

## BLOCK REPRESENTATIONS FOR CLASSES OF ISOMETRIC OPERATORS BETWEEN KREĀN SPACES

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*Abstract.* The behavior of isometric and unitary operators between KreĀn spaces is investigated by means of block decompositions. Therefore two types of isometric operators having a block representation, so-called archetypical isometric operators, are introduced. It is shown that interesting classes of isometric operators, in particular the class of unitary operators, can be expressed as a composition of archetypical isometric operators and bounded unitary operators. As a consequence of these block representations, useful information about the behavior of the isometric operators under consideration can be obtained. In particular, some results on (the Weyl families of) (quasi-) boundary triplets are presented.

### 1. Introduction

In [17] the author showed that the graphs of unitary operators (or relations) can be decomposed in a specific manner and that, as a consequence of that decomposition, unitary operators allow for an (operator) block representation. Here that investigation is continued by studying isometric operators which allow for a block representation. A key tool here will be the so-called *archetypical* isometric operators. For a KreĀn space  $\{\mathfrak{K}, [\cdot, \cdot]\}$  with fundamental symmetry  $j$  which contains a hyper-maximal neutral subspace  $\mathfrak{M}$ , these archetypical isometric (unitary) operators are the isometric (unitary) operators which have w.r.t. the decomposition  $\mathfrak{M} \oplus j\mathfrak{M}$  of  $\mathfrak{K}$  the following block representations:

$$\begin{pmatrix} I & 0 \\ ijS & I \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} B & 0 \\ 0 & jB^{-*}j \end{pmatrix},$$

where  $S$  is a symmetric operator in  $\{\mathfrak{M}, [j \cdot, \cdot]\}$  and  $B$  is any operator in  $\mathfrak{M}$  with  $\overline{\text{ran}} B = \mathfrak{M}$ . Note that these operators, in the bounded case, appear as so-called transformers in the work of Yu.L. Shmul'jan, see [16], and see also [12].

Archetypical isometric operators appear naturally in the study of isometric operators. For instance, if an isometric operator has a hyper-maximal neutral subspace in its domain and maps this subspace onto a neutral subspace with equal defect numbers, then it can be written as the composition of archetypical isometric operators and a bounded unitary operator. Essentially, that class of isometric operators is the abstract analogue

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of the quasi-boundary triplets introduced by J. Behrndt and M. Langer in [7]. Furthermore, it is also shown that unitary operators can always be written as a composition of either of the above displayed archetypical unitary operators and a bounded unitary operator.

Using this block representation approach to isometric and unitary operators, the behavior of isometric and unitary operators can be easily made explicit, thus providing useful information on it. For instance, it is shown that the composition of unitary operators can be an isometric operator which can not be extended to a unitary operator, that unitary operators can map hyper-maximal neutral subspaces onto closed neutral subspaces with nonzero, but equal, defect numbers, and that unitary operators cannot be distinguished by their domains (not even from isometric operators).

Furthermore, using this approach new necessary and sufficient conditions for an isometric operator to be (extendable to) a unitary operator are obtained; those conditions are in turn used to gain some insight into when the composition of an isometric and unitary operator is (extendable to) a unitary operator. As a further application of the obtained results, the already mentioned quasi-boundary triplets are investigated. It is shown that by composing them with an archetypical isometric operator, and closing up, they turn into generalized boundary triplets (which can be interpreted as unitary operators). This connection can be used to obtain necessary and sufficient conditions for quasi-boundary triplets to be boundary relations and, moreover, it can be used to connect their Weyl family to the class of bounded Nevanlinna functions (which are Weyl families of generalized boundary triplets). A third application concerns the generalization of some results on boundary relations for intermediate extensions obtained by V. Derkach, S. Hassi, M. Malamud and H.S.V. de Snoo in [12] to the Kreĭn spaces setting. This generalization makes it clear how to generalize some other results in [12] to the Kreĭn space setting, see also [6]. Finally, the obtained block representations of unitary operators are also used to show how a Weyl family of a boundary relation can be written as the transformation of a bounded and boundedly invertible Nevanlinna family.

The contents of this paper are now outlined. In Section 2 several basic facts about isometric and unitary relations in Kreĭn spaces are recalled. Archetypical isometric operators are introduced and analyzed in Section 3. Section 4 contains the main results; here isometric and unitary operators are investigated by means of the archetypical isometric operators. In Section 5 applications of the results from Section 3 and 4 to (quasi-) boundary relations are presented.

## 2. Preliminaries

Basic facts about Kreĭn spaces are recalled and some notation is introduced, see [3, 8] for details. This is followed by a short introduction to isometric and unitary relations between Kreĭn spaces. In the last section, special classes of isometric and unitary relations, namely the so-called quasi-boundary relations and boundary relations, are presented.

### 2.1. Kreĭn spaces

The notation  $\{\mathfrak{H}, (\cdot, \cdot)\}$  and  $\{\mathfrak{K}, [\cdot, \cdot]\}$  is always used to denote Hilbert and Kreĭn spaces, respectively. To distinguish different Hilbert and Kreĭn spaces subindexes are used:  $\mathfrak{H}_1, \mathfrak{K}_1, \mathfrak{H}_2, \mathfrak{K}_2$ , etc.. For a fundamental symmetry of  $\{\mathfrak{K}_i, [\cdot, \cdot]_i\}$  the notation  $j_i$  is reserved and  $\mathfrak{K}_i^+ [ + ] \mathfrak{K}_i^-$  denotes a canonical decompositions of  $\{\mathfrak{K}_i, [\cdot, \cdot]_i\}$ . Here the dimensions of  $\mathfrak{K}_i^+$  and  $\mathfrak{K}_i^-$  are independent of the canonical decomposition and are denoted by  $k_i^+$  and  $k_i^-$ , respectively.

A subspace of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  is always a linear subspace which is called closed if it is closed with respect to the topology induced by some (and hence every) fundamental symmetry of  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . The notions of (uniform) positivity, (uniform) negativity, (maximal) neutrality, (maximal) nonnegativity and (maximal) nonpositivity of subspaces of a Kreĭn spaces are as usual. Recall that a neutral subspace  $\mathcal{L}$  satisfies  $\mathcal{L} \subseteq \mathcal{L}^{[\perp]}$  and that if  $\mathcal{L} = \mathcal{L}^{[\perp]}$ , then  $\mathcal{L}$  is called hyper-maximal neutral, see [3, Ch. I: 4.19]. As a generalization of this concept, a closed semi-definite, i.e. a neutral, nonnegative or nonpositive, subspace  $\mathcal{L}$  of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  is called *hyper-maximal semi-definite* if  $\mathcal{L}^{[\perp]}$  is maximal neutral, see [17]. Equivalently, given any fundamental symmetry  $j$  of  $\{\mathfrak{K}, [\cdot, \cdot]\}$ , a closed semi-definite subspace  $\mathcal{L}$  is hyper-maximal semi-definite if and only if it induces an orthogonal decomposition of the space:

$$\mathfrak{K} = \mathcal{L} \oplus j\mathcal{L}^{[\perp]} = \mathcal{L}^{[\perp]} \oplus (\mathcal{L} \cap j\mathcal{L}) \oplus j\mathcal{L}^{[\perp]}, \tag{2.1}$$

where the orthogonality is w.r.t. the Hilbert space inner product  $[j\cdot, \cdot]$ . In particular, (2.1) shows that a closed semi-definite subspace  $\mathcal{L}$  is hyper-maximal semi-definite if and only if  $\mathcal{L}^{[\perp]}$  is hyper-maximal neutral in the Kreĭn space  $\{\mathfrak{K} \cap (\mathcal{L} \cap j\mathcal{L})^{[\perp]}, [\cdot, \cdot]\}$ .

Let  $\mathcal{L}$  be a neutral subspace of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and let  $\mathfrak{K}^+ [ + ] \mathfrak{K}^-$  be a canonical decomposition of  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . Then the defect numbers  $n_{\pm}(\mathcal{L})$  of  $\mathcal{L}$  are defined as

$$n_+(\mathcal{L}) = \dim(\mathcal{L}^{[\perp]} \cap \mathfrak{K}^-) \quad \text{and} \quad n_-(\mathcal{L}) = \dim(\mathcal{L}^{[\perp]} \cap \mathfrak{K}^+). \tag{2.2}$$

For the well-definedness of these numbers, i.e. the independence of the chosen canonical decomposition, see e.g. [3, Ch. I: Theorem 6.7].

### 2.2. Linear relations in Kreĭn spaces

Recall that a mapping  $H$  from a set  $X$  to set  $Y$  is called a multi-valued mapping if  $Hx := H(x)$  is a subset of  $Y$  for every  $x \in X$ . Using this concept  $H$  is called a *linear multi-valued operator* from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  if  $H$  is a linear multi-valued mapping from a subspace of  $\mathfrak{K}_1$ , called the domain of  $H$  or  $\text{dom}H$  for short, to  $\mathfrak{K}_2$  such that

$$H(f + cg) = Hf + cHg, \quad f, g \in \text{dom}H, c \in \mathbb{C},$$

see [10]. Here  $Hf + cHg$  is the sum of subspaces of  $\mathfrak{K}_2$ , i.e.  $Hf + cHg = \{f' + cg' : f' \in Hf \text{ and } g' \in Hg\}$ . For a multi-valued operator  $H$  and a subspace  $\mathcal{L} \subseteq \text{dom}H$ , the subspace  $H(\mathcal{L})$  of  $\mathfrak{K}_2$  is defined as

$$H(\mathcal{L}) = \{f' \in \mathfrak{K}_2 : \exists f \in \mathcal{L} \text{ s.t. } f' \in Hf\}.$$

Using this definition, the range, kernel and multi-valued part of a multi-valued operator  $H$  are defined as follows

$$\begin{aligned} \text{ran}H &= H(\text{dom}H) = \{f' \in \mathfrak{K}_2 : \exists f \in \text{dom}H \text{ s.t. } f' \in Hf\}, \\ \text{ker}H &= \{f \in \text{dom}H : Hf = \text{mul}H\}, \quad \text{mul}H = H0. \end{aligned}$$

Since a multi-valued operator is linear, there exists for every multi-valued operator  $H$  a non-unique single-valued operator  $H_o$  such that  $H = H_o + \text{mul}H$ . In particular, a multi-valued operator is a single-valued operator if and only if  $\text{mul}H = \{0\}$ . The graph of a multi-valued operator  $H$  is the subspace  $\text{gr}H$  of  $\mathfrak{K}_1 \times \mathfrak{K}_2$  defined as

$$\text{gr}H = \{\{f, f'\} \in \mathfrak{K}_1 \times \mathfrak{K}_2 : f \in \text{dom}H \text{ and } f' \in Hf\}.$$

Conversely, with each subspace of  $\mathfrak{K}_1 \times \mathfrak{K}_2$  one can associate a multi-valued operator. Recall that subspaces of  $\mathfrak{K}_1 \times \mathfrak{K}_2$  are also called (linear) relations from  $\mathfrak{K}_1$  to  $\mathfrak{K}_2$ , see [2]. Here, following [10], the term relation is used as a synonym for a multi-valued operator

The inverse of a relation  $H$  is the relation  $H^{-1}$  defined as

$$H^{-1}f' = \{f \in \mathfrak{K}_1 : f' \in Hf\}, \quad f' \in \text{dom}H^{-1} = \text{ran}H,$$

and the adjoint of a relation  $H$  is the relation  $H^{[*]}$  defined via

$$\text{gr}H^{[*]} = \{\{f, f'\} \in \mathfrak{K}_2 \times \mathfrak{K}_1 : [f', g]_1 = [f, g']_2, \forall \{g, g'\} \in \text{gr}H\}.$$

Here the adjoint of  $H$  is denoted by  $H^*$  if  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  and  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  are both Hilbert spaces. The definition of  $H^{[*]}$  implies that

$$(\text{dom}H)^{[\perp]_1} = \text{mul}H^{[*]} \quad \text{and} \quad (\text{ran}H)^{[\perp]_2} = \text{ker}H^{[*]}. \tag{2.3}$$

For relations  $G$  and  $H$  from  $\mathfrak{K}_1$  to  $\mathfrak{K}_2$ , the notation  $G \subseteq H$  is used to denote that  $H$  is an extension of  $G$ , i.e.  $\text{gr}G \subseteq \text{gr}H$ . In particular, with this notation

$$G = H \quad \text{if and only if} \quad G \subseteq H, \quad \text{ran}H \subseteq \text{ran}G, \quad \text{ker}H \subseteq \text{ker}G. \tag{2.4}$$

Furthermore, the following well-known statement holds, cf. e.g. [2]; for the last statement in Lemma 2.1 below see also [12, Lemma 2.9].

LEMMA 2.1. *Let  $\{\mathfrak{K}_i, [\cdot, \cdot]_i\}$ ,  $i = 1, 2, 3$ , be Kreĭn spaces and let  $G : \mathfrak{K}_1 \rightarrow \mathfrak{K}_2$  and  $H : \mathfrak{K}_2 \rightarrow \mathfrak{K}_3$  be linear relations. Then*

$$(H^{[*]})^{-1} = (H^{-1})^{[*]}, \quad (HG)^{-1} = G^{-1}H^{-1} \quad \text{and} \quad G^{[*]}H^{[*]} \subseteq (HG)^{[*]}.$$

Moreover, if  $G$  is closed,  $\text{ran}G$  is closed and  $\text{dom}H \subseteq \text{ran}G$  or  $H$  is closed,  $\text{dom}H$  is closed and  $\text{ran}G \subseteq \text{dom}H$ , then  $G^{[*]}H^{[*]} = (HG)^{[*]}$ .

### 2.3. Unitary relations in Kreĭn spaces

Here some basic facts about isometric and unitary relations are recalled from [17], see also [2, 3, 8, 11]. A relation  $U$  from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  is called *isometric* or *unitary* if

$$U^{-1} \subseteq U^{[*]} \quad \text{or} \quad U^{-1} = U^{[*]},$$

respectively. Combining this definition with (2.3) yields that isometric relations  $V$  and unitary relations  $U$  satisfy

$$\begin{aligned} \ker V &\subseteq (\text{dom } V)^{[\perp]_1}, & \text{mul } V &\subseteq (\text{ran } V)^{[\perp]_2}, \\ \ker U &= (\text{dom } U)^{[\perp]_1}, & \text{mul } U &= (\text{ran } U)^{[\perp]_2}. \end{aligned} \tag{2.5}$$

By definition a unitary relation is closed, a relation is isometric if and only if its closure is isometric, and a relation is isometric or unitary if and only if its inverse is isometric or unitary, respectively. Furthermore, a unitary relation has a closed domain if and only if its range is closed. In particular, if a unitary relation is surjective, then it is a bounded operator, cf. (2.5). A bounded unitary operator with a trivial kernel is an everywhere defined unitary operator (with everywhere defined inverse); such an operator is usually called a *standard unitary operator*.

Unitary relations are maximal isometric relations in a special sense: If  $U$  is isometric, then  $U$  is unitary if and only if for  $f \in \mathfrak{K}_1$  and  $f' \in \mathfrak{K}_2$

$$[f, g]_1 = [f', g']_2, \quad \forall \{g, g'\} \in \text{gr } U \implies \{f, f'\} \in \text{gr } U. \tag{2.6}$$

Further necessary and sufficient conditions for isometric relations to be unitary can be found in [17], see also the references therein.

Observe that Lemma 2.1 implies that the composition of isometric relations is an isometric relation. The composition of unitary relations is in general not a unitary relation. However, Lemma 2.1 shows that the composition  $U_2 U_1$  of the unitary relations  $U_1$  and  $U_2$  is unitary if  $\text{ran } U_1$  is closed and  $\text{dom } U_2 \subseteq \text{ran } U_1$  or  $\text{dom } U_2$  is closed and  $\text{ran } U_1 \subseteq \text{dom } U_2$ .

