

THE PAULSEN PROBLEM IN OPERATOR THEORY

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Abstract. The *Paulsen Problem* in Hilbert space frame theory has proved to be one of the most intractable problems in the field. We will help explain why by showing that this problem is equivalent to a fundamental, deep problem in operator theory. This answers a question posed by Bodmann and Casazza. We will also give generalizations of these problems and we will spell out exactly the *complementary versions* of the problem.

1. Introduction

The *Paulsen Problem* has proved to be one of the most intractable problems in frame theory (See Section 2 for definitions):

PROBLEM 1.1. (Paulsen Problem) *Find the function $h(\varepsilon, M, N)$ so that for any ε -nearly equal norm, ε -nearly Parseval frame $\{f_i\}_{i=1}^N$ for a M -dimensional Hilbert space \mathcal{H}_M , there is an equal norm Parseval frame $\{g_i\}_{i=1}^M$ for \mathcal{H}_M satisfying:*

$$\sum_{i=1}^N \|f_i - g_i\|^2 \leq h(\varepsilon, M, N).$$

A fundamental question here is whether the function $h(\varepsilon, M, N)$ actually depends upon N . We have no examples showing this at this time, although it is known that this function must depend upon M . For all examples we know at this time, we have

$$h(\varepsilon, M, N) \leq 16\varepsilon M.$$

For a dozen years there was no progress at all on the Paulsen Problem. Recently, some progress has been made on the problem. First, Bodmann and Casazza [7] used differential equations to give an estimate for the function $h(\varepsilon, M, N)$. This paper leaves open the case where M, N are not relatively prime. Using gradient descent of the *frame potential*, Casazza, Fickus and Mixon [9] gave a completely different solution for the

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Paulsen Problem which works in the case where M, N are relatively prime. The estimates in these two papers seem to be quite far from optimal since it is on the order of $M^2 N^9 \varepsilon$ and best evidence indicates the answer should be of the form $cM\varepsilon$ or at worst $cN\varepsilon$ for some universal, small c .

We will show why the Paulsen Problem has proved to so intractable by showing that it is equivalent to a fundamental, deep problem in operator theory. The fact that there must be a connection between these two problems was first observed in [7]. In effect, we are answering a problem left open in that paper.

PROBLEM 1.2. (Projection Problem) *Let \mathcal{H}_N be an N -dimensional Hilbert space with orthonormal basis $\{e_i\}_{i=1}^N$. Find the function $g(\varepsilon, M, N)$ satisfying the following. If P is a projection of rank M on \mathcal{H}_N satisfying*

$$(1 - \varepsilon) \frac{M}{N} \leq \|Pe_i\|^2 \leq (1 + \varepsilon) \frac{M}{N}, \quad \text{for all } i = 1, 2, \dots, N,$$

then there is a projection Q with $\|Qe_i\|^2 = \frac{M}{N}$ for all $i = 1, 2, \dots, N$ satisfying

$$\sum_{i=1}^N \|Pe_i - Qe_i\|^2 \leq g(\varepsilon, M, N).$$

In [7], it is shown that the Hilbert-Schmidt distance between an ε -nearly constant diagonal projection and its closest constant diagonal projection is less than or equal to $2h(\varepsilon, M, N)$. Here, we will show the full equivalence (up to a factor of 4) of $f(\varepsilon, M, N)$ and $g(\varepsilon, M, N)$. Analyzing the diagonal properties of projections has a long history. Kadison [23, 24] gave a complete characterization of the diagonals of projections for both the finite and infinite dimensional case. Analogous results on projections in type II_1 factors was given by Argerami and Massey [1]. For the more general problem of characterizing the diagonals of the unitary orbit of a self-adjoint operator, there is much more literature. This is equivalent in frame theory to characterizing the sequences which occur as the norms of a frame with a specified frame operator. We refer the reader to [2, 3, 4, 5, 8, 13, 14, 10, 19, 21, 25, 26, 27, 28] for a review of the work in this direction.

We will also consider the *Naimark complement* of nearly equal norm Parseval frames. We will show that the Paulsen function for a Parseval frame and its Naimark complement have a natural relationship. As a consequence of this, we will see that the Paulsen Problem only has to be solved for frames with a small number of elements relative to the dimension of the space. In particular, we only have to deal with the case of $N \leq 2M$.

This paper is organized as follows. In Section 2 we give the requisite background needed from frame theory. In Section 3 we will prove a sequence of results which give an exact relationship between nearly equal norm Parseval frames for \mathcal{H}_M and the distance between orthogonal projections P, Q of rank M on \mathcal{H}_N . As a tool here, we will relate our quantities to the *principal angles* between subspaces of a Hilbert space and the *chordal distance* between subspaces of a Hilbert space. In Section 4 we give an exact calculation relating the Paulsen Problem function and the function in the

Projection Problem. Section 5 contains generalizations of both problems and in Section 6 we will relate the Paulsen Problem function for a frame and its Naimark complement.

2. Frame theory

In this section we will give a brief introduction to frame theory containing the results used in the paper. For the basics on frame theory see [15].

