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THE SIMILARITY DEGREE OF APPROXIMATELY DIVISIBLE C*–ALGEBRAS

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Abstract. Let $\mathscr{A}$ be a unital approximately divisible C*-algebra. We show that the similarity degree of $\mathscr{A}$ is at most 5.

1. Introduction

In 1955, R. Kadison [6] formulated the following conjecture: If $\mathscr{A}$ is a unital C*-algebra and $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ ($\mathscr{H}$ is a Hilbert space) is a unital bounded homomorphism, then π is similar to a *-homomorphism, that is, there exists an invertible operator $S \in \mathscr{B}(\mathscr{H})$ such that $S^{-1}\pi(\cdot)S$ is a *-homomorphism.

This conjecture remains unproved, although many partial results are known. U. Haagerup [5] proved that π is similar to a *-homomorphism if and only if it is completely bounded. Moreover,

$$\|\pi\|_{cb} = \inf\{\|S\| \cdot \|S^{-1}\|\}$$

where the infimum runs over all invertible S such that $S^{-1}\pi(\cdot)S$ is a *-homomorphism. By definition, $\|\pi\|_{cb} = \sup_{n\geqslant 1} \|\pi_n\|$ where $\pi_n : \mathscr{M}_n(\mathscr{A}) \to \mathscr{M}_n(\mathscr{B}(\mathscr{H}))$ is the mapping taking n by n matrix $[a_{ij}]_{n\times n}$ to matrix $[\pi(a_{ij})]_{n\times n}$. U. Haagerup [5] also proved that π is similar to a *-homomorphism whenever π is finitely cyclic, i.e., there are vectors $e_1,\ldots,e_n \in \mathscr{H}$ such that $\pi(\mathscr{A})e_1 + \ldots + \pi(\mathscr{A})e_n$ is dense in $\mathscr{H}$.

G. Pisier [8] proved that if a unital C*-algebra $\mathscr{A}$ verifies Kadison's conjecture, then there is a number α for which there exists a constant K so that any bounded homomorphism $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ satisfies $\|\pi\|_{cb} \leqslant K\|\pi\|^{\alpha}$. Moreover, the smallest number α with the property is an integer denoted by $d(\mathscr{A})$ and called *the similarity degree*. It is clear that a C*-algebra $\mathscr{A}$ verifies Kadison's conjecture if and only if $d(\mathscr{A}) < \infty$.

REMARK 1.1. When determining $d(\mathscr{A})$, it is only necessary to consider unital bounded homomorphisms that are one-to-one. To see this, let π_0 be a unital *-isomorphism from $\mathscr{A}$ to $\mathscr{B}(\mathscr{K})$ for some Hilbert space $\mathscr{K}$. It is not difficult to see that $\pi \oplus \pi_0$ is one-to-one and $\|\pi \oplus \pi_0\| = \|\pi\|$ and $\|\pi \oplus \pi_0\|_{cb} = \|\pi\|_{cb}$.

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There are few concrete examples of C*-algebras known to verify Kadison's conjecture. We list them below, together with their respective degrees:

(1) $\mathscr{A}$ is nuclear if and only if $d(\mathscr{A}) = 2$ ([2], [3], [11]);

(2) if $\mathscr{A} = \mathscr{B}(\mathscr{H})$, then $d(\mathscr{A}) = 3$ ([10]);

(3) $d(\mathscr{A} \otimes \mathscr{K}(\mathscr{H})) \leqslant 3$ for any C^*-algebra $\mathscr{A}$ ([5], [9]);

(4) if $\mathscr{M}$ is a factor of type II_1 with property Γ, then $d(\mathscr{M}) = 3$ ([4]);

(5) if $\mathscr{A}$ is nuclear and contains unital matrix algebras of any order, then $d(\mathscr{A} \otimes \mathscr{B}) \leqslant 5$ for any unital C*-algebra $\mathscr{B}$ ([12]).

The class of approximately divisible C*-algebras was introduced by B. Blackadar, A. Kumjian and M. Rørdam [1], where they constructed a large class of simple C*-algebras having trivial non-stable K-theory. They showed that the class of approximately divisible C*-algebras contains all simple unital AF-algebras and most of the simple unital AH-algebras with real rank 0, including every nonrational noncommutative torus.

In this paper, we show that the similarity degree of every unital approximately divisible C*-algebra is at most 5.