### 2.4. Quasi-boundary relations in Kreĭn spaces

A relation  $S$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  is called *symmetric* or *selfadjoint* if  $S \subseteq S^{[*]}$  or  $S = S^{[*]}$ , respectively. In particular, a relation  $S$  in  $\{\mathfrak{H}, (\cdot, \cdot)\}$  is called *symmetric* or *selfadjoint* if  $S \subseteq S^*$  or  $S = S^*$ , respectively. On the Cartesian product spaces  $\mathfrak{K}^2$  and  $\mathfrak{H}^2$  define the sesqui-linear forms  $\ll \cdot, \cdot \gg$  and  $\langle \cdot, \cdot \rangle$  by

$$\begin{aligned} \ll \{f, f'\}, \{g, g'\} \gg &= -i([f', g] - [f, g']), \quad f, f', g, g' \in \mathfrak{K}, \\ \langle \{f, f'\}, \{g, g'\} \rangle &= -i((f', g) - (f, g')), \quad f, f', g, g' \in \mathfrak{H}. \end{aligned} \tag{2.7}$$

Then  $\{\mathfrak{K}^2, \ll \cdot, \cdot \gg\}$  and  $\{\mathfrak{H}^2, \langle \cdot, \cdot \rangle\}$  are Kreĭn spaces. For a relation  $K$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and a relation  $H$  in  $\{\mathfrak{H}, (\cdot, \cdot)\}$ ,  $(\text{gr } K)^{\ll \perp \gg}$  and  $(\text{gr } H)^{\langle \perp \rangle}$ , where the orthogonal complements are w.r.t.  $\ll \cdot, \cdot \gg$  and  $\langle \cdot, \cdot \rangle$ , coincide with  $\text{gr } K^{[*]}$  and  $\text{gr } H^*$ , respectively. In particular,  $S$  is a (closed) symmetric or selfadjoint relation in  $\{\mathfrak{K}, [\cdot, \cdot]\}$

$(\{\mathfrak{H}, (\cdot, \cdot)\})$  if and only if  $\text{gr}S$  is a (closed) neutral or hyper-maximal neutral subspace of  $\{\mathfrak{R}^2, \ll \cdot, \cdot \gg\}$  ( $\{\mathfrak{H}^2, \langle \cdot, \cdot \rangle\}$ ).

Using the above notation, the notion of a boundary relation for the adjoint of a symmetric relation in the Kreĭn space  $\{\mathfrak{K}, [\cdot, \cdot]\}$  can be defined, see [11, 4].

DEFINITION 2.2. Let  $S$  be a closed symmetric relation in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . Then  $\{\mathcal{H}, \Gamma\}$  is called a *boundary relation* for  $S^{[*]}$  if  $\{\mathcal{H}, (\cdot, \cdot)\}$  is a Hilbert space and  $\Gamma$  is a unitary relation from  $\{\mathfrak{R}^2, \ll \cdot, \cdot \gg\}$  to  $\{\mathcal{H}^2, \langle \cdot, \cdot \rangle\}$  with  $\ker \Gamma = \text{gr}S$ .

Note that the condition  $\ker \Gamma = \text{gr}S$  in Definition 2.2 implies that  $\overline{\text{dom}} \Gamma = \text{gr}S^{[*]}$ , see (2.5). If  $\Gamma$  maps onto  $\mathcal{H}^2$ , then  $\text{dom} \Gamma$  is closed and  $\{\mathcal{H}, \Gamma\}$  is called an *ordinary boundary triplet*, see [13]. In [7] the concept of an ordinary boundary triplet was generalized to the concept of a quasi-boundary triplet; below a natural generalization of that concept is presented. Therefore define  $\Gamma_0$  and  $\Gamma_1$  for a mapping  $\Gamma$  from  $\mathfrak{R}^2$  to  $\mathcal{H}^2$  via

$$\text{gr} \Gamma_i = \{ \{ \{ f, f' \}, g_i \} : \{ \{ f, f' \}, \{ g_0, g_1 \} \} \in \text{gr} \Gamma \}, \quad i = 0, 1. \tag{2.8}$$

DEFINITION 2.3. Let  $S$  be a closed symmetric relation in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . Then  $\{\mathcal{H}, \Gamma\}$  is called a *quasi-boundary relation* for  $S^{[*]}$  if  $\{\mathcal{H}, (\cdot, \cdot)\}$  is a Hilbert space and  $\Gamma$  is an isometric relation from  $\{\mathfrak{R}^2, \ll \cdot, \cdot \gg\}$  to  $\{\mathcal{H}^2, \langle \cdot, \cdot \rangle\}$  with  $\ker \Gamma = \text{gr}S$ ,  $\text{mul} \Gamma = (\text{ran} \Gamma)^{\langle \perp \rangle}$ , and  $\ker \Gamma_0$  is the graph of a selfadjoint relation in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ .

If  $\{\mathcal{H}, \Gamma\}$  is a quasi-boundary relation and  $\overline{\text{ran}} \Gamma = \mathcal{H}^2$ , then  $\text{mul} \Gamma = \{0\}$ , see (2.5). In that case, the quasi-boundary relation  $\{\mathcal{H}, \Gamma\}$  is called a *quasi-boundary triplet* in accordance with [7]. The conditions  $\text{mul} \Gamma = (\text{ran} \Gamma)^{\langle \perp \rangle}$  and  $\ker \Gamma_0 = (\ker \Gamma_0)^{[*]}$  together imply that  $\ker \Gamma = (\text{dom} \Gamma)^{\ll \perp \gg}$ , see [17, Corollary 3.23]. Therefore, as for boundary relations, the domain of a quasi-boundary relation is dense in the graph of  $S^{[*]}$ :  $\overline{\text{dom}} \Gamma = \text{gr}S^{[*]}$ .

By means of eigenspaces, a Weyl family can be associated with (quasi-) boundary relations, see [11, 7, 4]. Recall, that for a relation  $H$  in  $\mathfrak{K}$  the eigenspaces of  $H$  are denoted by  $\widehat{\mathfrak{N}}_\lambda(H) = \{ \{ f_\lambda, \lambda f_\lambda \} : f_\lambda \in \ker(H - \lambda) \}$ ,  $\lambda \in \mathbb{C}$ .

DEFINITION 2.4. Let  $S$  be a closed symmetric relation in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ , let  $\{\mathcal{H}, \Gamma\}$  be a (quasi-) boundary relation for  $S^{[*]}$  and let  $T$  be the relation such that  $\text{gr}T = \text{dom} \Gamma$ . Then the *Weyl family* associated with  $\{\mathcal{H}, \Gamma\}$  is the relation-valued function  $M(\lambda)$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , defined via

$$\text{gr}M(\lambda) = \{ \{ h, h' \} : \exists \{ f, f' \} \in \widehat{\mathfrak{N}}_\lambda(T) \text{ s.t. } \{ \{ f, f' \}, \{ h, h' \} \} \in \text{gr} \Gamma \}.$$

If  $M(\lambda)$  is an operator-valued function, then it will, as usual, be called the *Weyl function* associated with the (quasi-) boundary relation, see [11]. In that case the Weyl function can be defined via

$$M(\lambda)\Gamma_0\{f, f'\} = \Gamma_1\{f, f'\}, \quad \{f, f'\} \in \widehat{\mathfrak{N}}_\lambda(T).$$

Note that the Weyl family associated with a (quasi-) boundary relation is an operator-function, i.e. a Weyl function, if for instance  $\text{ran}\Gamma_0 = \mathcal{H}$ , since in that case  $\text{mul}\Gamma \cap (\{0\} \times \mathcal{H}) = \{0\}$  (because  $\text{mul}\Gamma = (\text{ran}\Gamma)^{\perp}$ , see (2.5)).

### 3. Archetypical unitary relations and their compositions

Two types of unitary operators having a simple block structure are introduced; they will be called *archetypical* unitary operators. Recall that, in the bounded case, archetypical unitary operators appear as so-called transformers in [16]. They also appear naturally in the framework of boundary relations, there they are used to renormalize the Weyl family associated to a boundary relation, see [12]; cf. Section 5.2 below. Here archetypical unitary operators and their composition are considered in the general case.

#### 3.1. Archetypical unitary relations

Let  $j$  be a fundamental symmetry of the Kreĭn space  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and let  $\mathfrak{M}$  be a hyper-maximal semi-definite subspace of  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . Then  $\mathfrak{M}$  induces a decomposition of  $\mathfrak{K}$ :  $\mathfrak{K} = \mathfrak{M}^{\perp} \oplus (\mathfrak{M} \cap j\mathfrak{M}) \oplus j\mathfrak{M}^{\perp}$ , see (2.1). Here  $\mathfrak{M} \cap j\mathfrak{M}$  is a uniformly definite subspace of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and the behavior of isometric operators on this subspace is easily understood, see e.g. [17, Corollary 3.13]. Hence, w.l.o.g. assume that  $\mathfrak{M}$  is hyper-maximal neutral and introduce for a relation  $S$  in (the Hilbert space)  $\{\mathfrak{M}, [j\cdot, \cdot]\}$ , the relation  $\Upsilon_1(S)$  in  $\mathfrak{K}$  as

$$\Upsilon_1(S)(f + jg) = f + j(iSf + g), \quad f \in \text{dom}S, g \in \mathfrak{M}.$$

Note that  $\Upsilon_1(S)$  is a relation (or, equivalently, has a non-trivial kernel) if and only if  $S$  is a relation, and that  $(\Upsilon_1(S))^{-1} = \Upsilon_1(-S)$ . If  $S$  is an operator, then  $\Upsilon_1(S)$  is an operator (with a trivial kernel) which has the following block representation:

$$\Upsilon_1(S) = \begin{pmatrix} I & 0 \\ j i S & I \end{pmatrix},$$

where the righthand side block decomposition is w.r.t. the decomposition  $\mathfrak{M} \oplus j\mathfrak{M}$  of  $\mathfrak{K}$ . As a consequence of its definition,  $\Upsilon_1(S)$  is an isometric operator or relation if and only if  $S$  is a symmetric operator or relation, respectively. Since  $\text{clos}(\Upsilon_1(S)) = \Upsilon_1(\text{clos}(S))$ ,  $\Upsilon_1(S)$  can be an operator whilst its closure is a relation. Proposition 3.2 below summarizes the above discussion and provides a characterization for  $\Upsilon_1(S)$  to be unitary, see [12, Example 2.11]. Here a short proof for the characterization of  $\Upsilon_1(S)$  to be unitary is included; it is based on the following useful lemma.

LEMMA 3.1. *Let  $U$  be an isometric relation from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and assume that there exist hyper-maximal neutral subspaces  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  in  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  and  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ , respectively, such that  $\mathfrak{M}_1 \subseteq \text{dom}U$  and  $U(j_1\mathfrak{M}_1 \cap \text{dom}U) = \mathfrak{M}_2$  for a fundamental symmetry  $j_1$  of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$ . Then  $U$  is a unitary relation.*

*Proof.* Let  $j_2$  be a fundamental symmetry of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and let  $k \in \mathfrak{K}_1$  and  $k' \in \mathfrak{K}_2$  be such that  $[f, k]_1 = [f', k']_2$  for all  $\{f, f'\} \in \text{gr}U$ . Then by the assumptions there exists  $\{h, h'\} \in \text{gr}U$  such that  $k - h \in j_1\mathfrak{M}_1$  and  $k' - h' \in j_2\mathfrak{M}_2$ . Clearly,

$$[f, k - h]_1 = [f', k' - h']_2, \quad \forall \{f, f'\} \in \text{gr}U. \tag{3.1}$$

Now, by the assumption that  $U(j_1\mathfrak{M}_1 \cap \text{dom}U) = \mathfrak{M}_2$ , there exists a  $g \in j_1\mathfrak{M}_1 \cap \text{dom}U$  such that  $\{g, j_2(k' - h')\} \in \text{gr}U$ . Therefore (3.1) implies that

$$0 = [g, k - h]_1 = [j_2(k' - h'), (k' - h')]_2.$$

This shows that  $k' - h' = 0$  and, hence,  $[f, k - h]_1 = 0$  for all  $f \in \text{dom}U$  by (3.1), i.e.  $k - h \in (\text{dom}U)^{\perp 1} \subseteq \mathfrak{M}_1^{\perp 1} = \mathfrak{M}_1$ . Since  $k - h \in j_1\mathfrak{M}_1$ , this implies that  $k - h = 0$ , i.e.  $\{k, k'\} = \{h, h'\} \in \text{gr}U$ . Consequently, (2.6) implies that  $U$  is a unitary relation.  $\square$

**PROPOSITION 3.2.** *Let  $j$  be a fundamental symmetry of  $\{\mathfrak{K}, [\cdot, \cdot]\}$ , assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and let  $S$  be a relation in  $\mathfrak{M}$ . Then  $\Upsilon_1(S)$  is a (closed) isometric relation or (extendable to) a unitary relation in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  if and only if  $S$  is a (closed) symmetric relation or (extendable to) a selfadjoint relation in the Hilbert space  $\{\mathfrak{M}, [j \cdot, \cdot]\}$ , respectively. Moreover,  $\Upsilon_1(S)$  is an isometric operator with a trivial kernel if and only if  $S$  is a symmetric operator and  $\Upsilon_1(S)$  is a standard unitary operator if and only if  $S$  is a symmetric operator with  $\text{dom}S = \mathfrak{M}$ , i.e. if and only if  $S$  is a bounded selfadjoint operator.*

*Proof.* Only the first equivalence is proven, the remaining statements follow directly from it and the definition of  $\Upsilon_1(S)$ . To prove that equivalence first note that if  $T$  is a symmetric extension of  $S$ , then  $\Upsilon_1(T)$  is an isometric extension of  $\Upsilon_1(S)$ . Hence, it suffices to prove that  $\Upsilon_1(S)$  is unitary if and only if  $S$  is selfadjoint.

If  $S$  is selfadjoint, then  $j\mathfrak{M} \subseteq \text{dom}\Upsilon_1(S)$  and  $\Upsilon_1(S)(\mathfrak{M} \cap \text{dom}\Upsilon_1(S)) = \{f + jif : f \in \text{dom}S\}$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}, [\cdot, \cdot]\}$ , see Proposition 3.6 below. Hence, Lemma 3.1 implies that  $\Upsilon_1(S)$  is unitary. To prove the converse assume that  $S$  is a maximal symmetric relation which is not selfadjoint, and that  $\Upsilon_1(S)$  is unitary. Then there exist  $\{f, f'\} \in \text{gr}S^*$  such that  $\text{Im}[jf, f'] \neq 0$ , and a direct calculation shows that  $[f, g] = [f + jif', g']$  for all  $\{g, g'\} \in \text{gr}(\Upsilon_1(S))$ , i.e.,  $\{f, f + jif'\} \in \text{gr}(\Upsilon_1(S))$  by (2.6). On the other hand,  $[f, f] = 0$  and, by assumption,  $[f + jif', f + jif'] = i([jf', f] - [f, jf']) \neq 0$ . Therefore  $\{f, f + jif'\}$  cannot belong to the graph of an isometric relation. This contradiction completes the proof.  $\square$

Observe that Proposition 3.2 yields basic examples of isometric operators which can not be extended to unitary operators (or relations); namely  $\Upsilon_1(S)$  for symmetric operators  $S$  in (the Hilbert space)  $\{\mathfrak{M}, [j \cdot, \cdot]\}$  with unequal defect numbers.

Next define for a relation  $B$  in the Hilbert space  $\{\mathfrak{M}, [j \cdot, \cdot]\}$ , with adjoint  $B^*$ , the relation  $\Upsilon_2(B)$  as

$$\Upsilon_2(B)(f + jg) = Bf + jB^{-*}g, \quad f \in \text{dom}B, g \in \text{dom}B^{-*}.$$

Here and what follows the notation  $B^{-*}$  is used to denote  $(B^*)^{-1} = (B^{-1})^*$ , see Lemma 2.1. A direct calculation shows that  $\Upsilon_2(B)$  is an isometric relation in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ , which is an operator if and only if  $\text{mul}B = \{0\}$  and  $\text{ker}B^* = (\text{ran}B)^\perp = \{0\}$ , and that  $\text{clos}(\Upsilon_2(B)) = \Upsilon_2(\text{clos}(B))$ . Hence, if  $B$  is a non-closable operator with  $\overline{\text{ran}}B = \mathfrak{M}$ , then  $\Upsilon_2(B)$  is an isometric operator whereas  $\text{clos}(\Upsilon_2(B))$  is an isometric relation. If  $\Upsilon_2(B)$  is an operator, then it has the following block representation w.r.t the decomposition  $\mathfrak{M} \oplus j\mathfrak{M}$  of  $\mathfrak{K}$ :

$$\Upsilon_2(B) = \begin{pmatrix} B & 0 \\ 0 & jB^{-*} \end{pmatrix}.$$

Note that  $\Upsilon_2(B)$  is an isometric operator with a trivial kernel if and only if  $B$  satisfies

$$\text{ker}B = \{0\}, \quad \overline{\text{dom}}B = \mathfrak{M}, \quad \text{mul}B = \{0\} \quad \text{and} \quad \overline{\text{ran}}B = \mathfrak{M}. \tag{3.2}$$

Furthermore, using (2.3), it follows that  $\Upsilon_2(B)$  and  $\text{clos}(\Upsilon_2(B))$  are both isometric operators with a trivial kernel if and only if  $B$  satisfies

$$\overline{\text{dom}}B^* = \mathfrak{M}, \quad \overline{\text{dom}}B = \mathfrak{M}, \quad \overline{\text{ran}}B^* = \mathfrak{M} \quad \text{and} \quad \overline{\text{ran}}B = \mathfrak{M}. \tag{3.3}$$

The conditions in (3.3) are equivalent to those in (3.2) if  $B$  is a closed operator. Proposition 3.3 below summarizes the above discussion and provides a characterization for  $\Upsilon_2(B)$  to be unitary.