DEFINITION 2.1. A family of vectors $\{f_i\}_{i=1}^N$ in an M -dimensional Hilbert space \mathcal{H}_M is a *frame* if there are constants $0 < A \leq B < \infty$ so that for all $f \in \mathcal{H}_M$ we have

$$A\|f\|^2 \leq \sum_{i=1}^M |\langle f, f_i \rangle|^2 \leq B\|f\|^2.$$

If $A = B$, this is a *tight frame* and if $A = B = 1$, it is a *Parseval frame*. If there is a constant c so that $\|f_i\| = c$, for all $i = 1, 2, \dots, N$ it is an *equal norm frame* and if $c = 1$, it is a *unit norm frame*.

If $\{f_i\}_{i=1}^N$ is a frame for \mathcal{H}_M , the *analysis operator* of the frame is the operator $T : \mathcal{H}_M \rightarrow \ell_2(N)$ given by

$$T(f) = \sum_{i=1}^N \langle f, f_i \rangle e_i,$$

where $\{e_i\}_{i=1}^N$ is the natural orthonormal basis of $\ell_2(N)$. The *synthesis operator* is T^* and satisfies

$$T^* \left(\sum_{i=1}^N a_i e_i \right) = \sum_{i=1}^N a_i f_i.$$

The *frame operator* is the positive, self-adjoint invertible operator $S = T^*T$ on \mathcal{H}_M and satisfies

$$S(f) =: T^*T(f) = \sum_{i=1}^N \langle f, f_i \rangle f_i.$$

A direct calculation shows that the frame $\{S^{-1/2} f_i\}_{i=1}^N$ is a Parseval frame called the *canonical Parseval frame* for the frame. Also, $\{f_i\}_{i=1}^N$ is a Parseval frame if and only if $S = I$. We say that two frames $\{f_i\}_{i \in I}, \{g_i\}_{i \in I}$ for \mathcal{H} are *isomorphic* if there is an invertible operator L on \mathcal{H} satisfying $Lf_i = g_i$, for all $i \in I$. It is known [11] that two frames are isomorphic if and only if their analysis operators have the same image, and two Parseval frames are isomorphic if and only if the isomorphism is a unitary operator. If $\{f_i\}_{i=1}^N$ is a frame with frame operator S having eigenvalues $\{\lambda_j\}_{j=1}^M$, then

$$\sum_{i=1}^N \|f_i\|^2 = \sum_{j=1}^M \lambda_j.$$

So if $\{f_i\}_{i=1}^N$ is an equal norm Parseval frame then

$$\|f_1\|^2 = \frac{1}{N} \sum_{i=1}^N \|f_i\|^2 = \frac{M}{N}.$$

We will need a distance function for frames and projections.

DEFINITION 2.2. If $\mathcal{F} = \{f_i\}_{i=1}^N$ and $\mathcal{G} = \{g_i\}_{i=1}^N$ are frames for \mathcal{H}_M , we define the distance between them by

$$d(\mathcal{F}, \mathcal{G}) = \sum_{i=1}^N \|f_i - g_i\|^2.$$

If P, Q are projections on $\ell_2(N)$, we define

$$d(P, Q) = \sum_{i=1}^N \|Pe_i - Qe_i\|^2,$$

where $\{e_i\}_{i=1}^N$ is the natural orthonormal basis for $\ell_2(N)$.

Of course, d is not really a distance function unless we square it in the above inequalities. For the Paulsen Problem, we define:

DEFINITION 2.3. A frame $\{f_i\}_{i=1}^N$ with frame operator S is ε -nearly Parseval if

$$(1 - \varepsilon)I \leq S \leq (1 + \varepsilon)I.$$

The frame is ε -nearly equal norm if

$$(1 - \varepsilon)\frac{M}{N} \leq \|f_i\|^2 \leq (1 + \varepsilon)\frac{M}{N}.$$

A reduction of the Paulsen Problem to the Parseval case is done in [7].

PROPOSITION 2.4. If $\mathcal{F} = \{f_i\}_{i=1}^N$ is an ε -nearly Parseval frame for \mathcal{H}_M then the Parseval frame $\mathcal{G} = \{S^{-1/2}f_i\}_{i=1}^N$ satisfies

$$d(\mathcal{F}, \mathcal{G}) \leq M(2 - \varepsilon - 2\sqrt{1 - \varepsilon}) \leq \frac{M\varepsilon^2}{4}.$$

It is also nearly equal norm with the bounds:

$$\frac{(1 - \varepsilon)^2 M}{1 + \varepsilon} \frac{1}{N} \leq \|S^{-1/2}f_i\|^2 \leq \frac{(1 + \varepsilon)^2 M}{1 - \varepsilon} \frac{1}{N}.$$

It is known [6, 12, 21] that the canonical Parseval frame is the closest Parseval frame (with the distance function above) to a given frame. It is also known that this constant is best possible in general. So we are not giving up anything by working with a simpler variation of the Paulsen Problem.

PROBLEM 2.5. (Parseval Paulsen Problem) Find the function $h(\varepsilon, M, N)$ so that whenever $\{f_i\}_{i=1}^N$ is an ε -nearly equal norm Parseval frame, then there is an equal norm Parseval frame \mathcal{G} so that

$$d(\mathcal{F}, \mathcal{G}) \leq h(\varepsilon, M, N).$$

Finally, we recall a fundamental result in frame theory – the classification theorem for Parseval frames [15, 18] – which will be used extensively here.

