Phi(PTP) = ZPTP + Z_{PTP}$, that is, the statements (1) and (3) hold.

Since $[\Phi(P), \Phi(PTP)] = [P, PTP] = 0$, by Lemma 2.6, we get

$$ZP\Phi(PTP)Q = 0. \tag{5}$$

Taking any $S \in \mathcal{U}$, by the surjectivity of Φ , there exists some element $X \in \mathcal{U}$ such that $\Phi(X) = PSQ$. Then we have $[\Phi(P), PSQ] = [\Phi(P), \Phi(X)] = [P, X]$. By Lemma 2.6, we get

$$ZPSQ = PXP + PXQ - PXP = PXQ. \tag{6}$$

Note that $[\Phi(PTP), PSQ] = [\Phi(PTP), \Phi(X)] = [PTP, X]$, that is,

$$P\Phi(PTP)PSQ - PSQ\Phi(PTP)Q = PTPXP + PTPXQ - PXPTP. \tag{7}$$

Combining Eq. (7) with Eq. (6), we get

$$P\Phi(PTP)PSQ - PSQ\Phi(PTP)Q - ZPTPSQ = PTPXP - PXPTP,$$

which implies that $P\Phi(PTP)PSQ - PSQ\Phi(PTP)Q - ZPTPSQ = 0$, that is,

$$(P\Phi(PTP)P - ZPTP)PSQ = PSQ(Q\Phi(PTP)Q) \quad \text{for all } S \in \mathcal{U}. \tag{8}$$

Now let us prove that there exist $Z_{PTP}(A) \in \mathcal{Z}(\mathcal{A})$ and $Z_{PTP}(B) \in \mathcal{Z}(\mathcal{B})$ such that

$$P\Phi(PTP)P - ZPTP = \begin{pmatrix} Z_{PTP}(A) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q\Phi(PTP)Q = \begin{pmatrix} 0 & 0 \\ 0 & Z_{PTP}(B) \end{pmatrix}.$$

In fact, let $P\Phi(PTP)P - ZPTP = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$, $PSQ = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and $Q\Phi(PTP)Q = \begin{pmatrix} 0 & 0 \\ 0 & B_0 \end{pmatrix}$.

By a direct matrix computation and using Eq. (8), we get

$$A_0M = MB_0 \quad \text{for all } M \in \mathcal{M}. \tag{9}$$

Thus, for any $A \in \mathcal{A}$, we have

$$(AA_0 - A_0A)M = AA_0M - A_0(AM) = AMB_0 - AMB_0 = 0.$$

Since \mathcal{M} is loyal, \mathcal{M} is a faithful left \mathcal{A} -module. Hence, the above equation implies $A_0 \in \mathcal{Z}(\mathcal{A})$. Write $A_0 = Z_{PTP}(A)$. By the assumption $P\mathcal{Z}(\mathcal{U})P = \mathcal{Z}(P\mathcal{U}P)$, there exists $Z_1 = \begin{pmatrix} Z_A & 0 \\ 0 & Z_B \end{pmatrix} \in \mathcal{Z}(\mathcal{U})$ such that $PZ_1P = \begin{pmatrix} Z_{PTP}(A) & 0 \\ 0 & 0 \end{pmatrix}$, and so $Z_A = Z_{PTP}(A)$. Since $\begin{pmatrix} Z_{PTP}(A) & 0 \\ 0 & Z_B \end{pmatrix} \in \mathcal{Z}(\mathcal{U})$, we have

$$Z_{PTP}(A)M = MZ_B \quad \text{for all } M \in \mathcal{M}. \tag{10}$$

This, together with Eq. (9), yields $MZ_B = MB_0$. Note that \mathcal{M} is also a faithful right \mathcal{B} -module. It follows that $Z_B = B_0$. So we can take $Z_{PTP}(B) = Z_B = B_0$.

By Eq. (10), it is clear that $Z_{PTP} = \begin{pmatrix} Z_{PTP}(A) & 0 \\ 0 & Z_{PTP}(B) \end{pmatrix} \in \mathcal{Z}(\mathcal{U})$. Hence we achieve that

$$\Phi(PTP) = ZPTP + Z_{PTP} + P\Phi(PTP)Q. \tag{11}$$

Finally, we show that $Z^2 = I$ and $P\Phi(PTP)Q = 0$. Indeed, for any $PTP, PSP \in P\mathcal{U}P$, since Φ preserves the strong commutativity, we have

$$\begin{aligned} [PTP, PSP] &= [\Phi(PTP), \Phi(PSP)] = [ZPTP + P\Phi(PTP)Q, ZPSP + P\Phi(PSP)Q] \\ &= Z^2[PTP, PSP] + ZPTP\Phi(PSP)Q - ZPSP\Phi(PTP)Q, \end{aligned}$$