**PROPOSITION 3.3.** *Let  $j$  be a fundamental symmetry of  $\{\mathfrak{K}, [\cdot, \cdot]\}$ , assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and let  $B$  be a relation in  $\mathfrak{M}$ . Then  $\Upsilon_2(B)$  and  $\Upsilon_2(\text{clos}(B)) = \text{clos}(\Upsilon_2(B))$  are an isometric and a unitary relation in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ , respectively. Moreover,  $\Upsilon_2(B)$  or  $\Upsilon_2(\text{clos}(B))$  is an isometric or unitary operator with a trivial kernel if and only if  $B$  satisfies (3.2) or (3.3), respectively, and  $\Upsilon_2(B)$  is a standard unitary operator if and only if  $B$  and  $B^{-1}$  are everywhere defined closed operators.*

*Proof.* It suffices to prove that  $\Upsilon_2(\text{clos}(B))$  is unitary. Let  $h, h', k, k' \in \mathfrak{M}$  be such that  $[h + jh', f + jg] = [k + jk', f' + jg']$  for all  $\{f, f'\} \in \text{grclos}(B)$  and  $\{g, g'\} \in \text{gr}B^{-*}$ . Then  $[jh', f] = [jk', f']$  for all  $\{f, f'\} \in \text{grclos}(B)$  and  $[h, jg] = [k, jg']$  for all  $\{g, g'\} \in \text{gr}B^{-*}$ , i.e.,  $\{h', k'\} \in \text{gr}B^{-*}$  and  $\{h, k\} \in \text{grclos}(B)$ . I.e.,  $\{h + jh', k + jk'\} \in \text{gr}(\Upsilon_2(\text{clos}(B)))$  and, hence, (2.6) implies that  $\Upsilon_2(\text{clos}(B))$  is unitary.  $\square$

Next it is shown that unitary operators of the type  $\Upsilon_2(B)$  can map a hyper-maximal neutral subspace onto a closed neutral subspace with equal, but nonzero, defect numbers. In light of Theorem 4.8 below, this provides a simple proof for [9, Lemma 4.4]. Recall that the defect numbers  $n_+(\mathcal{L})$  and  $n_-(\mathcal{L})$  of a neutral subspace  $\mathcal{L}$  are as defined in (2.2).

**PROPOSITION 3.4.** *Let  $j$  be a fundamental symmetry of  $\{\mathfrak{K}, [\cdot, \cdot]\}$ , assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and let  $B$  be a closed unbounded operator in the Hilbert space  $\{\mathfrak{M}, [j\cdot, \cdot]\}$  with  $\text{dom}B = \mathfrak{M} = \text{ran}B$  and  $\text{ker}B = \{0\}$ . Then for every  $0 \leq m \leq \aleph_0$  there exists a hyper-maximal neutral subspace  $\mathcal{L}$  of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  such that  $\mathcal{L} \subseteq \text{dom}(\Upsilon_2(B))$  and that  $\Upsilon_2(B)(\mathcal{L})$  is a closed neutral subspace of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  with defect numbers  $n_\pm(\Upsilon_2(B)(\mathcal{L})) = m$ .*

*Proof.* Since  $B^*$  is a densely defined unbounded operator with  $\text{ran} B^* = \mathfrak{M}$  and  $\text{ker} B^* = \{0\}$ , there exists an  $m$ -dimensional closed subspace  $\mathfrak{N}_m$  of  $\{\mathfrak{M}, [j, \cdot]\}$  such that  $\text{dom} B^* \cap \mathfrak{N}_m = \{0\}$  and  $\mathfrak{M} = \text{clos} \{B^{-1}f : f \in \mathfrak{M} \ominus \mathfrak{N}_m\}$ , see e.g. [9, Lemma 4.2]. Hence,

$$Cf = Bf, \quad f \in \text{dom} C = \{g \in \text{dom} B : Bg \in \mathfrak{M} \ominus \mathfrak{N}_m\},$$

considered as an operator from  $\mathfrak{M}$  to  $\mathfrak{M} \ominus \mathfrak{N}_m$  is a closed operator and satisfies  $\overline{\text{dom} C} = \mathfrak{M}$ ,  $\text{ran} C = \mathfrak{M} \ominus \mathfrak{N}_m$  and  $\text{ker} C = \{0\}$ . Now define the isometric operator  $U$  from  $\{\mathfrak{K}, [\cdot, \cdot]\}$  to  $\{\mathfrak{K} \ominus (\mathfrak{N}_m + j\mathfrak{N}_m), [\cdot, \cdot]\}$  as

$$U(f + jf') = Cf + jC^{-*}f', \quad f \in \text{dom} C, f' \in \mathfrak{M}.$$

Then by definition  $\text{dom} U \subseteq \text{dom} \Upsilon_2(B)$  and arguments as in Proposition 3.3 show that  $U$  is a unitary operator from  $\{\mathfrak{K}, [\cdot, \cdot]\}$  to  $\{\mathfrak{K} \ominus (\mathfrak{N}_m + j\mathfrak{N}_m), [\cdot, \cdot]\}$ . Let  $WK$  be the polar decomposition of  $C$ , then  $K$  is a (nonnegative) selfadjoint operator in  $\{\mathfrak{M}, [j, \cdot]\}$  with  $\text{dom} K = \text{dom} C$  and, hence,

$$\mathfrak{L} := \{f + jiKf : f \in \text{dom} K\}$$

is a hyper-maximal neutral subspace of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  contained in the domain of  $U$ .

By definition of  $K$ ,  $KC^{-1}$  is a closed operator from  $\{\mathfrak{M} \ominus \mathfrak{N}_m, [j, \cdot]\}$  to  $\{\mathfrak{M}, [j, \cdot]\}$  with domain  $\mathfrak{M} \ominus \mathfrak{N}_m$ . Moreover,  $KB^{-1}$  coincides with  $KC^{-1}$  when the latter is considered as a mapping in  $\{\mathfrak{M}, [j, \cdot]\}$ , because  $\text{dom} K = \text{dom} C$  and  $C \subseteq B$ . Therefore  $S := B^{-*}KB^{-1}$  is a closed symmetric operator with domain  $\mathfrak{M} \ominus \mathfrak{N}_m$ , i.e.  $S$  is a bounded symmetric operator with  $n_{\pm}(S) = m$ . Now the proof is completed by observing that  $\mathfrak{L} \subseteq \text{dom} (\Upsilon_2(B))$  and that  $\Upsilon_2(B)(\mathfrak{L}) = \{f + jiSf : f \in \text{dom} S\}$ .  $\square$

Henceforth, the introduced isometric (unitary) relations  $\Upsilon_1(S)$  and  $\Upsilon_2(B)$  will be called *archetypical* isometric (unitary) relations. Next they will be used to show that unitary operators are not characterized by their domain.

**EXAMPLE 3.5.** Let  $\{\mathfrak{K}, [\cdot, \cdot]\}$  be a Kreĭn space with fundamental symmetry  $j$  and assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ .

i) Let  $K$  be an unbounded selfadjoint operator in (the Hilbert space)  $\{\mathfrak{M}, [j, \cdot]\}$  with  $\text{ran} K = \mathfrak{M}$ . Then  $\text{dom} \Upsilon_1(K) = \text{dom} K \oplus j\mathfrak{M} = \text{dom} \Upsilon_2(K)$ . Moreover, Proposition 3.11 below yields that  $\Upsilon_1(K) (\Upsilon_2(K))^{-1} = \Upsilon_1(K) \Upsilon_2(K^{-1})$  is an unbounded unitary operator.

ii) Let  $S$  be a closed symmetric operator in  $\{\mathfrak{M}, [j, \cdot]\}$  with  $\overline{\text{dom} S} = \mathfrak{M}$  whose defect numbers are  $n_+(S) = 1$  and  $n_-(S) = 0$ , and let  $B$  be an everywhere defined bounded operator in  $\{\mathfrak{M}, [j, \cdot]\}$  with  $\text{ran} B = \text{dom} S$  and  $\text{ker} B = \{0\}$ , see [14, Theorem 1.1]. Then  $\text{dom} (\Upsilon_1(S)) = \text{dom} S \oplus j\mathfrak{M} = \text{dom} (\Upsilon_2(B^{-1}))$ , where  $\Upsilon_1(S)$  is a closed isometric operator in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ , which cannot be extended to a unitary operator, and  $\Upsilon_2(B^{-1})$  is a unitary operator in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ .

### 3.2. Standard unitary operators

Let  $j$  be a fundamental symmetry of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ , i.e.,  $\mathfrak{K} = \mathfrak{M} \oplus j\mathfrak{M}$ . If  $\mathcal{P}_{\mathfrak{M}}$  is the orthogonal projection onto  $\mathfrak{M}$  w.r.t.  $[j\cdot, \cdot]$ , then a subspace  $\mathfrak{L}$  of  $\mathfrak{K}$  can be represented by means of a relation  $A_{\mathfrak{L}}$  in  $\mathfrak{M}$  as follows:  $\mathfrak{L} = \{f \oplus jif' : \{f, f'\} \in \text{gr}A_{\mathfrak{L}}\}$ , where

$$\text{gr}A_{\mathfrak{L}} := \{\{\mathcal{P}_{\mathfrak{M}}f, -i\mathcal{P}_{\mathfrak{M}}jf'\} : f \in \mathfrak{L}\}. \tag{3.4}$$

In this manner (hyper-maximal) neutral subspaces of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  can be represented by relations in the Hilbert space  $\{\mathfrak{M}, [j\cdot, \cdot]\}$ . Note that this representation of a subspace by means of a relation is similar to the representation of subspaces by means of angular operators, only here the coordinates are chosen to be hyper-maximal neutral and not uniformly definite, cf. [3, Ch. I: Section 8]. The following lemma characterizes the relations which represent (hyper-maximal) neutral subspaces.

LEMMA 3.6. *Let  $j$  be a fundamental symmetry of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . Then  $\mathfrak{L}$  is a (closed) neutral or hyper-maximal neutral subspace of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  if and only if  $A_{\mathfrak{L}}$  as in (3.4) is a (closed) symmetric or selfadjoint relation in  $\{\mathfrak{M}, [j\cdot, \cdot]\}$ , respectively.*

*Proof.* Let  $\mathfrak{L}$  be a subspace and let  $A_{\mathfrak{L}}$  be as in (3.4). Then  $g + ijg' \in \mathfrak{L}^{[\perp]}$ ,  $g, g' \in \mathfrak{M}$ , if and only if

$$0 = [f + jif', g + jig'] = [f, jig'] + [jif', g] = i([jf', g] - [jf, g']), \quad \forall (f + jif) \in \mathfrak{L}.$$

Since  $(f + jif) \in \mathfrak{L}$  if and only if  $\{f, f'\} \in \text{gr}A_{\mathfrak{L}}$ , the above calculation shows that  $g + ijg' \in \mathfrak{L}^{[\perp]}$  if and only if  $\{g, g'\} \in \text{gr}A_{\mathfrak{L}}^*$ . From this observation the statement follows.  $\square$

Let  $\mathfrak{L}$  be a hyper-maximal neutral subspace of  $\{\mathfrak{K}, [\cdot, \cdot]\}$ , then Lemma 3.6 and [14, Theorem 1.1] imply that there exists a selfadjoint relation  $K$  in the Hilbert space  $\{\mathfrak{M}, [j\cdot, \cdot]\}$  and a closed operator  $B$  in  $\{\mathfrak{M}, [j\cdot, \cdot]\}$  with  $\text{dom}B = \mathfrak{M}$ ,  $\text{ker}B = \{0\}$  and  $\text{ran}B = \text{dom}K \oplus \text{mul}K$ , respectively, such that

$$\mathfrak{L} = \{\mathcal{P}_K Bf + j(i\mathcal{P}_K K Bf + (I - \mathcal{P}_K)Bf) : f \in \mathfrak{M}\}.$$

Here  $\mathcal{P}_K$  is the orthogonal projection onto  $\overline{\text{dom}K} = (\text{mul}K)^{\perp}$  in  $\{\mathfrak{M}, [j\cdot, \cdot]\}$ .

Using the above discussion, standard unitary operators can almost be decomposed in terms of (unbounded) archetypical unitary operators introduced in the previous subsection. In particular, Proposition 3.7 together with Theorem 4.8 below shows that to investigate compositions of unitary operators, it suffices to consider compositions of archetypical unitary operators.

PROPOSITION 3.7. *Let  $U$  be an isometric operator in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  with fundamental symmetry  $j$  and assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . Then  $U$  is a standard unitary operator if and only if there exist a closed*

subspace  $\mathfrak{N}$  of  $\mathfrak{M}$ , selfadjoint operators  $K_1$  and  $K_2$  in the Hilbert space  $\{\mathfrak{M}, [j, \cdot]\}$  with  $\text{dom}K_2 = \mathfrak{M}$  and  $\text{clos}(K_2^{-1} - K_1)$  being a selfadjoint relation in  $\{\mathfrak{M}, [j, \cdot]\}$ , a closed operator  $B$  in  $\{\mathfrak{M}, [j, \cdot]\}$  satisfying  $\text{dom}B = \mathfrak{M}$ ,  $\text{ran}B = \text{dom}K_1$ ,  $\text{ker}B = \{0\}$ ,  $\text{dom}\text{clos}(K_2B^{-*}) = \mathfrak{M}$ ,  $\text{mul}\text{clos}(K_2B^{-*}) = \{0\}$  and  $\text{ran}\text{clos}(K_2B^{-*}) = \text{dom}\text{clos}(K_1^{-1} - K_2)$  such that

$$U_{\mathfrak{N}}^{-1}U = \text{clos}(\Upsilon_1(K_1)j\Upsilon_1(K_2)j\Upsilon_2(B)). \tag{3.5}$$

Here, with  $\mathcal{P}_{\mathfrak{N}}$  the orthogonal projection onto  $\mathfrak{N}$  in  $\{\mathfrak{M}, [j, \cdot]\}$ ,  $U_{\mathfrak{N}}$  is the standard unitary operator in  $\{\mathfrak{K}, [j, \cdot]\}$  defined as

$$U_{\mathfrak{N}}(f + jf') = \mathcal{P}_{\mathfrak{N}}f + (I - \mathcal{P}_{\mathfrak{N}})f' + j((I - \mathcal{P}_{\mathfrak{N}})f + \mathcal{P}_{\mathfrak{N}}f'), \quad f, f' \in \mathfrak{M}.$$

*Proof.* If  $U$  is a standard unitary operator, then  $\mathfrak{M} \subseteq \text{dom}U$  and  $U(\mathfrak{M})$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}, [j, \cdot]\}$ . Hence, by the discussion preceding this statement, there exists a selfadjoint relation  $K$  in  $\{\mathfrak{M}, [j, \cdot]\}$  and a closed operator  $B$  in  $\{\mathfrak{M}, [j, \cdot]\}$  with  $\text{dom}B = \mathfrak{M}$ ,  $\text{ker}B = \{0\}$  and  $\text{ran}B = \text{dom}K \oplus \text{mul}B$  such that with  $\mathcal{P}_K$  the orthogonal projection onto  $\overline{\text{dom}K}$  in  $\{\mathfrak{M}, [j, \cdot]\}$

$$U \upharpoonright_{\mathfrak{M}} = \left( j(i\mathcal{P}_KKB + (I - \mathcal{P}_K)B) \right) = U_{\mathfrak{N}} \left( j \begin{pmatrix} B \\ \mathcal{P}_KKB \end{pmatrix} \right),$$

where  $\mathfrak{N} = \overline{\text{dom}K}$  and the block decomposition on the range is w.r.t. the decomposition  $\mathfrak{M} \oplus j\mathfrak{M}$  of  $\mathfrak{K}$ . Note that  $K_1 := \mathcal{P}_K K \oplus 0_{\text{mul}K}$  is a selfadjoint operator in  $\{\mathfrak{M}, [j, \cdot]\}$ . These observations show that there exist operators  $C$  and  $D$  in  $\mathfrak{M}$  with  $\text{dom}C = \mathfrak{M} = \text{dom}D$  such that w.r.t. decomposition  $\mathfrak{M} \oplus j\mathfrak{M}$  of  $\mathfrak{K}$ :

$$U_{\mathfrak{N}}^{-1}U = \begin{pmatrix} B & iCj \\ jiK_1B & jDj \end{pmatrix}. \tag{3.6}$$

Note that  $U_{\mathfrak{N}}^{-1}U$  being a standard unitary operator is bounded. Hence, (3.6) implies that  $C$  and  $D$  are also bounded. Since  $\text{dom}C = \mathfrak{M} = \text{dom}D$ , this implies that  $C$  and  $D$  are closed operators.