THEOREM 2.6. *A family $\{f_i\}_{i=1}^N$ is a Parseval frame for \mathcal{H}_M if and only if the analysis operator T for the frame is an isometry satisfying:*

$$Tf_i = Pe_i, \text{ for all } i = 1, 2, \dots, N.,$$

where $\{e_i\}_{i=1}^N$ is the natural orthonormal basis of $\ell_2(N)$ and P is the orthogonal projection of $\ell_2(N)$ onto $T(\mathcal{H}_M)$.

3. Preliminary results

Let us first outline the proof of the equivalence of the Paulsen Problem and the Projection Problem. This will explain the results we develop in this section.

First we will assume that the Parseval Paulsen Problem function $h(\varepsilon, M, N)$ is given and let P be a rank M projection on $\ell_2(N)$ with ε -nearly constant diagonal. We need to find a constant diagonal projection whose distance to P is on the order of $h(\varepsilon, M, N)$. To do this, we consider $\mathcal{F} = \{Pe_i\}_{i=1}^N$ a nearly equal norm Parseval frame for \mathcal{H}_M . It follows that there is a equal norm Parseval frame $\mathcal{G} = \{g_i\}_{i=1}^N$ for \mathcal{H}_M with

$$d(\mathcal{F}, \mathcal{G}) \leq h(\varepsilon, M, N).$$

Letting T_1 be the analysis operator for \mathcal{G} , we have the existence of a projection Q on $\ell_2(N)$ so that

$$T_1g_i = Qe_i, \text{ for all } i = 1, 2, \dots, N.$$

So it is the problem of finding $d(P, Q)$ we will address in this section.

Conversely, if we assume the Projection Problem function $g(\varepsilon, M, N)$ is given, we choose a nearly equal norm Parseval frame $\mathcal{F} = \{f_i\}_{i=1}^N$ with analysis operator $T : \mathcal{H}_M \rightarrow \ell_2(N)$ an isometry and satisfying

$$Tf_i = Pe_i, \text{ for all } i = 1, 2, \dots, N.$$

We need to find the closest equal norm Parseval frame to \mathcal{F} . By our assumption, P is a projection with nearly constant diagonal. By the Projection Problem, there is a projection Q on $\ell_2(N)$ with $d(P, Q) \leq g(\varepsilon, M, N)$. It follows that $\{Qe_i\}_{i=1}^N$ is a equal norm Parseval frame. We will be done if we can find an equal norm Parseval frame $\mathcal{G} = \{g_i\}_{i=1}^N$ for \mathcal{H}_M with analysis operator T_1 satisfying:

$$T_1g_i = Qe_i, \text{ and } d(\mathcal{F}, \mathcal{G}) \approx g(\varepsilon, M, N). \tag{1}$$

So it is the problem of finding \mathcal{G} we address in this section. This problem is made more difficult by the fact that there are many frames G satisfying Equation 1 and most of them are not close to \mathcal{F} . In particular, if $\mathcal{G} = \{g_i\}_{i=1}^N$ satisfies Equation 1, and U is any unitary operator on \mathcal{H}_M , then $U(\mathcal{G}) = \{Ug_i\}_{i=1}^N$ also satisfies Equation 1. To address this problem, we will introduce the *chordal distance* between subspaces of a

Hilbert space and give a computation of this distance in terms of our distance function. Using this, we will be able to construct the required frame \mathcal{G} .

We need a result from [7] and for completeness include its proof.

THEOREM 3.1. *Let $\mathcal{F} = \{f_i\}_{i \in I}, \mathcal{G} = \{g_i\}_{i \in I}$ be Parseval frames for \mathcal{H} with analysis operators T_1, T_2 respectively. If*

$$d(\mathcal{F}, \mathcal{G}) = \sum_{i \in I} \|f_i - g_i\|^2 < \varepsilon,$$

then

$$d(T_1(\mathcal{F}), T_2(\mathcal{G})) = \sum_{i \in I} \|T_1 f_i - T_2 g_i\|^2 < 4\varepsilon.$$

Proof. Note that for all $j \in I$,

$$T_1 f_j = \sum_{i \in I} \langle f_j, f_i \rangle e_i, \text{ and } T_2 g_j = \sum_{i \in I} \langle g_j, g_i \rangle e_i.$$

Hence,

$$\begin{aligned} \|T_1 f_j - T_2 g_j\|^2 &= \sum_{i \in I} |\langle f_j, f_i \rangle - \langle g_j, g_i \rangle|^2 \\ &= \sum_{i \in I} |\langle f_j, f_i - g_i \rangle + \langle f_j - g_j, g_i \rangle|^2 \\ &\leq 2 \sum_{i \in I} |\langle f_j, f_i - g_i \rangle|^2 + 2 \sum_{i \in I} |\langle f_j - g_j, g_i \rangle|^2. \end{aligned}$$