2. Notation and preliminaries

Let $\mathscr{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$ and $\mathscr{M}_k(\mathbb{C})^n$ be the direct sum of n copies of $\mathscr{M}_k(\mathbb{C})$. Suppose $\mathscr{B}$ is a finite-dimensional C*-algebra. Then there exist r positive integers $k_1, \ldots, k_r$ such that

$$\mathscr{B} \cong \mathscr{M}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathscr{M}_{k_r}(\mathbb{C}).$$

Define the *subrank* of $\mathscr{B}$ to be

$$\mathrm{SubRank}(\mathscr{B}) = \min\{k_1, \ldots, k_r\}.$$

Let $\{e_{ij}^{(s)} : 1 \leqslant i, j \leqslant k_s, 1 \leqslant s \leqslant r\}$ be a set of matrix units for $\mathscr{B}$. That is,

$$(e_{ij}^{(s)})^* = e_{ji}^{(s)}, \qquad \sum_{1 \leqslant s \leqslant r} \sum_{1 \leqslant i \leqslant k_s} e_{ii}^{(s)} = I$$

and

$$e_{ij}^{(s)} e_{i_1,j_1}^{(s_1)} = \begin{cases} e_{ij_1}^{(s)}, & \text{if } s = s_1, j = i_1 \\ 0, & \text{otherwise} \end{cases}.$$

We define approximately divisibility for (nonseparable) C*-algebras by removing the separability assumption in the definition 1.2 in [1].

DEFINITION 2.1. A unital C*-algebra $\mathscr{A}$ with unit $I_{\mathscr{A}}$ is approximately divisible if, for every $x_1, \ldots, x_n \in \mathscr{A}$ and $\varepsilon > 0$, there is a finite-dimensional C*-subalgebra $\mathscr{B}$ of $\mathscr{A}$ such that

(1) $I_{\mathscr{A}} \in \mathscr{B}$,

(2) $\mathrm{SubRank}(\mathscr{B}) \geqslant 2$,

(3) $\|x_i y - y x_i\| < \varepsilon$ for $i = 1, \ldots, n$ and all $y \in \mathscr{B}$ with $\|y\| \leqslant 1$.

The following proposition is taken from Theorem 1.3 and Corollary 2.10 in [1].

PROPOSITION 2.2. ([1]) Let $\mathscr{A}$ be a unital separable approximately divisible C*-algebra with the unit $I_{\mathscr{A}}$. Then there exists an increasing sequence $\{\mathscr{A}_m\}_{m=1}^{\infty}$ of subalgebras of $\mathscr{A}$ such that

(1) $\mathscr{A} = \overline{\cup_m \mathscr{A}_m}$,

(2) for any positive integer m, $\mathscr{A}_m' \cap \mathscr{A}_{m+1}$ contains a finite-dimensional C*-subalgebra $\mathscr{B}$ with $I_{\mathscr{A}} \in \mathscr{B}$ and $\mathrm{SubRank}(\mathscr{B}) \geqslant 2$,

(3) for any positive integers m and k, there is a finite-dimensional C*-subalgebra $\mathscr{B}$ of $\mathscr{A}_m' \cap \mathscr{A}$ with $I_{\mathscr{A}} \in \mathscr{B}$ and $\mathrm{SubRank}(\mathscr{B}) \geqslant k$.

3. Similarity degree

We will show that the similarity degree of every unital approximately divisible C^*-algebra is at most 5. To do that, we need the following lemmas.

LEMMA 3.1. *Let $\mathscr{A}$ be a C*-algebra with unit $I_{\mathscr{A}}$, $\mathscr{A}_0$ and $\mathscr{B}$ be commuting C*-subalgebras of $\mathscr{A}$ that contain $I_{\mathscr{A}}$. Suppose $\mathscr{B} \cong \mathscr{M}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathscr{M}_{k_r}(\mathbb{C})$ with $k_1, \ldots, k_r \geqslant n \geqslant 2$ for some positive integer n, and $\{e_{ij}^{(s)} : 1 \leqslant i,j \leqslant k_s, 1 \leqslant s \leqslant r\}$ is a set of matrix units for $\mathscr{B}$. If $\{a_{ij} : 1 \leqslant i,j \leqslant n\} \subseteq \mathscr{A}_0$, then*

$$\| \sum_{1\leqslant s\leqslant r} \sum_{1\leqslant i,j\leqslant n} a_{ij} e_{ij}^{(s)} \| = \| [a_{ij}]_{n\times n} \|.$$

Proof. Let $p_1 = I_{k_1} \oplus \cdots 0 \oplus 0, \ldots, p_r = 0 \oplus \cdots \oplus 0 \oplus I_{k_r}$ be the projections in $\mathscr{B}$, where I_{k_s} is the unit of $\mathscr{M}_{k_s}(\mathbb{C})$ $(1 \leqslant s \leqslant r)$. Then it is clear that $p_1 + \cdots + p_r = I_{\mathscr{A}}$ and for any $1 \leqslant s \leqslant r$, $1 \leqslant i,j \leqslant k_s$, $e_{ij}^{(s)} = p_s e_{ij}^{(s)}$.