Since  $(\Upsilon_1(K_1))^{-1} = \Upsilon_1(-K_1)$ , it follows that

$$(\Upsilon_1(K_1))^{-1}U_{\mathfrak{N}}^{-1}U = \Upsilon_1(-K_1) \begin{pmatrix} B & iCj \\ jiK_1B & jDj \end{pmatrix} = \begin{pmatrix} B & iCj \\ 0 & j(D + K_1C)j \end{pmatrix}. \tag{3.7}$$

Since  $U_{\mathfrak{N}}$  and  $U$  are standard unitary operators and  $(\Upsilon_1(K_1))^{-1}$  is a unitary operator, the righthand side of (3.7) is also a unitary operator, cf. Lemma 2.1. The isometry of that operator implies that  $(D + K_1C) \subseteq B^{-*}$  and the fact that  $j\mathfrak{M} \subseteq \text{ran}((\Upsilon_1(K_1))^{-1}U_{\mathfrak{N}}^{-1}U) = \text{dom}\Upsilon_1(K_1)$  implies that  $\text{ran}(D + K_1C) = \mathfrak{M}$ . Since  $\text{ker}B^{-*} = (\text{dom}B)^{\perp} = \{0\}$ , the preceding observations imply that  $(D + K_1C) = B^{-*}$ , see (2.4). Hence,  $\text{dom}B^* = \text{ran}B^{-*} = \mathfrak{M}$  and

$$\begin{pmatrix} B & iCj \\ 0 & j(D + K_1C)j \end{pmatrix} = \begin{pmatrix} I & iCB^*j \\ 0 & I \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & jB^{-*}j \end{pmatrix} = j\Upsilon_1(CB^*)j\Upsilon_2(B). \tag{3.8}$$

Since  $B$  is a closed operator satisfying  $\text{dom} B = \mathfrak{M} = \overline{\text{ran}} B$  and  $\text{ker} B = \{0\}$ ,  $\Upsilon_2(B)$  is a unitary operator with a trivial kernel. Hence, (3.8) implies that  $\Upsilon_1(CB^{-*})$  is isometric and, hence,  $K_2 := CB^*$  is a symmetric operator, see Proposition 3.2. Since  $\text{dom} B^* = \mathfrak{M}$  and  $\text{dom} C = \mathfrak{M}$ ,  $K_2$  is in fact an everywhere defined symmetric operator, i.e.,  $K_2$  is a (bounded) selfadjoint operator in  $\{\mathfrak{M}, [j, \cdot]\}$ . Combining (3.7) and (3.8) yields that  $U_t := I_{\text{ran } \Upsilon_1(K_1)} U_{\mathfrak{N}}^{-1} U$  can be written as

$$U_t = \Upsilon_1(K_1)j\Upsilon_1(K_2)j\Upsilon_2(B) = \begin{pmatrix} B & iK_2B^{-*}j \\ jiK_1Bj & (I - K_1K_2)B^{-*}j \end{pmatrix}. \tag{3.9}$$

Since  $\overline{\text{ran}}(\Upsilon_1(K_1)) = \mathfrak{K}$ , the closure of  $U_t$  coincides with  $U_{\mathfrak{N}}^{-1}U$  showing that (3.5) holds. As a consequence of (3.6), (3.9) and the proven closedness of  $C$ ,  $\text{clos}(K_2B^{-*}) = C$  which yields  $\text{dom} \text{clos}(K_2B^{-*}) = \mathfrak{M}$  and  $\text{mul} \text{clos}(K_2B^{-*}) = \{0\}$ . Moreover, since  $\text{clos}(U_t)$  is a standard unitary operator and  $\text{clos}(\text{dom} U_t \cap j\mathfrak{M}) = j\mathfrak{M}$ ,  $\text{clos}(U_t(\text{dom} U_t \cap j\mathfrak{M})) = \text{clos}(\{if + j(K_2^{-1} - K_1)f : f \in \text{ran}(K_2B^{-*})\})$  is a hyper-maximal neutral subspace. Hence, Proposition 3.6 implies that  $\text{clos}(K_2^{-1} - K_1)$  is a selfadjoint relation and that  $\text{dom} \text{clos}(K_2^{-1} - K_1) \subseteq \text{ran} \text{clos}((K_2B^{-*}))$ . Finally, from  $\overline{\text{dom}} U_t = \mathfrak{K}$ ,  $\text{dom}(\text{clos}(K_2B^{-*})) = \mathfrak{M}$  and (3.9), it follows that  $\text{dom} \text{clos}(K_2^{-1} - K_1) = \text{ran}(\text{clos}(K_2B^{-*}))$ .

Conversely, the assumptions imply that the closure of the righthand side of (3.9) is an everywhere defined isometric operator with dense range, i.e.  $U_{\mathfrak{N}}^{-1}U$  and, hence, also  $U$  is a standard unitary operator.  $\square$

For technical purposes the following property of standard unitary operators will be useful later.

**LEMMA 3.8.** *Let  $\{\mathfrak{K}, [\cdot, \cdot]\}$  be a Kreĭn space with fundamental symmetries  $j$  and  $j'$ , and assume that  $\mathfrak{M}$  and  $\mathfrak{M}'$  are hyper-maximal neutral subspaces in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . Then there exists a standard unitary operator  $U$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  such that  $U(\mathfrak{M}) = \mathfrak{M}'$  and  $U(j\mathfrak{M}) = j'\mathfrak{M}'$ .*

*Proof.* If the assumptions hold, then, clearly,  $\{\mathfrak{M}, [j, \cdot]\}$  and  $\{\mathfrak{M}', [j', \cdot]\}$  are Hilbert spaces of equal dimension. Let  $U_t$  be a (standard) unitary operator between these Hilbert spaces, then  $U$  defined by

$$U(f_0 + jf_1) = U_t f_0 + j'U_t f_1, \quad f_0, f_1 \in \mathfrak{M},$$

is a standard unitary operator which has the stated properties.  $\square$

### 3.3. Compositions of archetypical unitary operators

Let  $j$  be a fundamental symmetry of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . If  $K_1$  and  $K_2$  are selfadjoint relations in (the Hilbert space)  $\{\mathfrak{M}, [j, \cdot]\}$ , then

$$\Upsilon_1(K_1)\Upsilon_1(K_2) = \Upsilon_1(K_1 + K_2),$$

cf. [12, Example 2.11]. This composition is (extendable to) a unitary relation if and only if  $K_1 + K_2$  is (extendable to) a selfadjoint relation, see Proposition 3.2. Example 3.9 below provides an example of two selfadjoint operators  $K_1$  and  $K_2$  such that their sum cannot be extended to a selfadjoint relation, i.e.,  $\Upsilon_1(K_1 + K_2)$  can not be extended to a unitary relation.

EXAMPLE 3.9. In the Hilbert space  $L^2(\mathbb{R}_+)$  consider the differential expressions  $\ell_1 f = -f'' - 2if' - f$  and  $\ell_2 f = f'' + f$ . Both expressions can be interpreted as canonical differential systems which are definite on  $\mathbb{R}_+$ , see e.g. [5]. With

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Delta(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad H_1(t) = \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix}, \quad H_2(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

these systems are

$$\mathcal{J}F'(t) - H_i(t)F(t) = \lambda\Delta(t)F(t), \quad t \in \mathbb{R}_+ \text{ a.e., } \lambda \in \mathbb{C},$$

where  $F = (f_1, f_2)^T$  and  $i = 1, 2$ . With  $L^2_\Delta(\mathbb{R}_+)$  the Hilbert space (of equivalence classes) associated with  $\Delta$ , the minimal relations generated by the above canonical systems are symmetric operators in  $L^2_\Delta(\mathbb{R}_+)$  with defect numbers  $(1, 1)$ , which follows e.g. from [15, Proposition 5.25] together with the definiteness of the systems. In particular, for both systems 0 is a regular endpoint and  $\infty$  is an endpoint in the limit-point case. Therefore, properly understood,  $K_1$  and  $K_2$  defined by

$$\text{gr } K_i = \{ \{F, G\} \in L^2_\Delta(\mathbb{R}_+) \times L^2_\Delta(\mathbb{R}_+) : \ell_i f_1 = g_1, f_1(0) = 0 \}, \quad i = 1, 2,$$

where  $F = (f_1, f_2)^T$  and  $G = (g_1, g_2)^T$ , are selfadjoint operators in the Hilbert space  $L^2_\Delta(\mathbb{R}_+)$ , see [5, Section 4.1 and 5.3]. Moreover,  $\text{dom } K_2 \subseteq \text{dom } K_1$  and, hence, the sum of  $K_1$  and  $K_2$  is the symmetric operator  $S$ :

$$\text{gr } S = \{ \{F, G\} \in L^2_\Delta(\mathbb{R}_+) \times L^2_\Delta(\mathbb{R}_+) : F \in \text{dom } K_2, \ell_S f_1 = g_1, f_1(0) = 0 \},$$

where  $\ell_S f_1 = -2if'_1$ ,  $F = (f_1, f_2)^T$  and  $G = (g_1, g_2)^T$ . Hence, the closure of  $S$  is a well-known symmetric operator with defect numbers  $n_+(S) = 0$  and  $n_-(S) = 1$  corresponding to  $\ell_S$ .

In Example 3.10 below the selfadjoint operators from Example 3.9 are used to show that there exist unitary operators which map hyper-maximal neutral subspaces onto non-closed neutral subspaces which can not be extended to hyper-maximal neutral subspaces, cf. Proposition 3.4.

EXAMPLE 3.10. Let  $K_1$  and  $K_2$  be the selfadjoint operators in  $\mathfrak{H} := L^2_\Delta(\mathbb{R}_+)$  as in Example 3.9, moreover, define  $\mathfrak{j}$  and  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{K} := \mathfrak{H} \times \mathfrak{H}$  by

$$\mathfrak{j}\{f_0, f_1\} = i\{-f_1, f_0\} \quad \text{and} \quad \langle \{f_0, f_1\}, \{g_0, g_1\} \rangle = i[(f_0, g_1)_{\mathfrak{H}} - (f_1, g_0)_{\mathfrak{H}}].$$

Then  $\{\mathfrak{K}, \langle \cdot, \cdot \rangle\}$  is a Kreĭn space,  $\mathfrak{j}$  is a fundamental symmetry for  $\{\mathfrak{K}, \langle \cdot, \cdot \rangle\}$ ,  $\mathfrak{M} := \mathfrak{H} \times 0$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}, \langle \cdot, \cdot \rangle\}$ , and  $K_1$  and  $K_2$  can

be interpreted as selfadjoint operators in  $\{\mathfrak{M}, \langle j, \cdot, \cdot \rangle\}$ . Now  $\Upsilon_1(K_1)$  is a unitary operator in  $\{\mathfrak{K}, \langle \cdot, \cdot \rangle\}$  and  $\mathfrak{L} := \{\{f, K_2 f\} : f \in \text{dom} K_2\} = \text{gr} K_2$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}, \langle \cdot, \cdot \rangle\}$  such that  $\mathfrak{L} \subseteq \text{dom} \Upsilon_1(K_1) = \text{dom} K_1 \oplus j\mathfrak{M}$ , because  $\text{dom} K_2 \subseteq \text{dom} K_1$ . Moreover,  $\Upsilon_1(K_1)\mathfrak{L} = \text{gr}(K_1 + K_2)$  is a (non-closed) neutral subspace which can not be extended to a hyper-maximal neutral subspace, because  $K_1 + K_2$  is a symmetric operator which can not be extended to a selfadjoint operator, see Example 3.9 and Lemma 3.6.

Example 3.9 can also be used to show that there exists isometric operators which cannot be extended to unitary relations such that the closure of their composition with a unitary relation is (extendable to) a unitary relation. Another example of this is obtained by considering the composition of  $\Upsilon_1(K)$  and  $\Upsilon_1(S)$ , where  $K$  is a selfadjoint operator in  $\{\mathfrak{M}, [j, \cdot, \cdot]\}$  and  $S$  is a symmetric operator with unequal defect numbers in  $\{\mathfrak{M}, [j, \cdot, \cdot]\}$  such that  $\text{dom} S \cap \text{dom} K = \{0\}$ , cf. [14, Theorem 3.6]. Then, clearly,  $\Upsilon_1(K)\Upsilon_1(S) = I_{j\mathfrak{M}}$  can be extended to a unitary operator.

Different from the composition of the triangular archetypical unitary operators considered above, the composition of two archetypical unitary operators which have a diagonal representation (in the same coordinates), can always be extended to a unitary relation: Let  $B_1$  and  $B_2$  be two closed operators (or relations), then

$$\Upsilon_2(B_1)\Upsilon_2(B_2) = \begin{pmatrix} B_1 B_2 & 0 \\ 0 & j B_1^{-*} B_2^{-*} j \end{pmatrix} \subseteq \Upsilon_2(\text{clos}(B_1 B_2)).$$

Here it is used that  $B_1^{-*} B_2^{-*} \subseteq (B_1 B_2)^{-*}$ , see Lemma 2.1.

Next the composition of the different types of archetypical unitary relations is considered, i.e. compositions of the type  $\Upsilon_1(S)\Upsilon_2(B)$ . The following two statements give some conditions for when this composition is unitary.

**PROPOSITION 3.11.** *Let  $j$  be a fundamental symmetry of  $\{\mathfrak{K}, [ \cdot, \cdot ]\}$  and assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  in  $\{\mathfrak{K}, [ \cdot, \cdot ]\}$ . Moreover, let  $B$  be an operator in (the Hilbert space)  $\{\mathfrak{M}, [j, \cdot, \cdot]\}$  with  $\overline{\text{dom} B} = \mathfrak{M} = \text{ran clos}(B)$  and  $\ker \text{clos}(B) = \{0\}$ , and let  $S$  be a symmetric relation in  $\{\mathfrak{M}, [j, \cdot, \cdot]\}$ . Then  $\Upsilon_1(S)\Upsilon_2(\text{clos}(B))$  is (extendable to) a unitary relation in  $\{\mathfrak{K}, [ \cdot, \cdot ]\}$  if and only if  $S$  is (extendable to) a selfadjoint relation in  $\{\mathfrak{M}, [j, \cdot, \cdot]\}$ .*

*In particular,  $\Upsilon_1(S)\Upsilon_2(\text{clos} B)$  is a unitary operator if and only if  $S$  is a selfadjoint operator in  $\{\mathfrak{M}, [j, \cdot, \cdot]\}$  with  $\text{dom} S \cap \text{mul clos}(B) = \{0\}$ .*

*Proof.* Since the last equivalence is clear, it suffices to prove the first equivalence. Therefore note that if  $T$  is a symmetric extension of  $S$ , then  $\Upsilon_1(T)\Upsilon_2(\text{clos} B)$  is an isometric extension of  $\Upsilon_1(S)\Upsilon_2(\text{clos} B)$ . Hence, to prove the first equivalence it suffices to show that  $\Upsilon_1(S)\Upsilon_2(\text{clos} B)$  is unitary if and only if  $S$  is selfadjoint.