Summing over j and using the fact that our frames \mathcal{F} and \mathcal{G} are Parseval gives

$$\begin{aligned} \sum_{j \in I} \|T_1 f_j - T_2 g_j\|^2 &\leq 2 \sum_{j \in I} \sum_{i \in I} |\langle f_j, f_i - g_i \rangle|^2 + 2 \sum_{j \in I} \sum_{i \in I} |\langle f_j - g_j, g_i \rangle|^2 \\ &= 2 \sum_{i \in I} \sum_{j \in I} |\langle f_j, f_i - g_i \rangle|^2 + 2 \sum_{j \in I} \|f_j - g_j\|^2 \\ &= 2 \sum_{i \in I} \|f_i - g_i\|^2 + 2 \sum_{j \in I} \|f_j - g_j\|^2 \\ &= 4 \sum_{j \in I} \|f_j - g_j\|^2. \quad \square \end{aligned}$$

As we noted above, $d(T_1(\mathcal{F}), T_2(\mathcal{G}))$ need not be bounded by $d(\mathcal{F}, \mathcal{G})$ in general. We now show that there is at least one choice of \mathcal{G} which gives the correct bound. For this, we need to introduce *principle angles* and the *chordal distance* between subspaces of a Hilbert space. For notation, if \mathcal{H} is a Hilbert space, denote the unit sphere by $Sp_{\mathcal{H}}$.

DEFINITION 3.2. Given M -dimensional subspaces W_1, W_2 of a Hilbert space, define the M -tuple $(\sigma_1, \sigma_2, \dots, \sigma_M)$ as follows:

$$\sigma_1 = \max\{\langle f, g \rangle : f \in Sp_{W_1}, g \in Sp_{W_2}\} = \langle f_1, g_1 \rangle.$$

For $2 \leq i \leq M$,

$$\sigma_i = \max\{\langle f, g \rangle : \|f\| = \|g\| = 1, \langle f_j, f \rangle = 0 = \langle g_j, g \rangle, \text{ for } 1 \leq j \leq i-1\},$$

where

$$\sigma_i = \langle f_i, g_i \rangle.$$

The M -tuple $(\theta_1, \theta_2, \dots, \theta_M)$ with $\theta_i = \cos^{-1}(\sigma_i)$ is called the *principle angles* between W_1, W_2 . The *chordal distance* between W_1, W_2 is given by

$$d_c^2(W_1, W_2) = \sum_{i=1}^M \sin^2 \theta_i.$$

So by the definition, there exists orthonormal bases $\{a_j\}_{j=1}^M, \{b_j\}_{j=1}^M$ for W_1, W_2 respectively satisfying

$$\|a_j - b_j\| = 2 \sin \left(\frac{\theta}{2} \right), \text{ for all } j = 1, 2, \dots, M.$$

It follows that for $0 \leq \theta \leq \frac{\pi}{2}$,

$$\sin^2 \theta \leq 4 \sin^2 \left(\frac{\theta}{2} \right) = \|a_j - b_j\|^2 \leq 4 \sin^2 \theta, \text{ for all } j = 1, 2, \dots, M.$$

Hence,

$$d_c^2(W_1, W_2) \leq \sum_{j=1}^M \|a_j - b_j\|^2 \leq 4d_c^2(W_1, W_2). \tag{2}$$

We also need the following result [16].

LEMMA 3.3. *If \mathcal{H}_N is an N -dimensional Hilbert space and P, Q are rank M orthogonal projections onto subspaces W_1, W_2 respectively, then the chordal distance $d_c(W_1, W_2)$ between the subspaces satisfies*

$$d_c^2(W_1, W_2) = M - \text{Tr } PQ.$$

Next we give the precise connection between chordal distance for subspaces and the distance between the projections onto these subspaces. This result can be found in [16] in the language of Hilbert-Schmidt norms. We give our own proof for the sake of completeness.

PROPOSITION 3.4. *Let \mathcal{H}_N be an N -dimensional Hilbert space with orthonormal basis $\{e_i\}_{i=1}^N$. Let P, Q be the orthogonal projections of \mathcal{H}_N onto M -dimensional subspaces W_1, W_2 respectively. Then the chordal distance between W_1, W_2 satisfies*

$$d_c^2(W_1, W_2) = \frac{1}{2} \sum_{i=1}^N \|Pe_i - Qe_i\|^2.$$

In particular, there are orthonormal bases $\{g_i\}_{i=1}^M$ for W_1 and $\{h_i\}_{i=1}^M$ for W_2 satisfying

$$\frac{1}{2} \sum_{i=1}^N \|Pe_i - Qe_i\|^2 \leq \sum_{i=1}^M \|g_i - h_i\|^2 \leq 2 \sum_{i=1}^N \|Pe_i - Qe_i\|^2.$$

Proof. We compute:

$$\begin{aligned} \sum_{i=1}^N \|Pe_i - Qe_i\|^2 &= \sum_{i=1}^N \langle Pe_i - Qe_i, Pe_i - Qe_i \rangle \\ &= \sum_{i=1}^N \|Pe_i\|^2 + \sum_{i=1}^N \|Qe_i\|^2 - 2 \sum_{i=1}^N \langle Pe_i, Qe_i \rangle \\ &= 2M - 2 \sum_{i=1}^N \langle QPe_i, e_i \rangle \\ &= 2M - 2 \operatorname{Tr} QP \\ &= 2M - 2[M - d_c^2(W_1, W_2)] \\ &= 2d_c^2(W_1, W_2). \end{aligned}$$

This combined with Equation 2 completes the proof. \square

Now we are ready to answer the second problem we need to address in this section.