Define

$$\pi : \mathscr{M}_{k_1}(p_1\mathscr{A}_0) \oplus \cdots \oplus \mathscr{M}_{k_r}(p_r\mathscr{A}_0) \to C^*(\mathscr{A}_0, \mathscr{B})$$

by

$$\pi([p_1 a_{ij}^{(1)}]_{k_1\times k_1} \oplus \cdots \oplus [p_r a_{ij}^{(r)}]_{k_r\times k_r}) = \sum_{s=1}^{r} \sum_{i,j=1}^{k_s} a_{ij}^{(s)} e_{ij}^{(s)},$$

for any $a_{ij}^{(s)} \in \mathscr{A}_0$. It is clear that π is a *-isomorphism.

Thus, in $\mathscr{M}_n(\mathscr{A})$,

$$\begin{aligned} \|[a_{ij}]_{n\times n}\| &= \| \sum_{s=1}^{r} \begin{pmatrix} p_s & & 0 \\ & \ddots & \\ 0 & & p_s \end{pmatrix} [a_{ij}]_{n\times n} \| \\ &= \max\{\|[p_s a_{ij}]_{n\times n}\| : 1 \leqslant s \leqslant n\}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
&\Big\| \sum_{1\leqslant s\leqslant r} \sum_{1\leqslant i,j\leqslant n} a_{ij} e_{ij}^{(s)} \Big\| \\
&= \Big\| \pi\left(\begin{pmatrix} [p_1 a_{ij}]_{n\times n} & 0 \\ 0 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} [p_r a_{ij}]_{n\times n} & 0 \\ 0 & 0 \end{pmatrix} \right) \Big\| \\
&= \max\{ \|[p_s a_{ij}]_{n\times n}\| : 1 \leqslant s \leqslant r \}. \quad \square
\end{aligned}$$

LEMMA 3.2. *Suppose $\mathscr{A}$ is a unital approximately divisible C*-algebra and $E \subseteq \mathscr{A}$ is countable. Then there is a unital separable approximately divisible C*-subalgebra $\mathscr{D}$ of $\mathscr{A}$ such that $E \subseteq \mathscr{D}$.*

Proof. Suppose $\mathscr{W}$ is a unital separable C*-subalgebra of $\mathscr{A}$. Choose a countable dense subset S of $\mathscr{W}$ and let Λ be the (countable) set of all pairs (F,k) with $F \subseteq S$ finite and $k \in \mathbb{N}$. It follows from the approximate divisibility of $\mathscr{A}$ that there is a finite-dimensional unital C*-subalgebra $\mathscr{B}_\lambda$ of $\mathscr{A}$ such that

$$\|xy - yx\| < 1/k$$

for all $x \in F$ and $y \in \mathscr{B}_\lambda$ with $\|y\| \leqslant 1$. We define $\widehat{\mathscr{W}} = C^*(\mathscr{W} \cup \cup_{\lambda\in\Lambda} \mathscr{B}_\lambda)$. If we define $\mathscr{D}_0 = C^*(E)$ and, for each positive integer n, we define $\mathscr{D}_{n+1} = \widehat{\mathscr{D}_n}$, then it is clear that $\mathscr{D} = [\cup_{n=1}^{\infty} \mathscr{D}_n]^-$ is the required separable approximately divisible C*-subalgebra. $\square$

THEOREM 3.3. *If $\mathscr{A}$ is a unital approximately divisible C*-algebra, then*

$$d(\mathscr{A}) \leqslant 5.$$

Proof. Note that given a bounded homomorphism π, there is a countable subset E of $\mathscr{A}$ that determines both $\|\pi\|$ and $\|\pi\|_{cb}$. By Lemma 3.2, there is a unital separable approximately divisible C*-subalgebra $\mathscr{D}$ of $\mathscr{A}$ such that $E \subseteq \mathscr{D}$. Therefore, without loss of generality, we can assume that $\mathscr{A}$ is separable.

Let $\mathscr{A} = \overline{\cup_m \mathscr{A}_m}$ with $\mathscr{A}_m$ defined in Proposition 2.2. By Remark 1.1, let $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ be a one-to-one unital bounded homomorphism, where $\mathscr{H}$ is a Hilbert space. It is sufficient to prove that

$$\|\pi|_{\cup_m \mathscr{A}_m}\|_{cb} \leqslant K \|\pi\|^5$$

for some constant K.