If  $S$  is selfadjoint, then the fact that  $\Upsilon_1(S)\Upsilon_2(\text{clos} B)$  is unitary follows from Lemma 3.1 as in Proposition 3.2. To prove the converse assume that  $S$  is a maximal symmetric relation which is not selfadjoint, and that  $\Upsilon_1(S)\Upsilon_2(\text{clos} B)$  is unitary. Then there exists  $\{f, f'\} \in \text{gr} S^*$  such that  $\text{Im}[f, f'] \neq 0$  and by the assumptions on  $B$  there exists a  $h \in \text{dom clos}(B)$  such that  $\{h, f\} \in \text{gr}(\text{clos} B)$ . Now a direct calculation shows

that  $[h, g] = [f + jif', g']$  for all  $\{g, g'\} \in \text{gr}U$ , i.e.  $\{h, f + jif'\} \in \text{gr}U$  by (2.6). On the other hand,  $[h, h] = 0$ , because  $h \in \mathfrak{M}$ , and  $[f + jif', f + jif'] = i([jf', f] - [f, jf']) \neq 0$ , by the assumption on  $\{f, f'\}$ . This shows that  $\{h, f + jif'\}$  cannot be contained in the graph of an isometric relation. This contradiction completes the proof.  $\square$

**COROLLARY 3.12.** *Let  $j$  be a fundamental symmetry of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . Moreover, let  $B$  be a closed operator in (the Hilbert space)  $\{\mathfrak{M}, [j, \cdot, \cdot]\}$  with  $\overline{\text{dom}B} = \mathfrak{M} = \text{ran}B$  and  $\ker B = \{0\}$ , and let  $S$  be a symmetric operator in  $\{\mathfrak{M}, [j, \cdot, \cdot]\}$ . Then  $\Upsilon_1(S)\Upsilon_2(B^{-1})$  is (extendable to) a unitary operator in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  if and only if  $B^{-*}SB^{-1}$  is (extendable to) a selfadjoint operator in  $\{\mathfrak{M}, [j, \cdot, \cdot]\}$ .*

*Proof.* Note that

$$\begin{aligned} \Upsilon_1(S)\Upsilon_2(B^{-1}) &= \begin{pmatrix} B^{-1} & 0 \\ j i S B^{-1} & j B^* j \end{pmatrix} = \begin{pmatrix} B^{-1} & 0 \\ 0 & j B^* j \end{pmatrix} \begin{pmatrix} I & 0 \\ j i B^{-*} S B^{-1} & I \end{pmatrix} \\ &= \Upsilon_2(B^{-1})\Upsilon_1(B^{-*}SB^{-1}) = (\Upsilon_1(-B^{-*}SB^{-1})\Upsilon_2(B))^{-1}. \end{aligned}$$

Here the second equality holds, because the assumptions on  $B$  imply that  $\text{ran}B^* = \mathfrak{M}$ . Since an isometric relation is unitary if and only if its inverse is unitary, the above equality together with Proposition 3.11 shows that the statement holds.  $\square$

Example 3.13 below shows that  $S$  in Corollary 3.12 need not be a selfadjoint operator nor even a symmetric operator with equal defect numbers for  $B^{-*}SB^{-1}$  to be selfadjoint and, hence,  $\Upsilon_1(S)\Upsilon_2(B^{-1})$  to be unitary.

**EXAMPLE 3.13.** Let  $j$  be a fundamental symmetry of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . Moreover, let  $S$  be a closed symmetric operator in the Hilbert space  $\{\mathfrak{M}, [j, \cdot, \cdot]\}$  with  $\overline{\text{dom}S} = \mathfrak{M}$  and defect numbers  $n_{\pm}(S) = n_{\pm}$ , where  $n_{\pm} \leq \aleph_0$ , and let  $B$  be a closed operator in  $\{\mathfrak{M}, [j, \cdot, \cdot]\}$  with  $\text{dom}B = \text{dom}S$ ,  $\ker B = \{0\}$  and  $\text{ran}B = \mathfrak{M}$ , see [14]. Then  $K := B^{-*}SB^{-1}$  is a symmetric operator with  $\text{dom}K = \mathfrak{M}$ , i.e.  $K$  is a selfadjoint operator in  $\{\mathfrak{M}, [j, \cdot, \cdot]\}$ .

**REMARK 3.14.** Note that if  $S$  and  $B$  are as in Example 3.13, then the unitary operator  $\Upsilon_1(S)\Upsilon_2(B^{-1})$  maps the hyper-maximal neutral subspace  $\mathfrak{M}$  onto the closed neutral subspace  $\{f + jSf : f \in \text{dom}S\}$  with defect numbers  $n_+$  and  $n_-$ . Hence, unitary relations may map hyper-maximal neutral subspaces onto closed neutral subspaces with nonzero defect numbers ( $\leq \aleph_0$ ).

### 4. Unitary relations and their block decompositions

Isometric operators containing a hyper-maximal semi-definite subspace in their domain are here studied. Such isometric operators are shown to be representable by compositions of archetypical isometric operators. Furthermore, necessary and sufficient conditions for isometric operators to be unitary are presented and it is shown that,

w.r.t. suitable coordinates, every unitary operator can be written as the composition of either of the archetypical unitary operators with a bounded unitary operator. Finally, the obtained characterizations of unitary operators are used to give conditions for when the composition of unitary operators is (extendable to) a unitary operator.

**4.1. Isometric operators and hyper-maximal subspaces**

Let  $j_2$  be a fundamental symmetry for  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ . Then here isometric operators  $V$  from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  are studied for which there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that  $V^{-1}(j_2\mathfrak{M} \cap \text{ran} V)$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$ . In other words, isometric operators are studied which have a hyper-maximal neutral subspace in their domain and map it onto a neutral subspace which can be extended to a hyper-maximal neutral subspace. Note that unitary relations essentially satisfy the preceding condition, see Theorem 4.8 below.

Theorem 4.2 below furnishes a block representation for isometric operators which satisfy the above condition and have a dense range. To prove that statement the following lemma will be used.

LEMMA 4.1. *Let  $V$  be an isometric operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\overline{\text{ran}} V = \mathfrak{K}_2$  and assume that there exists a hyper-maximal neutral subspace  $\mathcal{L}$  in  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  with  $\mathcal{L} \subseteq \text{dom} V$ . Then there exists a bounded unitary operator  $U_t$  from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  onto  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\text{dom} V \subseteq \text{dom} U_t$  such that  $VU_t^{-1}$  is an isometric operator in  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\overline{\text{dom}}(VU_t^{-1}) = \mathfrak{K}_2 = \overline{\text{ran}}(VU_t^{-1})$ .*

*In particular, if  $\mathfrak{M}$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and  $j_1$  and  $j_2$  are fundamental symmetries of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  and  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ , respectively, then  $U_t$  can be taken such that  $U_t(\mathcal{L}) = \mathfrak{M}$  and  $U_t(j_1\mathcal{L} \cap \text{dom} U_t) = j_2\mathfrak{M}$ .*

*Proof.* It is a direct consequence of the assumptions that  $\ker V = (\text{dom} V)^{[\perp]_1}$ , see [17, Corollary 3.8]. Hence, with  $\mathfrak{K}_3 := (\overline{\text{dom}} V)/\ker V$ ,  $U_1 : \overline{\text{dom}} V \subseteq \mathfrak{K}_1 \mapsto \mathfrak{K}_3$ ,  $f \mapsto f + [\ker V]$ , is a bounded unitary operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  onto the Krein space  $\{\mathfrak{K}_3, [\cdot, \cdot]_1\}$ , see [17, Lemma 3.10]. Therefore  $VU_1^{-1}$  is an isometric operator from  $\{\mathfrak{K}_3, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  which satisfies  $\overline{\text{dom}}(VU_1^{-1}) = \mathfrak{K}_3$  and  $\overline{\text{ran}}(VU_1^{-1}) = \mathfrak{K}_2$ .

Since  $\mathcal{L}$  is a hyper-maximal neutral subspace and  $U_1$  is a bounded unitary operator,  $U_1(\mathcal{L})$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_3, [\cdot, \cdot]_1\}$ . In particular,  $k_3^+ = k_3^-$ , see e.g. [3, Ch. I: Remark 4.16]. Since  $VU_1^{-1}$  maps  $U_1(\mathcal{L})$  injectively onto a neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ ,  $k_3^\pm \leq k_2^\pm$ . Moreover, the fact that  $VU_1^{-1}$  is an injective operator together with  $\overline{\text{dom}}(VU_1^{-1}) = \mathfrak{K}_3$  and  $\overline{\text{ran}}(VU_1^{-1}) = \mathfrak{K}_2$  yields that  $k_3^+ + k_3^- = k_2^+ + k_2^-$ . The preceding arguments together show that  $k_3^\pm = k_2^\pm$ . Therefore there exists a standard unitary operator  $U_2$  from  $\{\mathfrak{K}_3, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and the first statement holds with  $U_t := U_2U_1$ .

Since  $U_2U_1$  is a bounded unitary operator,  $U_2U_1(\mathcal{L})$  and  $U_2U_1(j_1\mathcal{L} \cap \text{dom} U_1)$  are hyper-maximal neutral subspaces of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and there exists a fundamental symmetry  $j_2'$  of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that  $U_2U_1(j_1\mathcal{L} \cap \text{dom} V) = j_2'U_2U_1(\mathcal{L})$ . Therefore, by Lemma 3.8, there exists a standard unitary operator  $U_3$  in  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that

$U_3(U_2U_1(\mathfrak{L})) = \mathfrak{M}$  and  $U_3(U_2U_1(j_1\mathfrak{L} \cap \text{dom}U_1)) = U_3(j_2'U_2U_1(\mathfrak{L})) = j_2\mathfrak{M}$ . Hence, the final statement holds with  $U_t := U_3U_2U_1$   $\square$

Theorem 4.2 below contains a representation for the isometric operators  $V$  for which  $V^{-1}(j_2\mathfrak{M} \cap \text{ran}V)$  is a hyper-maximal neutral subspace. It is shown that such operators have, up to a bounded unitary transformation, a triangular representation which can be expressed in terms of archetypical isometric operators. Note that the isometric operators considered in Theorem 4.2 below are a coordinate free version of quasi-boundary triplets, see Definition 2.3. To better see this connection, note that  $V^{-1}(j_2\mathfrak{M} \cap \text{ran}V) = \ker(\mathcal{P}_{\mathfrak{M}}V)$ , where  $\mathcal{P}_{\mathfrak{M}}$  is the orthogonal projection onto  $\mathfrak{M}$  w.r.t.  $[\cdot, \cdot]_2$ .

**THEOREM 4.2.** *Let  $V$  be an isometric operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\overline{\text{ran}V} = \mathfrak{K}_2$ , let  $j_2$  be a fundamental symmetry of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and, moreover, assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  in  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that  $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}}V)$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$ . Then there exists an operator  $B$  in the Hilbert space  $\{\mathfrak{M}, [\cdot, \cdot]_2\}$  with  $\overline{\text{dom}B} = \mathfrak{M} = \text{ranclos}(B)$  and  $\ker\text{clos}(B) = \{0\}$ , a symmetric operator  $S$  in  $\{\mathfrak{M}, [\cdot, \cdot]_2\}$  with  $\text{dom}S = \text{ran}B$  and  $\text{dom}S^* \cap \text{mulclos}(B) = \{0\}$ , and a bounded unitary operator  $U_t$  from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  onto  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\text{dom}V \subseteq \text{dom}U_t$ , mapping  $\mathfrak{L}$  onto  $j_2\mathfrak{M}$ , such that*

$$VU_t^{-1} = \begin{pmatrix} B & 0 \\ j_2iSB & j_2B^{-*}j_2 \end{pmatrix} = \Upsilon_1(S)\Upsilon_2(B). \tag{4.1}$$

Furthermore,  $\text{mulclos}(B) = \{0\}$  if and only if  $\text{clos}(V(\mathfrak{L})) = j_2\mathfrak{M}$ .

*Proof.* Note first that if (4.1) holds, then  $j_2V(\mathfrak{L}) = \text{dom}B^*$ . This together with  $\overline{\text{dom}B^*} = (\text{mulclos}(B))^\perp$ , see (2.3), shows that the final assertion holds. Next note that Lemma 4.1 implies the existence of a bounded unitary operator  $U_t$  as in the statement. Then  $W := VU_t^{-1}$  is an isometric operator in  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\overline{\text{dom}W} = \mathfrak{K}_2 = \overline{\text{ran}W}$ ,  $j_2\mathfrak{M} \subseteq \text{dom}W$  and  $W(j_2\mathfrak{M}) = V(\mathfrak{L}) \subseteq j_2\mathfrak{M}$ .

*Step 1:* Since  $j_2\mathfrak{M} \subseteq \text{dom}W$  and  $W(j_2\mathfrak{M}) \subseteq j_2\mathfrak{M}$ ,  $W$  has w.r.t. the decomposition  $\mathfrak{M} \oplus j_2\mathfrak{M}$  of  $\mathfrak{K}_2$  the following block representation:

$$W = \begin{pmatrix} B & 0 \\ j_2iC & j_2Dj_2 \end{pmatrix},$$

where  $B, C$  and  $D$  are operators in (the Hilbert space)  $\{\mathfrak{M}, [\cdot, \cdot]_2\}$  which satisfy  $\text{dom}D = \mathfrak{M}$ ,  $\ker D = \{0\}$  and  $\text{dom}B = \text{dom}C$ . Direct calculations shows that the fact that  $W$  is isometric implies that  $D \subseteq B^{-*}$  and that  $C = SB$  for a symmetric operator  $S$  with  $\text{dom}S = \text{ran}B$ , cf. Proposition 3.7.

*Step 2:* Next observe that  $\overline{\text{dom}B} = \mathfrak{M}$  and  $\overline{\text{ran}B} = \mathfrak{M}$ , because  $\overline{\text{dom}W} = \mathfrak{K}_2 = \overline{\text{ran}W}$ . Since  $\text{dom}D = \mathfrak{M}$ , see Step 1, and  $\text{mul}B^{-*} = (\text{ran}B)^\perp = \{0\}$ , equality must hold in the inclusion  $D \subseteq B^{-*}$  by (2.4):  $D = B^{-*}$ . Consequently,  $\text{ran}B^* = \text{dom}D = \mathfrak{M}$  and combining this with  $\overline{\text{ran}B} = \mathfrak{M}$  yields  $\text{ranclos}(B) = \mathfrak{M}$ . Moreover,  $\text{ran}B^* = \mathfrak{M}$  also yields  $\ker\text{clos}(B) = \{0\}$ , see (2.3).

*Step 3:* The arguments from step 1 and step 2 show that the asserted representation for  $W = VU_t^{-1}$  holds. Therefore  $\text{ran } V = \{f + j_2 i S f : f \in \text{dom } S\} + j_2 \text{dom } B^*$ . Since  $\overline{\text{ran } V} = \mathfrak{K}_2$ , it now follows that

$$\begin{aligned} \{0\} &= (\text{ran } V)^{\perp\perp} = \{f + j_2 i S f : f \in \text{dom } S\}^{\perp\perp} \cap (j_2 \text{dom } B^*)^{\perp\perp} \\ &= \{f + j_2 i S^* f : f \in \text{dom } S^*\} \cap (\text{mulclos}(B) \oplus j\mathfrak{M}), \end{aligned}$$

i.e.  $\text{dom } S^* \cap \text{mulclos}(B) = \{0\}$ . This completes the proof.  $\square$

REMARK 4.3. (i): Let  $j_1$  be any fundamental symmetry of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$ . Then note that  $U_t$  in Theorem 4.2 could have been chosen such that additionally  $U_t(j_1 \mathfrak{L} \cap \text{dom } U_t) = \mathfrak{M}$ , see Lemma 4.1. With that choice of  $U_t$ , (4.1) yields

$$V(j_1 \mathfrak{L} \cap \text{dom } V) = VU_t^{-1}(\mathfrak{M} \cap \text{dom } VU_t^{-1}) = \{f + j_2 i S f : f \in \text{dom } S\}.$$

In view of Proposition 3.11 and 3.6, this shows that the isometric operator in Theorem 4.2 is unitary if and only if  $S$  is a selfadjoint operator or, equivalently, if and only if  $V(j_1 \mathfrak{L} \cap \text{dom } V)$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ .

(ii): Using [17, Corollary 3.13], Theorem 4.2 can be extended to the case that  $\mathfrak{M}$  is a hyper-maximal semi-definite subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that  $\mathfrak{M} \cap j_2 \mathfrak{M} \subseteq \text{ran } V$  and  $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}^{\perp\perp}} V) = V^{-1}(j_2 \mathfrak{M} \cap \text{ran } V)$  is a hyper-maximal semi-definite subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$ . Namely in that case there exist  $S$  and  $B$  as in Theorem 4.2 (with  $\mathfrak{M}$  therein replaced by  $\mathfrak{M}^{\perp\perp}$ ) and a bounded unitary operator  $U_t$  from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\text{dom } V \subseteq \text{dom } U_t$ , mapping  $\mathfrak{L}$  onto  $j_2 \mathfrak{M}$ , such that w.r.t. the decomposition  $\mathfrak{M}^{\perp\perp} \oplus j_2 \mathfrak{M}^{\perp\perp} \oplus (j_2 \mathfrak{M} \cap \mathfrak{M})$  of  $\mathfrak{K}$

$$VU_t^{-1} = \begin{pmatrix} B & 0 & 0 \\ j_2 i S B & j_2 B^{-*} j_2 & 0 \\ 0 & 0 & I_{\mathfrak{M} \cap j_2 \mathfrak{M}} \end{pmatrix} = \Upsilon_1(S) \Upsilon_2(B) \oplus I_{\mathfrak{M} \cap j_2 \mathfrak{M}}.$$

Next two consequences of Theorem 4.2 are stated: The first shows that isometric operators as in Theorem 4.2 are closely connected to unitary relations and the second shows how the representation in Theorem 4.2 simplifies if it is assumed that  $V$  maps  $\ker(\mathcal{P}_{\mathfrak{M}} V)$  onto a hyper-maximal neutral subspace.