THEOREM 3.5. *Let P, Q be projections of rank M on \mathcal{H}_N and let $\{e_i\}_{i=1}^N$ be the natural orthonormal basis of \mathcal{H}_N . Further assume that there is a Parseval frame $\{f_i\}_{i=1}^N$ for \mathcal{H}_M with analysis operator T satisfying $Tf_i = Pe_i$, for all $i = 1, 2, \dots, N$. If*

$$\sum_{i=1}^M \|Pe_i - Qe_i\|^2 < \varepsilon,$$

then there is a Parseval frame $\{g_i\}_{i=1}^N$ for \mathcal{H}_M with analysis operator T_1 satisfying

$$T_1 g_i = Qe_i, \text{ for all } i = 1, 2, \dots, N,$$

and

$$\sum_{i=1}^N \|f_i - g_i\|^2 < 2\varepsilon.$$

Moreover, if $\{Qe_i\}_{i=1}^N$ is equal norm, then $\{g_i\}_{i=1}^N$ may be chosen to be equal norm.

Proof. By Proposition 3.4, there are orthonormal bases $\{a_j\}_{j=1}^M$ and $\{b_j\}_{j=1}^M$ for W_1, W_2 respectively satisfying

$$\sum_{j=1}^M \|a_j - b_j\|^2 < 2\varepsilon.$$

Let A, B be the $N \times M$ matrices whose j^{th} columns are a_j, b_j respectively. Let a_{ij}, b_{ij} be the (i, j) entry of A, B respectively. Finally, let $\{f'_i\}_{i=1}^N, \{g'_i\}_{i=1}^N$ be the i^{th} rows of A, B respectively. Then we have

$$\begin{aligned} \sum_{i=1}^N \|f'_i - g'_i\|^2 &= \sum_{i=1}^N \sum_{j=1}^M |a_{ij} - b_{ij}|^2 \\ &= \sum_{j=1}^M \sum_{i=1}^N |a_{ij} - b_{ij}|^2 \\ &= \sum_{i=1}^M \|a_j - b_j\|^2 \\ &\leq 2\varepsilon. \end{aligned}$$

Since the columns of A form an orthonormal basis for W_1 , we know that $\{f'_i\}_{i=1}^N$ is a Parseval frame which is isomorphic to $\{f_i\}_{i=1}^N$. Thus there is a unitary operator $U : \mathcal{H}_M \rightarrow \mathcal{H}_M$ with $Uf'_i = f_i$. Now let $\{g_i\}_{i=1}^N = \{Ug'_i\}_{i=1}^N$. Then

$$\sum_{i=1}^N \|f_i - Ug'_i\|^2 = \sum_{i=1}^N \|U(f'_i) - U(g'_i)\|^2 = \sum_{i=1}^N \|f'_i - g'_i\|^2 \leq 2\varepsilon.$$

Finally, if T_1 is the analysis operator for the Parseval frame $\{g_i\}_{i=1}^N$, then T_1 is an isometry and since $\{T_1(g_i)\}_{i=1}^N = \{Qe_i\}_{i=1}^N$, for all $i = 1, 2, \dots, N$, if Qe_i is equal norm, so is $T_1(g_i)$ and hence so is $\{g_i\}_{i=1}^N$. \square

4. The equivalence of our problems

Now we can show that the Paulsen Problem and the Projection Problem are equivalent in the sense that their functions $h(\varepsilon, M, N), g(\varepsilon, M, N)$, respectively, are within a factor of four of one another.

THEOREM 4.1. *If $h(\varepsilon, M, N)$ is the function for the Paulsen Problem and $g(\varepsilon, M, N)$ is the function for the Projection Problem, then*

$$h(\varepsilon, M, N) \leq 4g(\varepsilon, M, N) \leq 8h(\varepsilon, M, N).$$

Proof. First, assume that Problem 1.2 holds with function $g(\varepsilon, M, N)$. Let $\{f_i\}_{i=1}^N$ be a Parseval frame for \mathcal{H}_M satisfying

$$(1 - \varepsilon) \frac{M}{N} \leq \|f_i\|^2 \leq (1 + \varepsilon) \frac{M}{N}.$$

Let T be the analysis operator of $\{f_i\}_{i=1}^N$ and let P be the projection of \mathcal{H}_N onto range T . So, $Tf_i = Pe_i$, for all $i = 1, 2, \dots, N$. By our assumption that that Problem 1.2 holds, there is a projection Q on \mathcal{H}_N with constant diagonal so that

$$\sum_{i=1}^N \|Pe_i - Qe_i\|^2 \leq g(\varepsilon, M, N).$$

By Theorem 3.5, there is a Parseval frame $\{g_i\}_{i=1}^N$ for \mathcal{H}_M with analysis operator T_1 so that $T_1 g_i = Q e_i$ and

$$\sum_{i=1}^N \|f_i - g_i\|^2 \leq 2g(\varepsilon, M, N).$$

Since T_1 is a co-isometry and $\{T_1 g_i\}_{i=1}^N$ is equal norm, it follows that $\{g_i\}_{i=1}^N$ is an equal norm Parseval frame satisfying the Paulsen Problem.