which implies that

$$(Z^2 - I)[PTP, PSP] = 0. \tag{12}$$

Since \mathcal{A} is noncommutative, there exist $T_0, S_0 \in \mathcal{U}$ such that $[PT_0P, PS_0P] \neq 0$. Write $U_0 = [PT_0P, PS_0P] = \begin{pmatrix} A'_0 & 0 \\ 0 & 0 \end{pmatrix}$ and $Z^2 - I = \begin{pmatrix} Z'_A & 0 \\ 0 & Z'_B \end{pmatrix}$. If $Z^2 - I \neq 0$, then $Z'_A \neq 0$. By Eq. (12), we have $\begin{pmatrix} Z'_A & 0 \\ 0 & Z'_B \end{pmatrix} \begin{pmatrix} A'_0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Z'_A A'_0 & 0 \\ 0 & 0 \end{pmatrix} = 0$. Hence $Z'_A A'_0 = 0$. For nonzero Z'_A , by the assumption $P\mathcal{Z}(\mathcal{U})P = \mathcal{Z}(P\mathcal{U}P)$, there exists a nonzero $Z'' = \begin{pmatrix} Z''_A & 0 \\ 0 & Z''_B \end{pmatrix} \in \mathcal{Z}(\mathcal{U})$ such that $PZ''P = \begin{pmatrix} Z'_A & 0 \\ 0 & 0 \end{pmatrix}$. So $Z'_A = Z''_A$. It follows that $Z'_A M = Z''_A M = MZ''_B$ for all $M \in \mathcal{M}$. Then we get

$$0 = Z'_A A'_0 M = A'_0 Z'_A M = A'_0 M Z''_B.$$

Since $Z''_B \neq 0$, we must have $A'_0 = 0$ as \mathcal{M} is loyal. Thus $U_0 = 0$, a contradiction. Therefore, $Z^2 = I$. It follows from Eq. (5) that $P\Phi(PTP)Q = 0$. Thus, by Eq. (11), we have $\Phi(PTP) = ZPTP + Z_{PTP}$. So the statements (1) and (3) are true.

Step 2. There exists $Z_{QTQ} \in \mathcal{Z}(\mathcal{U})$ such that $\Phi(QTQ) = ZQTQ + Z_{QTQ}$, that is, the statement (4) holds.

By a similar argument to that of Step 1, one can easily check that Step 2 is true.

Step 3. There exists $Z_{PTQ} \in \mathcal{Z}(\mathcal{U})$ such that $\Phi(PTQ) = ZPTQ + Z_{PTQ}$, that is, the statement (2) holds.

For any $PSP \in P\mathcal{U}P$, by Step 1, we have

$$[PSP, PTQ] = [\Phi(PSP), \Phi(PTQ)] = [ZPSP, \Phi(PTQ)] = [PSP, Z\Phi(PTQ)],$$

that is, $[PSP, Z\Phi(PTQ) - PTQ] = 0$. Since S is arbitrary, it follows that

$$Z\Phi(PTQ) - PTQ = \begin{pmatrix} Z'_{PTQ}(A) & 0 \\ 0 & B_{PTQ} \end{pmatrix} \tag{13}$$

for some $Z'_{PTQ}(A) \in \mathcal{Z}(\mathcal{A})$ and $B_{PTQ} \in \mathcal{B}$.

Similarly, by Step 2, one can prove that

$$Z\Phi(PTQ) - PTQ = \begin{pmatrix} A_{PTQ} & 0 \\ 0 & Z'_{PTQ}(B) \end{pmatrix} \tag{14}$$

for some $A_{PTQ} \in \mathcal{A}$ and $Z'_{PTQ}(B) \in \mathcal{Z}(\mathcal{B})$. Comparing Eq. (13) with Eq. (14), and noting that $Z^2 = I$, one gets

$$\Phi(PTQ) = ZPTQ + \begin{pmatrix} Z''_{PTQ}(A) & 0 \\ 0 & Z''_{PTQ}(B) \end{pmatrix},$$

where $Z''_{PTQ}(A) \in \mathcal{Z}(\mathcal{A})$ and $Z''_{PTQ}(B) \in \mathcal{Z}(\mathcal{B})$. For $Z''_{PTQ}(A)$, by the assumption $P\mathcal{Z}(\mathcal{U})P = \mathcal{Z}(P\mathcal{U}P)$, there exists a nonzero $Z_{PTQ} = \begin{pmatrix} Z_{PTQ}(A) & 0 \\ 0 & Z_{PTQ}(B) \end{pmatrix} \in \mathcal{Z}(\mathcal{U})$ such that $PZ_{PTQ}P = \begin{pmatrix} Z''_{PTQ}(A) & 0 \\ 0 & 0 \end{pmatrix}$, which implies that $Z''_{PTQ}(A) = Z_{PTQ}(A)$. So

$$\Phi(PTQ) = ZPTQ + Z_{PTQ} + \begin{pmatrix} 0 & 0 \\ 0 & f_B(PTQ) \end{pmatrix}, \tag{15}$$

where $f_B(PTQ) = Z''_{PTQ}(B) - Z_{PTQ}(B)$. To complete the proof of the step, we have to show that $f_B(PTQ) = 0$.