For any positive integer n, let $\{a_{ij} : 1 \leqslant i,j \leqslant n\}$ be a family of elements in $\cup_m \mathscr{A}_m$. Then there exists some positive integer m_0 such that $\{a_{ij} : 1 \leqslant i,j \leqslant n\}$ is in $\mathscr{A}_{m_0}$. From Proposition 2.2, there exists a finite-dimensional C*-subalgebra $\mathscr{B}$ containing the unit of $\mathscr{A}$ with $\mathrm{SubRank}(\mathscr{B}) \geqslant n$ and $\mathscr{B} \subset \mathscr{A}'_{m_0} \cap \mathscr{A}$. Let $\{e_{ij}^{(s)} : 1 \leqslant i,j \leqslant k_s, 1 \leqslant s \leqslant r\}$ be a set of matrix units for $\mathscr{B}$.

Since $\mathscr{B}$ is finite-dimensional, it follows that $\mathscr{B}$ is nuclear. Therefore, from [5], there exists an invertible operator S in $\mathscr{B}(\mathscr{H})$, such that $\|S\| \cdot \|S^{-1}\| \leqslant C\|\pi\|^2$ for some constant C, and $S^{-1}\pi|_{\mathscr{B}}(\cdot)S$ is a *-isomorphism. Let $\rho = S^{-1}\pi(\cdot)S$. Then $\{\rho(e_{ij}^{(s)}) : 1 \leqslant i, j \leqslant k_s, 1 \leqslant s \leqslant r\}$ is a set of matrix units for the C*-algebra $\rho(\mathscr{B})$. Hence, by Lemma 3.1,

$$\|\rho(\sum_{1\leqslant s\leqslant r}\sum_{1\leqslant i,j\leqslant n} a_{ij}e_{ij}^{(s)})\| \leqslant \|\rho\| \cdot \|\sum_{1\leqslant s\leqslant r}\sum_{1\leqslant i,j\leqslant n} a_{ij}e_{ij}^{(s)}\| = \|\rho\| \cdot \|[a_{ij}]_{n\times n}\|.$$

On the other hand, by Lemma 3.1,

$$\begin{aligned}
&\|\rho(\sum_{1\leqslant s\leqslant r}\sum_{1\leqslant i,j\leqslant n} a_{ij}e_{ij}^{(s)})\| \\
&= \|\sum_{s=1}^{r}\sum_{1\leqslant i,j\leqslant n} \rho(a_{ij})\rho(e_{ij}^{(s)})\| \\
&= \|[\rho(a_{ij})]_{n\times n}\|.
\end{aligned}$$

Therefore we get

$$\|[\rho(a_{ij})]_{n\times n}\| \leqslant \|\rho\| \cdot \|[a_{ij}]_{n\times n}\| \leqslant \|S\| \cdot \|S^{-1}\| \cdot \|\pi\| \cdot \|[a_{ij}]_{n\times n}\| \leqslant C\|\pi\|^3\|[a_{ij}]_{n\times n}\|.$$

That means that $\|\rho|_{\cup_m \mathscr{A}_m}\|_{cb} \leqslant C\|\pi\|^3$, then

$$\|\pi|_{\cup_m \mathscr{A}_m}\|_{cb} = \|S\rho|_{\cup_m \mathscr{A}_m} S^{-1}\|_{cb} \leqslant \|S^{-1}\| \cdot \|S\| \cdot \|\rho|_{\cup_m \mathscr{A}_m}\|_{cb} \leqslant C^2\|\pi\|^5. \quad \square$$

F. Pop [12] proved that if $\mathscr{A}$ is a unital C*-algebra, $\mathscr{B}$ is a unital nuclear C*-algebra and contains unital matrix algebras of any order, then the similarity degree of $\mathscr{A} \otimes \mathscr{B}$ is at most 5. Below we give a generalization of F. Pop's result.

To prove our result, we need the following lemma (Corollary 2.3 in [12]).

LEMMA 3.4. *Let $\mathscr{A}$ and $\mathscr{B}$ be unital C*-algebras and $\mathscr{B}$ nuclear. If π is a unital bounded homomorphism of $\mathscr{A} \otimes \mathscr{B}$ such that $\pi|_{\mathscr{A}}$ is completely bounded and $\pi|_{\mathscr{B}}$ is a *-homomorphism, then π is completely bounded and $\|\pi\|_{cb} \leqslant \|\pi|_{\mathscr{A}}\|_{cb}$.*

Using Lemma 3.4 and the idea in the proof of Theorem 3.3, we can get the following theorem:

THEOREM 3.5. *Let $\mathscr{A}$ be a unital nuclear C*-algebra such that for any positive integer N, there is a finite-dimensional subalgebra in $\mathscr{A}$ containing the unit of $\mathscr{A}$ with subrank at least N. Then for any unital C*-algebra $\mathscr{B}$, $d(\mathscr{A} \otimes \mathscr{B}) \leqslant 5$.*

Note. The original version of this paper was included in [7].

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