COROLLARY 4.4. *Let  $V$  be an isometric operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\overline{\text{ran } V} = \mathfrak{K}_2$ , let  $j_2$  be a fundamental symmetry of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and assume that  $\mathfrak{M}$  is a hyper-maximal semi-definite subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that  $\mathfrak{M} \cap j_2 \mathfrak{M} \subseteq \text{ran } V$  and that  $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}^{\perp\perp}} V)$  is a hyper-maximal semi-definite subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$ . Then there exists a symmetric operator  $T$  in the Hilbert space  $\{\mathfrak{M}^{\perp\perp}, [j_2 \cdot, \cdot]_2\}$  with  $\text{dom } T = \mathfrak{M}^{\perp\perp}$  such that the closure of  $(\Upsilon_1(T) \oplus I_{\mathfrak{M} \cap j_2 \mathfrak{M}}) V$  is a unitary relation from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ .*

*Proof.* W.l.o.g. assume that  $\mathfrak{L}$  and  $\mathfrak{M}$  are hyper-maximal neutral subspaces, see Remark 4.3 (ii). Then by Theorem 4.2  $VU_t^{-1} = \Upsilon_1(S) \Upsilon_2(B)$ . Since  $(\Upsilon_1(S))^{-1} = \Upsilon_1(-S)$  and  $\text{dom } \Upsilon_1(S) = \mathfrak{K}_2$ , the statement holds with  $T = -S$ .  $\square$

COROLLARY 4.5. *Let  $V$  be an isometric operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\overline{\text{ran}}V = \mathfrak{K}_2$  and let  $\mathfrak{L} \subseteq \text{dom}V$  be a hyper-maximal semi-definite subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  such that  $V(\mathfrak{L})$  is a hyper-maximal semi-definite subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ . Then for every fundamental symmetry  $j_2$  of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and with  $\mathfrak{M} := j_2V(\mathfrak{L})$ , there exists a symmetric operator  $S$  in the Hilbert space  $\{\mathfrak{M}^{\perp j_2}, [j_2 \cdot, \cdot]_2\}$  with  $\overline{\text{dom}}S = \mathfrak{M}^{\perp j_2}$  and a bounded unitary operator  $U_t$  from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  onto  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\text{dom}V \subseteq \text{dom}U_t$ , mapping  $\mathfrak{L}$  onto  $j_2\mathfrak{M}$ , such that  $VU_t^{-1} = \Upsilon_1(S) \oplus I_{\mathfrak{M} \cap j_2\mathfrak{M}}$ .*

*Proof.* W.l.o.g. assume that  $\mathfrak{L}$  and  $\mathfrak{M}$  are hyper-maximal neutral subspaces, see Remark 4.2. Then the conditions of Theorem 4.2 are satisfied, i.e.  $VU_t^{-1} = \Upsilon_1(S)\Upsilon_2(B)$ . Moreover, the assumption that  $V(\mathfrak{L}) (\subseteq j_2\mathfrak{M})$  is hyper-maximal neutral implies that  $\text{ran}B^{*-} = j_2V(\mathfrak{L}) = \mathfrak{M}$ . This, together with the other properties of  $B$ , see Theorem 4.2, implies that  $\text{clos}(B)$  is an operator with a trivial kernel satisfying  $\text{dom}\text{clos}(B) = \mathfrak{M} = \text{ran}\text{clos}(B)$ . Consequently,  $\Upsilon_2(\text{clos}(B)) = \text{clos}(\Upsilon_2(B))$  is a standard unitary operator and, hence,  $\Upsilon_2(\text{clos}(B))U_t$  is a bounded unitary operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  onto  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ . This observation together with (4.1) shows that the statement holds with  $S$  as in Theorem 4.2.  $\square$

If  $V$  is as in Theorem 4.2 or, more generally, if  $V = \Upsilon_1(S)\Upsilon_2(B)U_t$  for a symmetric operator  $S$ , an operator  $B$  and a bounded unitary operator  $U_t$ , then  $\ker(\mathcal{P}_{j_2\mathfrak{M}}V) = V^{-1}(\mathfrak{M} \cap \text{ran}V)$  is also a neutral subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$ ,  $\ker V = \ker(\mathcal{P}_{\mathfrak{M}}V) \cap \ker(\mathcal{P}_{j_2\mathfrak{M}}V)$  and, moreover,

$$\ker(P_{j_2\mathfrak{M}}(VU_t^{-1})) = \{f + j_2B^*(-S)Bf : f \in \text{dom}(B^*SB)\} \tag{4.2}$$

Hence,  $\ker(P_{j_2\mathfrak{M}}(VU_t^{-1}))$ , and therefore also  $\ker(P_{j_2\mathfrak{M}}V)$ , is a hyper-maximal neutral subspace if and only if  $B^*SB$  is a selfadjoint relation in  $\{\mathfrak{M}, [j_2 \cdot, \cdot]_2\}$ , see Lemma 3.6. Equation (4.2) also shows that  $\ker(\mathcal{P}_{j_2\mathfrak{M}}V) = \ker V$  if and only if  $\text{dom}(B^*SB) = \{0\}$ . Next an example of a unitary operator  $U$  with  $\ker(\mathcal{P}_{j_2\mathfrak{M}}U) = \ker U$  is presented, cf. [11, Example 6.6].

EXAMPLE 4.6. Let  $j$  be a fundamental symmetry of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  such that  $\{\mathfrak{M}, [j \cdot, \cdot]\}$  is a separable Hilbert space. Next let  $K$  be a selfadjoint operator in  $\{\mathfrak{M}, [j \cdot, \cdot]\}$  with  $\text{ran}K \neq \mathfrak{M}$  and  $\overline{\text{ran}}K = \mathfrak{M}$ . Then there exists a unitary operator  $W$  in  $\{\mathfrak{M}, [j \cdot, \cdot]\}$  such that  $\text{ran}(WK) \cap \text{ran}K = \{0\}$ , see e.g. [14, Theorem 3.6]. Now let  $C$  be a closed operator such that  $\text{ran}C = \text{ran}(WK)$ ,  $\text{dom}C = \mathfrak{M}$  and  $\ker C = \{0\}$ , see [14]. Then  $B = C^{-*}$  is a closed operator with  $\overline{\text{dom}}B = \mathfrak{M} = \text{ran}B$ ,  $\ker B = \{0\}$  and  $\text{dom}B^* \cap \text{ran}K = \{0\}$ . Now  $\text{dom}(B^*KB) = \{0\}$  and, hence,  $U := \Upsilon_1(K)\Upsilon_2(B)$  is a unitary operator such that  $\ker(\mathcal{P}_{j_2\mathfrak{M}}U) = \ker U$ , see Proposition 3.11 and (4.2).

Furthermore, if  $V$  is as above, then

$$\ker(\mathcal{P}_{\mathfrak{M}}V) + \ker(\mathcal{P}_{j_2\mathfrak{M}}V) = \text{dom}V \quad \text{if and only if} \quad \text{ran}(SB) \subseteq \text{dom}B^*. \tag{4.3}$$

Example 4.7 (i) below shows that for two hyper-maximal neutral subspaces  $\mathfrak{L}_0$  and  $\mathfrak{L}_1$  there always exists a unitary operator  $U$  such that  $\mathfrak{L}_0 = \ker(\mathcal{P}_{\mathfrak{M}}U)$ ,  $\mathfrak{L}_1 = \ker(\mathcal{P}_{j_2\mathfrak{M}}U)$

and  $\ker(\mathcal{P}_{\mathfrak{M}}U) + \ker(\mathcal{P}_{j\mathfrak{M}}U) = \text{dom}U$ . Also an isometric operator, which can not be extended to a unitary operator, with the same properties is given, see Example 4.7 (ii) below. Recall that if the sum of  $\mathfrak{L}_0$  and  $\mathfrak{L}_1$  is closed and coincides with the orthogonal complement  $\mathfrak{L}^{\perp}$  of a closed neutral subspace  $\mathfrak{L}$ , then  $\mathfrak{L}_0$  and  $\mathfrak{L}_1$  are traditionally called transversal extensions of  $\mathfrak{L}$ . For such cases it is well known that there exists a bounded unitary operator  $U$  such that  $\mathfrak{L}_0 = \ker(\mathcal{P}_{\mathfrak{M}}U)$  and  $\mathfrak{L}_1 = \ker(\mathcal{P}_{j\mathfrak{M}}U)$ , see [13, Proposition 1.3].

EXAMPLE 4.7. Let  $j$  be a fundamental symmetry of  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and assume that there exists a hyper-maximal neutral subspace  $\mathfrak{M}$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ .

(i) Let  $\mathfrak{L}$  be hyper-maximal neutral subspace of  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . Then, by Lemma 3.6, there exists a selfadjoint relation  $K$  in  $\{\mathfrak{M}, [j\cdot, \cdot]\}$  such that  $\mathfrak{L} = \{f + ijf' : \{f, f'\} \in \text{gr}K\}$ . Now a direct calculation shows that the unitary relation  $U := j\Upsilon_1(K^{-1})j$  is such that  $\mathfrak{M} = \ker(\mathcal{P}_{j\mathfrak{M}}U)$ ,  $\mathfrak{L} = \ker(\mathcal{P}_{\mathfrak{M}}U)$  and  $\ker(\mathcal{P}_{\mathfrak{M}}U) + \ker(\mathcal{P}_{j\mathfrak{M}}U) = \text{dom}U$ .

(ii) Let  $S$  be a symmetric operator in the Hilbert space  $\{\mathfrak{M}, [j\cdot, \cdot]\}$  with unequal defect numbers and  $\overline{\text{dom}}S = \mathfrak{M} = \overline{\text{ran}}S$ , and let  $B$  be a closed operator with  $\overline{\text{dom}}B = \mathfrak{M} = \text{ran}B$  and  $\ker B = \{0\}$  such that  $\text{dom}B^* = \text{ran}S$ . Then  $K := B^*SB$  is a selfadjoint operator in  $\{\mathfrak{M}, [j\cdot, \cdot]\}$ , because by the assumptions  $\text{ran}K = \mathfrak{M}$ . Now  $V := U_1(S)\Upsilon_2(B)$  is an isometric operator in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  which cannot be extended to a unitary operator, see Proposition 3.11, while  $\mathfrak{L}_0 := \ker(\mathcal{P}_{\mathfrak{M}}V) = j\mathfrak{M}$  and  $\mathfrak{L}_1 := \ker(\mathcal{P}_{j\mathfrak{M}}V) = \{f + jiB^*(-S)Bf : f \in \text{dom}(B^*SB)\} = \{f - jiKf : f \in \text{dom}K\}$  are hyper-maximal neutral subspaces of  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . Finally, note that  $\text{dom}V = \mathfrak{L}_0 + \mathfrak{L}_1$  by (4.3), because  $\text{ran}(SB) = \text{dom}B^*$  by construction.

### 4.2. Block representations of unitary relations

Next necessary and sufficient conditions for an isometric operator to be unitary are presented. Note that Theorem 4.8 below weakens the conditions in [17, Theorem 5.6] and is an inverse to Lemma 3.1, see also Corollary 4.9 below. Theorem 4.8 below also shows how one can choose coordinates such that, up to a bounded unitary transformation, the unitary operator has a diagonal block representation in those coordinates. In particular, that shows that the unbounded part of a unitary operator can always be represented by a block diagonal unitary operator (in certain coordinates).

THEOREM 4.8. *Let  $U$  be an isometric relation from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ . Then  $U$  is unitary if and only if there exists a hyper-maximal semi-definite subspace  $\mathfrak{L} \subseteq \text{dom}U$  of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  and a fundamental symmetry  $j_1$  of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  such that*

(i)  *$U(\mathfrak{L}^{\perp})$  is a neutral subspace with equal defect numbers in the Kreĭn space  $\{\mathfrak{K}_2 \cap (U(\mathfrak{L} \cap j_1\mathfrak{L}))^{\perp}, [\cdot, \cdot]_2\}$ ;*

(ii)  *$U(j_1\mathfrak{L} \cap \text{dom}U)$  is a hyper-maximal semi-definite subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ .*

*In particular, if the above conditions hold and  $U$  is an operator,  $\mathfrak{M} := U(j_1\mathfrak{L} \cap \text{dom}U)$  and  $j_2$  is a fundamental symmetry of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ , then there exists a closed operator  $B$  in the Hilbert space  $\{\mathfrak{M}^{\perp}, [j_2\cdot, \cdot]\}$  with  $\text{dom}B = \mathfrak{M}^{\perp} = \text{ran}B$  and  $\ker B = \{0\}$ ,*

and a bounded unitary operator  $U_t$  from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  onto  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\text{dom} U \subseteq \text{dom} U_t$ , mapping  $j_1 \mathfrak{L} \cap \text{dom} U_t$  onto  $\mathfrak{M}$ , such that

$$UU_t^{-1} = \Upsilon_2(B) \oplus I_{\mathfrak{M} \cap j_2 \mathfrak{M}}.$$

*Proof.* The existence of a subspace  $\mathfrak{L}$  with the stated conditions follows from [17, Theorem 5.6] and the sufficiency of the conditions is the contents of Lemma 3.1. Therefore to complete the proof, only the last statement needs to be proven which w.l.o.g. is only done in case  $\mathfrak{L}$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$ , see Remark 4.3 (ii). Then, by Lemma 4.1, there exists a standard unitary operator  $U_h$  from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\text{dom} U \subseteq \text{dom} U_h$  mapping  $\mathfrak{L}$  onto  $j_2 \mathfrak{M}$  and  $j_1 \mathfrak{L} \cap \text{dom} U_t$  onto  $\mathfrak{M}$ . Therefore the unitary operator  $UU_h^{-1}$  (with a trivial kernel) has w.r.t. the decomposition  $\mathfrak{M} \oplus j_2 \mathfrak{M}$  of  $\mathfrak{K}_2$ , the block representation

$$UU_h^{-1} = \begin{pmatrix} B & iCj_2 \\ 0 & j_2 D j_2 \end{pmatrix},$$

where  $B$  is a closed operator satisfying  $\overline{\text{dom} B} = \text{clos}(U_h(j_1 \mathfrak{L} \cap \text{dom} U)) = \mathfrak{M}$ ,  $\text{ran} B = U(j_1 \mathfrak{L} \cap \text{dom} U) = \mathfrak{M}$ ,  $\text{ker} B = \{0\} = \text{mul} B$ , and  $C$  and  $D$  are operators satisfying  $\text{dom} C = \text{dom} D = U_h(\mathfrak{L} \cap \text{dom} U) = U_h(\mathfrak{L}) = j_2 \mathfrak{M}$ . Now the arguments as in Theorem 4.2 show that  $D = B^{-*}$  and that  $C = SB^{-*}$  for a symmetric operator  $S$  in  $\{\mathfrak{M}, [j_2 \cdot, \cdot]\}$ . This implies that  $UU_h^{-1}$  can be written as

$$UU_h^{-1} = \begin{pmatrix} B & iSB^{-*}j_2 \\ 0 & j_2 B^{-*}j_2 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & j_2 B^{-*}j_2 \end{pmatrix} \begin{pmatrix} I & iB^{-1}SB^{-*}j_2 \\ 0 & I \end{pmatrix}.$$

Here the second equality holds because  $\text{ran} B = \mathfrak{M}$ . Next observe that  $K := B^{-1}SB^{-*}$  is a symmetric operator, because  $\text{mul}(UU_h^{-1}) = \{0\}$ , with  $\text{dom} K = \mathfrak{M}$ , because  $\text{dom}(SB^{-*}) = \text{dom} C = \mathfrak{M}$  and  $\text{ran} B = \mathfrak{M}$ . This shows that  $K$  is a everywhere defined selfadjoint operator and, hence,  $\Upsilon_1(K)$  is a standard unitary operator. Therefore the statement holds with  $U_t = j_2 \Upsilon_1(K) j_2 U_h$ .  $\square$

**COROLLARY 4.9.** *Let  $U$  be an isometric relation from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ . Then  $U$  is unitary if and only if there exists a hyper-maximal semi-definite subspace  $\mathfrak{L} \subseteq \text{dom} U$  of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  and a fundamental symmetry  $j_1$  of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  such that  $U(j_1 \mathfrak{L} \cap \text{dom} U)$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ .*

*Proof.* The necessity of the condition follows from Theorem 4.8 and the sufficiency of the condition is the contents of Lemma 3.1.  $\square$