In fact, for any $PTQ, PSQ \in P\mathcal{U}P$, write $PTQ = \begin{pmatrix} 0 & M_T \\ 0 & 0 \end{pmatrix}$ and $PSQ = \begin{pmatrix} 0 & M_S \\ 0 & 0 \end{pmatrix}$.

By Eq. (15), we have

$$0 = [PSQ, PTQ] = [\Phi(PSQ), \Phi(PTQ)] = \begin{pmatrix} 0 & M_S f_B(PTQ) - M_T f_B(PSQ) \\ 0 & 0 \end{pmatrix},$$

and so

$$M_S f_B(PTQ) - M_T f_B(PSQ) = 0 \quad \text{for all } M_S, M_T \in \mathcal{M}. \tag{16}$$

By Eq. (16), to prove $f_B(PTQ) = 0$ is equivalent to show the following assertion.

ASSERTION. Let $f : \mathcal{M} \rightarrow \mathcal{B}$ be any map satisfying $M_1 f(M_2) - M_2 f(M_1) = 0$ for all $M_1, M_2 \in \mathcal{M}$. If \mathcal{A} is noncommutative and \mathcal{M} is a loyal (A, B) -bimodule, then $f(M) = 0$ for all $M \in \mathcal{M}$.

Taking any $A_1, A_2 \in \mathcal{A}$, by assumption, we get

$$\begin{aligned} A_1 A_2 M_1 f(M_2) &= A_1 M_2 f(A_2 M_1) = A_2 M_1 f(A_1 M_2) \\ &= A_2 A_1 M_2 f(M_1) = A_2 A_1 M_1 f(M_2). \end{aligned}$$

That is, $(A_1 A_2 - A_2 A_1) M_1 f(M_2) = 0$ for all $M_1 \in \mathcal{M}$. Since \mathcal{A} is noncommutative, there exist two elements A_1 and A_2 such that $A_1 A_2 - A_2 A_1 \neq 0$. It follows from the loyalty of \mathcal{M} that $f(M_2) = 0$. Since M_2 is arbitrary, the assertion is true.

Therefore, the statement (2) is true. \square

LEMMA 2.8. *There exists a map $f : \mathcal{U} \rightarrow \mathcal{Z}(\mathcal{U})$ such that $\Phi(T) = ZT + f(T)$ for all $T \in \mathcal{U}$, here Z is the element as in Lemma 2.6. Therefore, the “only if” part of the theorem is true.*

Proof. By Lemma 2.7(1), we have $Z^2 = I$. Now, for any $T \in \mathcal{U}$, by Lemma 2.7, we get

$$\begin{aligned} &\Phi(PTP) + \Phi(PTQ) + \Phi(QTQ) \\ &= ZPTP + Z_{PTP} + ZPTQ + Z_{PTQ} + ZQTQ + Z_{QTQ} \\ &= ZT + (Z_{PTP} + Z_{PTQ} + Z_{QTQ}). \end{aligned} \tag{17}$$

On the other hand, by Lemma 2.5, there exists $Z_T \in \mathcal{Z}(\mathcal{U})$ such that

$$\begin{aligned} & \Phi(T) - (\Phi(PTP) + \Phi(PTQ) + \Phi(QTQ)) \\ &= \Phi(PTP + PTQ + QTQ) - \Phi(PTP) - \Phi(PTQ) - \Phi(QTQ) = Z_T. \end{aligned} \tag{18}$$

Define a map $f : \mathcal{U} \rightarrow \mathcal{Z}(\mathcal{U})$ by $f(T) = Z_{PTP} + Z_{PTQ} + Z_{QTQ} + Z_T$ for each $T \in \mathcal{U}$. Then by Eqs.(17)-(18), we get $\Phi(T) = ZT + f(T)$, completing the proof. \square

REMARK 2.9. Before concluding the paper, we remark that the condition of Theorem 2.1 that \mathcal{A} or \mathcal{B} is noncommutative can not be deleted simply. To see this we give two counterexamples here.