Conversely, assume the Parseval Paulsen Problem has a positive solution with function $h(\varepsilon, M, N)$. Let P be an orthogonal projection on \mathcal{H}_N satisfying

$$(1 - \varepsilon) \frac{M}{N} \leq \|P e_i\|^2 \leq (1 + \varepsilon) \frac{M}{N}.$$

Then $\{P e_i\}_{i=1}^N$ is a Parseval frame for \mathcal{H}_M and by the Parseval Paulsen problem, there is an equal norm Parseval frame $\{g_i\}_{i=1}^N$ so that

$$\sum_{i=1}^N \|f_i - g_i\|^2 < h(\varepsilon, M, N).$$

Let T_1 be the analysis operator of $\{g_i\}_{i=1}^N$. Letting Q be the projection onto the range of T_1 , we have that $Q e_i = T_1 g_i$, for all $i = 1, 2, \dots, N$. By Theorem 3.1, we have that

$$\sum_{i=1}^N \|P e_i - T_1 g_i\|^2 = \sum_{i=1}^N \|P e_i - Q e_i\|^2 \leq 4h(\varepsilon, M, N).$$

Since T_1 is a co-isometry and $\{g_i\}_{i=1}^N$ is equal norm, it follows that Q is a constant diagonal projection. \square

5. Generalizations of the Paulsen Problem

In this section we will look at some recent generalizations of the Paulsen Problem.

DEFINITION 5.1. We say a sequence of numbers $\{a_i\}_{i=1}^N$ is a *Parseval admissible sequence* for \mathcal{H}_M if there is a Parseval frame $\{f_i\}_{i=1}^N$ for \mathcal{H}_M satisfying $\|f_i\|^2 = a_i^2$, for all $i = 1, 2, \dots, N$.

The following classification of Parseval admissible sequences can be found in [10].

THEOREM 5.2. A sequence of numbers $\{a_i\}_{i=1}^N$ is a *Parseval admissible sequence* for \mathcal{H}_M if and only if both of the following hold:

1. $\sum_{i=1}^N a_i^2 = M$.
- (2) $a_i \leq 1$, for every $i = 1, 2, \dots, N$.

Now we give a generalization of the Paulsen Problem.

PROBLEM 5.3. (Generalized Paulsen Problem) Let $\{a_i\}_{i=1}^N$ be a Parseval admissible sequence for \mathcal{H}_M . If $\{f_i\}_{i=1}^N$ is a Parseval frame for \mathcal{H}_M satisfying

$$(1 - \varepsilon)a_i^2 \leq \|f_i\|^2 \leq (1 + \varepsilon)a_i^2,$$

find the closest Parseval frame $\{g_i\}_{i=1}^N$ satisfying: $\|g_i\| = a_i$, for all $i = 1, 2, \dots, N$.

The work in this paper can be re-done to show that the Generalized Paulsen Problem is equivalent to a Generalized Projection Problem.

PROBLEM 5.4. (Generalized Projection Problem) If P is a rank M orthogonal projection on $\ell_2(N)$, $\{a_i\}_{i=1}^N$ satisfies Theorem 5.2 and

$$(1 - \varepsilon)a_i^2 \leq \|Pe_i\|^2 \leq (1 + \varepsilon)a_i^2,$$

find the closest projection Q to P satisfying $\|Qe_i\| = a_i$, for all $i = 1, 2, \dots, N$.

We end with a further generalization of the Paulsen Problem to frame operators.

DEFINITION 5.5. If S is a positive, self-adjoint invertible operator on \mathcal{H}_M , we say that a sequence of numbers $\{a_i\}_{i=1}^N$ is an S -admissible sequence if there exists a frame $\{f_i\}_{i=1}^N$ for \mathcal{H}_M having S as its frame operator and so that $\|f_i\|^2 = a_i^2$, for all $i = 1, 2, \dots, N$.

The classification of S -admissible sequences goes back to Horn and Johnson [20]. The simplest proof of this result is due Casazza and Leon [14].

THEOREM 5.6. Let S be a positive self-adjoint operator on a N -dimensional Hilbert space H_N . Let $\lambda_1 \geq \lambda_2 \geq \dots \lambda_N > 0$ be the eigenvalues of S . Fix $M \geq N$ and real numbers $a_1 \geq a_2 \geq \dots \geq a_M > 0$. The following are equivalent:

- (1) There is a frame $\{\varphi_j\}_{j=1}^M$ for H_N with frame operator S and $\|\varphi_j\| = a_j$, for all $j = 1, 2, \dots, M$.
- (2) For every $1 \leq k \leq N$,

$$\sum_{i=1}^k a_i^2 \leq \sum_{i=1}^k \lambda_i, \quad \text{and} \quad \sum_{i=1}^M a_i^2 = \sum_{i=1}^N \lambda_i. \tag{3}$$

Our final generalization of the Paulsen Problem is:

PROBLEM 5.7. If S is a positive, self-adjoint invertible operator on \mathcal{H}_M , $\{a_i\}_{i=1}^N$ is an S -admissible sequence, and $\{f_i\}_{i=1}^N$ is a frame with frame operator S and satisfying

$$(1 - \varepsilon)a_i \leq \|f_i\|^2 \leq (1 + \varepsilon)a_i,$$

then find the closest frame $\{g_i\}_{i=1}^N$ with frame operator S so that $\|g_i\|^2 = a_i$, for all $i = 1, 2, \dots, N$.