**REMARK 4.10.** Among other things, Theorem 4.8 implies that if there exists a hyper-maximal neutral subspace in the domain of a unitary operator  $U$  which is mapped onto a neutral subspace with equal defect numbers, then, up to a standard unitary transformation on its range,  $U$  can be interpreted as a so-called generalized boundary triplet, see [13, Definition 6.1]. This implies that the graph of every Weyl family associated

with such a boundary relation, see Definition 2.4, is a linear transformation of the graph of a bounded Weyl family. More specifically, let  $\{\mathcal{H}, \Gamma\}$  be a boundary relation for the adjoint of the closed symmetric relation  $S$  in  $\{\mathfrak{H}, (\cdot, \cdot)\}$  with  $\text{mul}\Gamma = \{0\}$  and assume that  $A$  is a selfadjoint relation in  $\{\mathfrak{H}, (\cdot, \cdot)\}$  such that  $\text{gr}A \subseteq \text{dom}\Gamma$  and that  $\Gamma(\text{gr}A)$  is extendable to a hyper-maximal neutral subspace of  $\{\mathcal{H}^2, \langle \cdot, \cdot \rangle\}$ , see Section 2.4 for the notation. If  $j$  is a fundamental symmetry of  $\{\mathfrak{H}^2, \langle \cdot, \cdot \rangle\}$ , then by Remark 4.3 (i)  $\mathfrak{M} := \Gamma(j\text{gr}A \cap \text{dom}\Gamma)$  is a hyper-maximal neutral subspace of  $\{\mathcal{H}, (\cdot, \cdot)\}$ . Hence, if  $U_t$  is a standard unitary operator in  $\{\mathcal{H}^2, \langle \cdot, \cdot \rangle\}$  which maps  $\mathfrak{M}$  onto  $\mathcal{H} \times \{0\}$ , see Lemma 3.8, then by Theorem 4.8 there exists a bounded unitary operator  $\Gamma'$  from  $\{\mathfrak{H}^2, \langle \cdot, \cdot \rangle\}$  to  $\{\mathcal{H}^2, \langle \cdot, \cdot \rangle\}$  (with  $\ker\Gamma' = \ker\Gamma$ ) and a closed operator  $B$  in  $\{\mathcal{H}, (\cdot, \cdot)\}$  with  $\overline{\text{dom}B} = \mathcal{H} = \text{ran}B$  and  $\ker B = \{0\}$  such that

$$U_t\Gamma = \begin{pmatrix} B & 0 \\ 0 & B^{-*} \end{pmatrix} \Gamma'. \tag{4.4}$$

Since  $\Gamma'$  is a bounded unitary relation with  $\text{mul}\Gamma' = \{0\}$ ,  $\{\mathcal{H}, \Gamma'\}$  is an ordinary boundary triplet for  $S^*$  and, hence, its Weyl family  $M'(\lambda)$  is a bounded and boundedly invertible Weyl function, see Section 2.4. From (4.4) it now follows, that the Weyl family  $M(\lambda)$  associated with  $\{\mathcal{H}, \Gamma\}$  is given by

$$\text{gr}M(\lambda) = U_t \text{gr}(B^{-*}M'(\lambda)B^{-1}), \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Theorem 4.8 shows that the domain and range of a unitary relation contain a hyper-maximal semi-definite subspace, cf. [9, Ch. IV.3]. In fact, Theorem 4.8 combined with Proposition 3.4 shows that the domain of a unitary relation contains a hyper-maximal semi-definite subspace which is mapped onto a hyper-maximal semi-definite subspace. Combing this observation with Corollary 4.5 and Proposition 3.2 yields a second block representation for unitary operators.

**COROLLARY 4.11.** *Let  $U$  be a unitary operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and let  $j_2$  be a fundamental symmetry of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ . Then there exists a hyper-maximal semi-definite subspace  $\mathfrak{M}$  of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ , a selfadjoint operator  $K$  in  $\{\mathfrak{M}^{[\perp]_2}, [j_2 \cdot, \cdot]_2\}$  and a bounded unitary operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  onto  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\text{dom}U \subseteq \text{dom}U_t$  such that  $UU_t^{-1} = \Upsilon_1(K) \oplus I_{\mathfrak{M} \cap j_2\mathfrak{M}}$ .*

Next some further necessary and sufficient conditions for an isometric relation to be unitary are stated; note that the following result extends [11, Lemma 5.5]<sup>1</sup>.

**THEOREM 4.12.** *Let  $U$  be an isometric operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and let  $j_2$  be a fundamental symmetry of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ . Then  $U$  is unitary if and only if there exists a hyper-maximal semi-definite subspace  $\mathfrak{M}$  of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that*

- (i)  $\mathfrak{M} = \mathcal{P}_{\mathfrak{M}}\text{ran}U$  and  $\mathfrak{M} \cap j_2\mathfrak{M} \subseteq \text{ran}U$ ;
- (ii)  $\ker(\mathcal{P}_{\mathfrak{M}^{[\perp]_2}}U)$  is a hyper-maximal semi-definite subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$ .

---

<sup>1</sup>Note that in [11, Lemma 5.5]  $A_0$  should be selfadjoint.

In particular, if (i) and (ii) hold, then there exists a closed operator  $B$  in the Hilbert space  $\{\mathfrak{M}^{[\perp]_2}, [j_2 \cdot, \cdot]_2\}$  with  $\overline{\text{dom}B} = \mathfrak{M}^{[\perp]_2} = \text{ran}B$  and  $\ker B = \{0\}$ , a selfadjoint operator  $K$  in  $\{\mathfrak{M}^{[\perp]_2}, [j_2 \cdot, \cdot]_2\}$  with  $\text{dom}K = \mathfrak{M}^{[\perp]_2}$  and a bounded unitary operator  $U_t$  from  $\{\mathfrak{K}_1, [\cdot, \cdot]_2\}$  onto  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\text{dom}U \subseteq \text{dom}U_t$ , mapping  $\ker(P_{\mathfrak{M}^{[\perp]_2}}U)$  onto  $j_2\mathfrak{M}$ , such that

$$UU_t^{-1} = \Upsilon_1(K)\Upsilon_2(B) \oplus I_{\mathfrak{M} \cap j_2\mathfrak{M}}. \tag{4.5}$$

*Proof.* If  $U$  is unitary, then  $\mathfrak{M}$  as in Theorem 4.8 satisfies (i)-(ii). In fact, in that case  $\mathfrak{M} \subseteq \text{ran}U$ . To prove sufficiency of the conditions (i)-(ii), it suffices to show that  $U$  has the indicated block decomposition if (i)-(ii) hold, see Proposition 3.11. Since  $\mathfrak{M} \cap j_2\mathfrak{M} \subseteq \text{ran}U$ , that is w.l.o.g. only done in case that  $\mathfrak{L}$ , and hence also  $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}^{[\perp]_2}}U)$ , is a hyper-maximal neutral subspace.

*Step 1:* Note that the assumption that  $\mathfrak{L}$  is hyper-maximal neutral implies that

$$\mathfrak{L} + \text{dom}U \cap \mathfrak{K}_1^+ = \text{dom}U = \mathfrak{L} + \text{dom}U \cap \mathfrak{K}_1^-. \tag{4.6}$$

Recall also that by assumption  $U(\mathfrak{L}) \subseteq j_2\mathfrak{M}$ . Next it is shown that  $U(\mathfrak{L})$  is under the assumptions (i) and (ii) in fact dense in  $j_2\mathfrak{M}$ . To see this let  $f_o \in j_2\mathfrak{M} \ominus \text{clos}(U(\mathfrak{L}))$ , then by the assumption (i) together with (4.6) there exists an  $f \in \text{dom}U \cap \mathfrak{K}_1^+$  such that  $\mathcal{P}_{\mathfrak{M}}Uf = j_2f_o$ . Consequently,  $[Uf, Ug]_2 = 0$  for every  $g \in \mathfrak{L}$  and, hence,  $[f, g]_1 = 0$  for every  $g \in \mathfrak{L}$ . Since  $f \in \text{dom}U \cap \mathfrak{K}_1^+$  and  $\mathfrak{L}$  is hyper-maximal neutral, the preceding equality can only hold if  $f = 0$ . Consequently,  $\text{clos}(U(\mathfrak{L})) = j_2\mathfrak{M}$ . That equality together with the assumption (i) and (4.6) yields that  $\overline{\text{ran}U} = \mathfrak{K}_2$ .

*Step 2:* Now by Theorem 4.2 there exists an operator  $B$  in  $\{\mathfrak{M}, [j_2 \cdot, \cdot]_2\}$  with  $\text{dom}B = \mathfrak{M} = \text{ran} \text{clos}(B)$  and  $\ker \text{clos}(B) = \{0\}$ , a symmetric operator  $K$  in  $\{\mathfrak{M}, [j_2 \cdot, \cdot]_2\}$  with  $\text{dom}K = \text{ran}B$  and a standard unitary operator  $U_3$  in  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\text{dom}U_i \subseteq \text{dom}U_3$ , mapping  $\ker(P_{\mathfrak{M}}U_i)$  onto  $j_2\mathfrak{M}$ , such that

$$U_a U_3^{-1} = \Upsilon_1(K)\Upsilon_2(B) = \begin{pmatrix} B & 0 \\ j_2 iKB & j_2 B^{-*} j_2 \end{pmatrix}. \tag{4.7}$$

Now the assumption (i) implies that  $\mathfrak{M} = \text{ran}B$  and, hence,  $\text{dom}K = \mathfrak{M}$ , i.e.  $K$  is a bounded selfadjoint operator and  $\text{ran}B = \mathfrak{M}$  together with  $\ker(\text{clos}(B)) = \{0\}$  implies that  $B$  is closed, see (2.4). This shows that (4.5) holds.  $\square$

Note that the second part of Theorem 4.12 yields a representation for the Weyl family  $M_\Gamma$  associated to a generalized boundary triplet  $\{\mathcal{H}, \Gamma\}$ , see [13, Definition 6.1]. Namely, (4.5) shows that for such a Weyl family there exists a bounded and boundedly invertible Weyl family  $M(\lambda)$  in  $\{\mathcal{H}, (\cdot, \cdot)\}$ , a selfadjoint operator  $K$  in  $\{\mathcal{H}, (\cdot, \cdot)\}$  and a closed operator  $B$  with  $\overline{\text{dom}B} = \mathcal{H} = \text{ran}B$  and  $\ker B = \{0\}$  such that

$$M_\Gamma(\lambda) = K + B^{-*}M(\lambda)B^{-1}.$$

The two conditions in Theorem 4.12 are independent of each other, i.e. there exists unitary operator such that either only (i) holds or only (ii) as is shown by the following example.

EXAMPLE 4.13. Let  $\{\mathfrak{H}^2, \langle \cdot, \cdot \rangle\}$  be the Kreĭn space associated to the Hilbert space  $\{\mathfrak{H}, (\cdot, \cdot)\}$  as in Section 2.4.

i) Let  $S$  be a closed symmetric operator in  $\{\mathfrak{H}, (\cdot, \cdot)\}$  with defect numbers  $n_+(S) = 1$  and  $n_-(S) = 0$  such that  $\overline{\text{dom}S} = \mathfrak{H} = \overline{\text{ran}S}$ . Moreover, let  $B$  be a closed operator in  $\{\mathfrak{H}, (\cdot, \cdot)\}$  with  $\text{dom}B = \mathfrak{H}$ ,  $\text{ran}B = \text{ran}S$  and  $\ker B = \{0\}$ , see [14], then  $K := B^{-1}SB^{-*}$  is a selfadjoint operator in  $\{\mathfrak{H}, (\cdot, \cdot)\}$  with  $\text{ran}K = \mathfrak{H}$ . Now  $U$  defined as

$$U = \begin{pmatrix} KB^{-*} & -B \\ B^{-*} & 0 \end{pmatrix},$$

where the block representation is w.r.t. the decomposition  $\mathfrak{H} \times \mathfrak{H}$  of  $\mathfrak{H}^2$ , is a unitary operator in  $\{\mathfrak{H}^2, \langle \cdot, \cdot \rangle\}$ , see e.g. Lemma 3.1. Clearly,  $\mathcal{P}_{\mathfrak{H} \times \{0\}}U \supseteq \text{ran}K = \mathfrak{H}$ , while on the other hand

$$\begin{aligned} \ker(\mathcal{P}_{\mathfrak{H} \times \{0\}}U) &= \{\{f, f'\} \in \text{dom}U : KB^{-*}f + Bf' = 0\} \\ &= \{\{f, f'\} \in \text{dom}U : f' = -B^{-1}KB^{-*}f\} \\ &= \{\{f, -Sf\} : f \in \text{dom}S\}. \end{aligned}$$

Since  $S$  is by assumption not selfadjoint in  $\{\mathfrak{H}, (\cdot, \cdot)\}$ , the above calculation shows that  $\ker(\mathcal{P}_{\mathfrak{H} \times \{0\}}U)$  is not hyper-maximal neutral, see Proposition 3.6.

(ii) Let  $B$  be a closed operator in  $\{\mathfrak{H}, (\cdot, \cdot)\}$  with  $\overline{\text{dom}B} = \mathfrak{H} = \text{ran}B$  and  $\ker B = \{0\}$ , and let  $K$  be an unbounded selfadjoint operator in  $\{\mathfrak{H}, (\cdot, \cdot)\}$ . Then  $U$  defined as

$$U = \begin{pmatrix} B & 0 \\ KB & B^{-*} \end{pmatrix},$$

where the block representation is w.r.t. the decomposition  $\mathfrak{H} \times \mathfrak{H}$  of  $\mathfrak{H}^2$ , is a unitary operator in  $\{\mathfrak{H}^2, \langle \cdot, \cdot \rangle\}$  with  $\text{dom}U = \{0\}$ , see Proposition 3.11. Clearly,  $\ker(\mathcal{P}_{\mathfrak{H} \times \{0\}}U) = \{0\} \times \text{dom}B^{-*} = \{0\} \times \mathfrak{H}$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}, \langle \cdot, \cdot \rangle\}$ . On the other hand,  $\mathcal{P}_{\mathfrak{H} \times \{0\}}U = (\text{ran}B \cap \text{dom}K) = \text{dom}K$ . Since  $K$  is by assumption unbounded, this implies that  $\mathcal{P}_{\mathfrak{H} \times \{0\}}U \neq \mathfrak{H}$ .

Corollary 4.14 below contains conditions for the unitary operator in (4.5) to be a bounded unitary operator which differ from the usual condition that the range of the unitary operator is onto.

COROLLARY 4.14. *Let  $U$  be a unitary operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ , let  $j_2$  be any fundamental symmetry of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and let  $\mathfrak{M}$  be a hyper-maximal semi-definite subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that Theorem 4.12 (i) and (ii) hold. Then  $U$  is a bounded unitary operator if and only if  $j_2\mathfrak{M} = \mathcal{P}_{j_2\mathfrak{M}}\text{ran}U$  and  $\ker(\mathcal{P}_{\mathfrak{M}}U) + \ker(\mathcal{P}_{j_2\mathfrak{M}}U) = \text{dom}U$ .*

*Proof.* By assumption  $U$  has the representation in (4.5). In fact, since  $K$  is a bounded selfadjoint operator,  $\Upsilon_1(K)$  is a standard unitary operator therein. Moreover, since  $B$  is closed and  $\text{ran}B = \mathfrak{M}^{[\perp]_2} = \overline{\text{dom}B}$  in (4.5),  $U$  is a bounded unitary operator, i.e.  $\text{ran}U = \mathfrak{K}_2$ , if and only if  $\text{dom}B^* = \mathfrak{M}^{[\perp]_2}$ . It is clear (see e.g. (4.7)) that  $\text{dom}B^* =$

$\mathfrak{M}^{\perp_2}$  if and only if  $\text{ran}(KB) \subseteq \text{dom}B^*$  and  $\mathcal{P}_{j_2\mathfrak{M}}\text{ran}U = j_2\mathfrak{M}$ . This observation together with (4.3) shows that the equivalence holds.  $\square$

Combining Theorem 4.8 with results from the previous section yields the following condition for an isometric relation to be (extendable to) a unitary relation. In particular, this yields a condition for a quasi-boundary triplet to be a boundary relation, see Theorem 5.2 below.