A most simple example is 2×2 upper triangular matrix algebras. Let $\mathcal{T} = \mathcal{T}(1,1)$ be the 2×2 upper triangular matrix algebra. It is clear that $\mathcal{A} = \mathcal{B} = \mathbb{C}$ is commutative and that the center $\mathcal{Z}(\mathcal{T}) = \mathbb{F}I$. For any $\lambda, \gamma \in \mathbb{F}$ with $\lambda \neq 0$, let $\Phi : \mathcal{T} \rightarrow \mathcal{T}$ be the bijective linear map defined by

$$\Phi\left(\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\right) = \begin{pmatrix} \lambda a_{11} + \gamma a_{12} & \frac{1}{\lambda} a_{12} \\ 0 & \lambda a_{22} \end{pmatrix} = \lambda \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} + \begin{pmatrix} \gamma a_{12} & (\frac{1}{\lambda} - \lambda)a_{12} \\ 0 & 0 \end{pmatrix}.$$

It is easily checked that Φ preserves the strong commutativity. However, Φ is not of the form stated in Theorem 2.1.

A more complicated example is the following. Let $\mathcal{A} = \mathcal{B} = \{M_\phi : \phi \in L_\infty[0, 1]\} \subset \mathcal{B}(L_2[0, 1])$ and $\mathcal{M} = \mathcal{B}(L_2[0, 1])$. Here M_ϕ denotes the operator defined by $M_\phi \xi = \phi \xi$ for any $\xi \in L_2[0, 1]$ and $\mathcal{B}(L_2[0, 1])$ denotes the algebra of all bounded linear operators acting on $L_2[0, 1]$. Let $\lambda \in \mathbb{F}$ be a nonzero scalar such that $\lambda^{-1} \neq \lambda$. Let $\Phi : \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) \rightarrow \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the bijective linear map defined by

$$\Phi\left(\begin{pmatrix} M_\phi & W \\ 0 & M_\psi \end{pmatrix}\right) = \begin{pmatrix} \lambda M_\phi & \lambda^{-1}W \\ 0 & \lambda M_\psi \end{pmatrix}.$$

Note that

$$\begin{aligned} & \left[\begin{pmatrix} \lambda M_{\phi_1} & \lambda^{-1}W_1 \\ 0 & \lambda M_{\psi_1} \end{pmatrix}, \begin{pmatrix} \lambda M_{\phi_2} & \lambda^{-1}W_2 \\ 0 & \lambda M_{\psi_2} \end{pmatrix} \right] \\ &= \begin{pmatrix} 0 & M_{\phi_1}W_2 + W_1M_{\psi_2} - M_{\phi_2}W_1 - W_2M_{\psi_1} \\ 0 & 0 \end{pmatrix} \\ &= \left[\begin{pmatrix} M_{\phi_1} & W_1 \\ 0 & M_{\psi_1} \end{pmatrix}, \begin{pmatrix} M_{\phi_2} & W_2 \\ 0 & M_{\psi_2} \end{pmatrix} \right] \end{aligned}$$

holds for any $M_{\phi_i} \in \mathcal{A}$, $M_{\psi_i} \in \mathcal{B}$ and $W_i \in \mathcal{M}$, $i = 1, 2$. Therefore Φ preserves the strong commutativity. We'll check that Φ is not of the form in Theorem 2.1. It is clear that the center

$$\mathcal{Z} = \mathcal{Z}(\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})) = \left\{ \begin{pmatrix} M_\mu & 0 \\ 0 & M_\mu \end{pmatrix} : \mu \in \mathbb{F} \right\} = \mathbb{F}I.$$

If Φ has the form in Theorem 2.1, then there exist a $\mu_0 \in \mathbb{F}$ and a map $h : \mathcal{A} \times \mathcal{M} \times \mathcal{B} \rightarrow \mathbb{F}$ such that

$$\begin{aligned} & \begin{pmatrix} \lambda M_\phi & \lambda^{-1} W \\ 0 & \lambda M_\psi \end{pmatrix} \\ &= \begin{pmatrix} M_{\mu_0} & 0 \\ 0 & M_{\mu_0} \end{pmatrix} \begin{pmatrix} M_\phi & W \\ 0 & M_\psi \end{pmatrix} + \begin{pmatrix} M_{h(M_\phi, W, M_\psi)} & 0 \\ 0 & M_{h(M_\phi, W, M_\psi)} \end{pmatrix} \\ &= \begin{pmatrix} M_{\mu_0} M_\phi + M_{h(M_\phi, W, M_\psi)} & M_{\mu_0} W \\ 0 & M_{\mu_0} M_\psi + M_{h(M_\phi, W, M_\psi)} \end{pmatrix}. \end{aligned}$$

It follows that $\mu_0 = \lambda^{-1}$ and $(\lambda - \lambda^{-1})M_\phi = (\lambda - \lambda^{-1})M_\psi$ holds for all $\phi, \psi \in L_\infty[0, 1]$, which is impossible.

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