6. The Paulsen Problem and Naimark complements

In this section we will use Naimark complements to show that we only need to solve the Paulsen Problem for $N \leq 2M$. If $\{f_i\}_{i=1}^N$ is a Parseval frame for \mathcal{H}_M with analysis operator T which is an isometry and satisfies

$$Tf_i = Pe_i, \text{ for all } i = 1, 2, \dots, N,$$

where $\{e_i\}_{i=1}^N$ is the natural orthonormal basis for $\ell_2(N)$ and P is the orthogonal projection of $\ell_2(N)$ onto $T(\mathcal{H}_M)$, the Naimark complement of $\{f_i\}_{i=1}^M$ is the Parseval frame $\{(I - P)e_i\}_{i=1}^N$ for \mathcal{H}_{N-M} . Now we will compare the Paulsen function for a Parseval frame to the Paulsen function for its Naimark complement.

THEOREM 6.1. *If $h(\varepsilon, M, N)$ is the Paulsen constant then*

$$h(\varepsilon, M, N) \leq 8g\left(\varepsilon \frac{M}{N - M}, N - M, N\right).$$

Proof. Assume that $\mathcal{F} = \{f_i\}_{i=1}^N$ is a ε -nearly equal norm Parseval frame for \mathcal{H}_N with analysis operator T which is a co-isometry. Then there is a projection P on $\ell_2(N)$ so that $Pe_i = Tf_i$, for all $i = 1, 2, \dots, N$. It follows that $\{(I - P)e_i\}_{i=1}^N$ is a Parseval frame and

$$\begin{aligned} \|(I - P)e_i\|^2 &= 1 - \|Pe_i\|^2 \\ &\leq 1 - (1 - \varepsilon) \frac{M}{N} \\ &= \left(1 + \varepsilon \frac{M}{N - M}\right) \left(1 - \frac{M}{N}\right). \end{aligned}$$

Similarly,

$$\|(I - P)e_i\|^2 \geq \left(1 - \varepsilon \frac{M}{N - M}\right) \left(1 - \frac{M}{N}\right).$$

Choose a Parseval frame $\{g_i\}_{i=1}^N$ for \mathcal{H}_{N-M} with analysis operator T_1 satisfying $T_1g_i = (I - P)e_i$. Since T_1 is an isometry, it follows that $\mathcal{G} = \{g_i\}_{i=1}^N$ is a $\varepsilon \frac{M}{N - M}$ -nearly equal norm Parseval frame. Hence, there is an equal norm Parseval frame $\mathcal{H} = \{h_i\}_{i=1}^N$ for \mathcal{H}_{N-M} with

$$d(\mathcal{G}, \mathcal{H}) \leq h\left(\varepsilon \frac{M}{N - M}, N - M, N\right),$$

where h is the Paulsen function for N vectors in \mathcal{H}_{N-M} . Let T_2 be the analysis operator for \mathcal{H} . Applying Theorem 3.1, we have that

$$d(T_1(\mathcal{G}), T_2(\mathcal{H})) \leq 4h\left(\varepsilon \frac{M}{N - M}, N - M, N\right).$$

Let $I - Q$ be the orthogonal projection onto $T_2(\mathcal{H})$. Now we check

$$\begin{aligned} d(\{Pe_i\}_{i=1}^N, \{Qe_i\}_{i=1}^N) &= \sum_{i=1}^N \|Pe_i - Qe_i\|^2 \\ &= \sum_{i=1}^N \|(I - P)e_i - (I - Q)e_i\|^2 \\ &= d(T_1(\mathcal{G}), T_2(\mathcal{H})) \\ &\leq 4h\left(\varepsilon \frac{M}{N - M}, N - M, N\right). \end{aligned}$$

By Theorem 3.5, we can choose a equal norm Parseval frame $\mathcal{K} = \{k_i\}_{i=1}^N$ for \mathcal{H}_{N-M} with analysis operator T_3 satisfying $T_3k_i = Qe_i$, for all $i = 1, 2, \dots, N$ and

$$d(\mathcal{F}, \mathcal{K}) \leq 8h\left(\varepsilon \frac{M}{N - M}, N - M, N\right). \quad \square$$

Given $N \geq M$, then either $N \leq 2M$ or $N \leq 2(N - M)$. So we have

COROLLARY 6.2. *To solve the Paulsen Problem, it suffices to solve it for ε -nearly equal norm Parseval frames $\{f_i\}_{i=1}^N$ for \mathcal{H}_M with $N \leq 2M$.*

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REFERENCES

- [1] M. ARGERAMI AND P. MASSEY, *Towards the Carpenter's theorem*, Proc. Amer. Math. Soc. **137** (2009), 3679–3687.
- [2] J. ANTEZANA, P. MASSEY, M. RUIZ, D. STOJANOFF, *The Schur-Horn theorem for operators and frames with prescribed norms and frame operator*, Illinois J. Math. **51** (2007), 537–560.
- [3] M. ARGERAMI, P. MASSEY, *A Schur-Horn theorem in Π_1 factors*, Indiana Univ. Math. J. **56** (2007), 2051–2059.
- [4] W. ARVESON, *Diagonals of normal operators with finite spectrum*, Proc. Natl. Acad. Sci. USA **104** (2007), 1152–1158.
- [5] W. ARVESON, R. KADISON, *Diagonals of self-adjoint operators*, Operator theory, operator algebras, and applications, 247–263, Contemp. Math., **414**, Amer. Math. Soc., Providence, RI, 2006.
- [6] R. BALAN, *Equivalence relations and distances between Hilbert frames*, Proc. Amer. Math. Soc. **127**, 8 (1999), 2353–2366.
- [7] B. BODMANN AND P. G. CASAZZA, *The road to equal norm Parseval frames*, Journal of Functional Analysis **258**, 2 (2010), 397–420.
- [8] M. BOWNIK, J. JASPER, *Characterization of sequences of frame norms*, J. Reine Angew. Math., to appear.
- [9] P. G. CASAZZA, M. FICKUS AND D. MIXON, *Auto-tuning unit norm tight frames*, Appl. and Comp. Harmonic Anal. **32** (2012), 1–15.
- [10] P. G. CASAZZA, M. FICKUS, M. LEON, J. KOVAČEVIĆ, J. C. TREMAIN, *A physical interpretation for finite tight frames*, Appl. and Comp. Harmonic Anal. (2006), 51–78.
- [11] P. G. CASAZZA AND J. J. KOVAČEVIĆ, *Uniform tight frames with erasures*, Advances in Computational Mathematics **18**, 2–4 (2003), 93–116.

- [12] P. G. CASAZZA AND G. KUTYNIOK, *A generalization of Gram-Schmidt orthogonalization generating all Parseval frames*, *Advances in Computational math.* **18** (2007), 65–78.
- [13] P. G. CASAZZA AND M. LEON, *Existence and construction of finite tight frames*, *J. Concr. Appl. Math.* **4**, 3 (2006), 277–289.
- [14] P. G. CASAZZA AND M. LEON, *Existence and construction of finite frames with a given frame operator*, *International Journal of Pure and Applied Mathematics* **63**, 2 (2010), 149–158.
- [15] O. CHRISTENSEN, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston (2003).
- [16] J. H. CONWAY, R. H. HARDIN, N. J. A. SLOANE, *Packing lines, planes, etc.: packings in Grassmannian spaces*, *Experiment. Math.* **5**, 2 (1996), 139–159.
- [17] I. C. GOHBERG AND A. S. MARKUS, *Some relations between eigenvalues and matrix elements of linear operators*, *Mat. Sb. (N.S.)* **64** (1964), 481–496.
- [18] D. HAN AND D. LARSON, *Frames, Bases and Group Representations*, *Mem. Amer. Math Soc.* **147** (2000).
- [19] A. HORN, *Doubly stochastic matrices and the diagonal of a rotation matrix*, *Amer. J. Math.* **76** (1954), 620–630.
- [20] R. A. HORN AND C. R. JOHNSON, *Matrix analysis*, Cambridge University Press, 1985.
- [21] A. J. E. M. JANSSEN, *Zak transforms with few zeroes and the tie*, In *Advances in Gabor Analysis*, H.G. Feichtinger and T. Strohmer (eds.), pages 31–70, Birkhäuser, Boston (2002).
- [22] J. JASPER, *The Schur-Horn theorem for operators with three point spectrum*, Preprint.
- [23] R. KADISON, *The Pythagorean theorem. I. The finite case*, *Proc. Natl. Acad. Sci. USA* **99** (2002), 4178–4184.
- [24] R. KADISON, *The Pythagorean theorem. II. The infinite discrete case*, *Proc. Natl. Acad. Sci. USA* **99** (2002), 5217–5222.
- [25] K. KORNELSON, D. LARSON, *Rank-one decomposition of operators and construction of frames*, *Wavelets, frames and operator theory*, 203–214, *Contemp. Math.*, **345**, Amer. Math. Soc., Providence, RI, 2004.
- [26] V. KAFTAL AND G. WEISS, *An infinite dimensional Schur-Horn theorem and majorization theory*, *J. Funct. Anal.*, to appear.
- [27] A. NEUMANN, *An infinite-dimensional version of the Schur-Horn convexity theorem*, *J. Funct. Anal.* **161** (1999), 418–451.
- [28] I. SCHUR, *Über eine Klasse von Mittelbildungen mit Anwendungen auf die Determinantentheorie*, *Sitzungsber. Berl. Math. Ges.* **22** (1923), 9–20.

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