**THEOREM 4.15.** *Let  $U$  be an isometric relation from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and let  $j_i$  be a fundamental symmetry of  $\{\mathfrak{K}_i, [\cdot, \cdot]_i\}$ , for  $i = 1, 2$ . Moreover, assume that  $\mathfrak{M}$  is a hyper-maximal semi-definite subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that  $\mathfrak{M} \cap j_2\mathfrak{M} \subseteq \text{ran}U$  and that  $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}^{\perp_2}}U)$  is a hyper-maximal semi-definite subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$ . Then  $U$  is (extendable to) a unitary relation if and only if  $U(j_1\mathfrak{L} \cap \text{dom}U)$  is (extendable to) a hyper-maximal semi-definite subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ .*

*In particular, if the above assumptions hold and  $U$  is a unitary operator, then there exists an operator  $B$  in  $\{\mathfrak{M}^{\perp_2}, [j_2\cdot, \cdot]_2\}$  with  $\overline{\text{dom}B} = \mathfrak{M}^{\perp_2} = \text{ran} \text{clos}(B)$  and  $\ker \text{clos}(B) = \{0\}$ , a selfadjoint operator  $K$  in  $\{\mathfrak{M}^{\perp_2}, [j_2\cdot, \cdot]_2\}$ , and a bounded unitary operator  $U_t$  from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  onto  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\text{dom}U \subseteq \text{dom}U_t$ , mapping  $\mathfrak{L}$  onto  $j_2\mathfrak{M}$ , such that*

$$UU_t^{-1} = \Upsilon_1(K)\Upsilon_2(\text{clos}(B)) \oplus I_{\mathfrak{M} \cap j_2\mathfrak{M}}.$$

*Proof.* W.l.o.g. the statement is only proven in case that  $\mathfrak{M}$  and  $\mathfrak{L}$  are hyper-maximal neutral subspaces, see Remark 4.3 (ii). It can also be assumed that  $U$  is closed, because if  $U$  is not closed, then  $\text{clos}(U)$  clearly satisfies the same conditions. Moreover,  $U$  can also w.l.o.g. be assumed to be an operator with a trivial kernel, see [17, Corollary 3.11]. Now arguments as in step 1 of the proof of Theorem 4.2 show that w.r.t. to the decomposition  $\mathfrak{L} \oplus j_1\mathfrak{L}$  of  $\mathfrak{K}_1$  and the decomposition  $\mathfrak{M} \oplus j_2\mathfrak{M}$  of  $\mathfrak{K}_2$ ,  $U$  has the following block decomposition

$$U = \begin{pmatrix} 0 & Cj_1 \\ j_2B & j_2iSCj_1 \end{pmatrix},$$

where  $B$  and  $C$  are operators from  $\{\mathfrak{L}, [j_1\cdot, \cdot]_1\}$  to  $\{\mathfrak{M}, [j_2\cdot, \cdot]_2\}$  with  $\text{dom}B = \mathfrak{L}$ ,  $\ker B = \{0\} = \ker C$  and  $C \subseteq B^*$ , and  $S$  is a symmetric operator in  $\{\mathfrak{M}, [j_2\cdot, \cdot]_2\}$  with  $\text{dom}S = \text{ran}C$ . Moreover, since  $U$  is by assumption closed,  $B$  needs to be closed. The above representation shows that  $U(j_1\mathfrak{L} \cap \text{dom}U)$  is (extendable to) a hyper-maximal neutral subspace if and only if  $S$  is (extendable to) a selfadjoint relation  $K$  in  $\{\mathfrak{M}, [j_2\cdot, \cdot]_2\}$ , cf. Lemma 3.6.

Now assume that  $U(j_1\mathfrak{L} \cap \text{dom}U)$  is (extendable to) a hyper-maximal neutral subspace, then there exists a selfadjoint extension  $K$  of  $S$ . Then  $U_a$  defined via

$$\text{gr}U_a = \{ \{f + j_1g, B^{-*}g + j_2(Bf + iKB^{-*}g)\} : f \in \mathfrak{M} \text{ and } g \in \text{dom}(KB^{-*}) \}$$

is an extension of  $U$  which is a unitary relation by Lemma 3.1, because  $\mathfrak{L} \subseteq \text{dom}U_a$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  and  $U_a(j_1\mathfrak{L} \cap \text{dom}U_a) = \{f + j_2iKf : f \in \text{dom}K\}$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ . Note that here it was

used that  $\text{ran } B^{-*} = \mathfrak{M}$ . Hence, if  $U(j_1 \mathfrak{L} \cap \text{dom } U)$  is (extendable to) a hyper-maximal neutral subspace, then  $U$  is (extendable to) a unitary relation.

To prove the converse assume that  $U$  is a unitary operator (see the discussion at the beginning of the proof), then, in particular,  $\overline{\text{ran}} U = \mathfrak{K}_2$ , see (2.5). Hence, Remark 4.3 (i) implies that  $U(j_1 \mathfrak{L} \cap \text{dom } U)$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ . This arguments show that if  $U$  is (extendable to) a unitary relation, then  $U(j_1 \mathfrak{L} \cap \text{dom } U)$  is (extendable to) a hyper-maximal neutral subspace.  $\square$

### 4.3. Compositions of unitary operators

As an application of the block representations presented in the preceding subsections, here conditions for the composition of a unitary operator with an isometric operator to be (extendable to) a unitary operator are given. Two cases are considered: The composition of unitary operators with closed isometric operators with a trivial kernel and the composition of unitary operators with bounded unitary operators with a kernel.

**PROPOSITION 4.16.** *Let  $U$  be a unitary operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ , let  $j_2$  a fundamental symmetry of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ , let  $\mathfrak{M}$  be a hyper-maximal neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that  $\ker(\mathcal{P}_{\mathfrak{M}}U)$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  and let  $V$  be a closed isometric operator in  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\ker V = \{0\}$ . Moreover, let  $B, K$  and  $U_t$  be as in Theorem 4.15 such that*

$$UU_t^{-1} = \Upsilon_1(K)\Upsilon_2(B). \tag{4.8}$$

*Then  $VU$  can be extended to a unitary operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\ker(\mathcal{P}_{\mathfrak{M}}VU) = \ker(\mathcal{P}_{\mathfrak{M}}U)$  if and only if there exists a closed relation  $D$  in the Hilbert space  $\{\mathfrak{M}, [j_2 \cdot, \cdot]_2\}$  such that  $D^{-*}B^{-*}$  is a closed operator satisfying  $\text{dom}(D^{-*}B^{-*}) = \mathfrak{M}$  and  $\ker(D^{-*}B^{-*}) = \{0\}$ , and a symmetric operator  $S$  in  $\{\mathfrak{M}, [j_2 \cdot, \cdot]_2\}$  which possesses a selfadjoint extension  $K_S$  satisfying  $\text{dom } K_S \cap (\text{ran}(D^{-*}B^{-*}))^\perp = \{0\}$ , such that  $V$  is an extension of*

$$\Upsilon_1(S)\Upsilon_2(D)\Upsilon_1(-K).$$

*In particular,  $\text{clos}(VU)$  is a unitary operator if and only if  $V$  is an isometric extension of  $\Upsilon_1(S)\Upsilon_2(D)\Upsilon_1(-K)$  as above and, additionally,  $\text{clos}(S)$  is selfadjoint and  $\text{clos}(DI_{\text{dom } K}B) = (D^{-*}B^{-*})^{-*}$ .*

*Proof.* If  $VU$  can be extended to a unitary operator and  $\ker(\mathcal{P}_{\mathfrak{M}}VU) = \ker(\mathcal{P}_{\mathfrak{M}}U)$ , then  $VUU_t^{-1}$ , where  $U_t$  is as in (4.8), is an isometric operator in  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that  $\ker(\mathcal{P}_{\mathfrak{M}}VUU_t^{-1}) = j_2\mathfrak{M}$ . Hence, as in step 1 of the proof of Theorem 4.2, there exist operators  $B_1$  and  $C$  in  $\{\mathfrak{M}, [j_2 \cdot, \cdot]_2\}$  with  $B_1 \subseteq C^{-*}$ ,  $\text{dom } C = \mathfrak{M}$ ,  $\ker C = \{0\} = \text{mul } C$  and a symmetric operator  $T$  in  $\{\mathfrak{M}, [j_2 \cdot, \cdot]_2\}$  with  $\text{dom } T = \text{ran } B_1$  such that

$$VUU_t^{-1} = \begin{pmatrix} B_1 & 0 \\ j_2 i T B_1 & j_2 C j_2 \end{pmatrix} = \Upsilon_1(T) \begin{pmatrix} B_1 & 0 \\ 0 & j_2 C j_2 \end{pmatrix}. \tag{4.9}$$

Since  $VU$ , and hence also  $VUU_t^{-1}$ , is extendable to a unitary operator, it follows that  $\text{mul clos } C = \{0\}$ . This observation together with  $\text{dom } C = \mathfrak{M}$  yields that  $C$  is a closed

operator. Moreover, since  $VU$  is extendable to a unitary operator,  $T$  is extendable to a selfadjoint operator  $K_S$  such that  $\text{dom}K_S \cap (\text{ran}C)^\perp = \{0\}$ , see Remark 4.2 (i) and step 3 of the proof of Theorem 4.2.

Combining (4.8) and (4.9) yields

$$V \upharpoonright_{\text{ran}U} = \Upsilon_1(T) \begin{pmatrix} B_1 B^{-1} & 0 \\ 0 & j_2 C B^* j_2 \end{pmatrix} \Upsilon_1(-K). \tag{4.10}$$

Since  $V$  is by assumption closed, the closure of the righthand side of (4.10) is contained in  $V$ . Hence, the assumption that  $V$  is an operator with a trivial kernel implies that the operator  $E := CB^*$  satisfies  $\text{ker} \text{clos}(E) = \{0\} = \text{mul} \text{clos}(E)$ . Hence,  $D := E^{-*}$  is a relation which satisfies the stated conditions, because

$$D^{-*} B^{-*} = \text{clos}(E) B^{-*} = C + \{0\} \times \text{mul} \text{clos}(E) = C.$$

Hence, by taking  $S$  to be a restriction of  $T$  to  $\text{ran}(B_1 B^{-1})$  the necessity of the conditions is clear.

Conversely, let  $D$  and  $S$  be as in the statement, then with  $\Delta := \text{dom}K \oplus j_2 \mathfrak{M}$

$$\Upsilon_1(S) \Upsilon_2(D) \Upsilon_1(-K) U U_t^{-1} = \Upsilon_1(S) \Upsilon_2(D) I_\Delta \Upsilon_2(B).$$

Now observe that

$$\Upsilon_2(D) I_\Delta \Upsilon_2(B) = \begin{pmatrix} D I_{\text{dom}K} B & 0 \\ 0 & j_2 D^{-*} B^{-*} j_2 \end{pmatrix} \subseteq \Upsilon_2((D^{-*} B^{-*})^{-*}).$$

By the assumptions  $E := (D^{-*} B^{-*})^{-*}$  is a (closed) relation satisfying  $\overline{\text{dom}E} = \mathfrak{M} = \text{ran}E$  and  $\text{ker}E = \{0\}$ . Hence, if  $K_S$  is a selfadjoint extension of  $S$  such that  $\text{dom}K_S \cap \text{mul}E = \{0\}$ , then, by the above calculations,  $\Upsilon_1(S) \Upsilon_2(D) \Upsilon_1(-K) U U_t^{-1}$  can be extended to the unitary operator  $\Upsilon_1(K_S) \Upsilon_2(E)$ , see Proposition 3.11, i.e.,  $VU$  can be extended to the unitary operator  $\Upsilon_1(K_S) \Upsilon_2(E) U_t$ .

The final equivalence is clear by the above observations.  $\square$

Note that the isometric operator  $\Upsilon_1(S) \Upsilon_2(D) \Upsilon_1(-K)$  in Proposition 4.16 need not be extendable to a unitary operator. Consider for instance the case that  $D = I$ , and that  $S$  and  $-K$  are the selfadjoint operators  $K_1$  and  $K_2$  from Example 3.9. However, in the case that  $U$  and  $VU$  in Proposition 4.16 are the abstract equivalents of generalized boundary triplets, then  $V$  must be a unitary operator.

**COROLLARY 4.17.** *Let  $U$  be a unitary operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ , let  $j_2$  be a fundamental symmetry of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and let  $\mathfrak{M}$  be a hyper-maximal neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that  $\text{ker}(\mathcal{P}_{\mathfrak{M}}U)$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  and that  $\mathcal{P}_{\mathfrak{M}} \text{ran}U = \mathfrak{M}$ . Moreover, let  $V$  be a closed isometric operator in  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  with  $\text{ker}V = \{0\}$  such that  $\text{ker}(\mathcal{P}_{\mathfrak{M}}VU) = \text{ker}(\mathcal{P}_{\mathfrak{M}}U)$  and  $\mathcal{P}_{\mathfrak{M}} \text{ran}(VU) = \mathfrak{M}$ . Then  $VU$  is a unitary operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and  $V$  is a unitary relation in  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ .*

*Proof.* The assumptions on  $VU$  imply by Theorem 4.12 that  $VU$  is a unitary operator. Moreover, Theorem 4.12 implies that  $K$  and  $T$  in (the proof of) Proposition 4.16 are bounded selfadjoint operators in  $\{\mathfrak{M}, [j_2 \cdot, \cdot]\}$  and, hence,  $\Upsilon_1(\text{clos}(S))$  and  $\Upsilon_1(-K)$  are standard unitary operators in  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ . From this it follows that  $\text{clos}(\Upsilon_1(S)\Upsilon_2(D)\Upsilon_1(-K)) = \Upsilon_1(\text{clos}(S))\Upsilon_2(\text{clos}(D))\Upsilon_1(-K)$  is a unitary relation in  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ . Since  $\Upsilon_1(S)\Upsilon_2(D)\Upsilon_1(-K) \subseteq V$  and  $V$  is by assumption closed, this implies that  $V$  itself is a unitary operator in  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ .  $\square$

In Proposition 4.16 the composition of a unitary operator with a closed isometric operator with a trivial kernel was considered. Next the composition of a unitary operator with a bounded unitary operator with a non-trivial kernel is considered.

**PROPOSITION 4.18.** *Let  $U$  be a unitary operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and let  $j_2$  be a fundamental symmetry of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ . Let  $\mathfrak{M}$  be a hyper-maximal neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that  $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}}U)$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  and let  $U_b$  be a bounded unitary operator from  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  onto  $\{\mathfrak{K}_3, [\cdot, \cdot]_3\}$  such that  $j_2\mathfrak{M} \subseteq \text{dom}U_b$  or, equivalently,  $\ker U_b \subseteq j_2\mathfrak{M}$ . Then  $U_bU$  is an isometric operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_3, [\cdot, \cdot]_3\}$  which can be extended to a unitary relation. In particular,  $U_bU$  is a unitary operator if and only if there exists a fundamental symmetry  $j_1$  of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  such that  $U(j_1\mathfrak{L} \cap \text{dom}U) \cap \text{dom}U_b + \ker U_b$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ .*

*Proof.* Note first that if  $j_2\mathfrak{M} \subseteq \text{dom}U_b$ , then  $\ker U_b = (\text{dom}U_b)^{\perp} \subseteq (j_2\mathfrak{M})^{\perp} = j_2\mathfrak{M}$  and, conversely, if  $\ker U_b \subseteq j_2\mathfrak{M}$ , then  $j_2\mathfrak{M} = (j_2\mathfrak{M})^{\perp} \subseteq (\ker U_b)^{\perp} = \text{dom}U_b = \text{dom}U_b$ , where in the last step the boundedness of  $U_b$  is used.

Since  $U(\mathfrak{L}) \subseteq j_2\mathfrak{M} (\subseteq \text{dom}U_b)$  is a neutral subspace with equal defect numbers and  $U_b$  is a bounded unitary operator,  $U_b(U(\mathfrak{L}))$  is a neutral subspace with equal defect numbers. Hence, by Theorem 4.15,  $U_bU$  is (extendable to) a unitary relation if and only if  $U_bU((j_1\mathfrak{L} \cap \text{dom}U) \cap \text{dom}U_b)$  is (extendable to) a hyper-maximal neutral subspace of  $\{\mathfrak{K}_3, [\cdot, \cdot]_3\}$ . Since  $U_b$  is a bounded unitary operator, this last condition is equivalent to  $U(j_1\mathfrak{L} \cap \text{dom}U) \cap \text{dom}U_b (+\ker U_b)$  being (extendable to) a hyper-maximal neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ . But that follows immediately from the fact that  $U(j_1\mathfrak{L} \cap \text{dom}U) \cap \text{dom}U_b$  is a restriction of  $U(j_1\mathfrak{L} \cap \text{dom}U)$  which is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  by Theorem 4.15, because  $U$  is unitary and  $\mathfrak{L} := \ker(\mathcal{P}_{\mathfrak{M}}U)$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$ .  $\square$

Not every composition of a unitary operator with a unitary operator with closed domain can be extended to a unitary operator as the following example shows.

**EXAMPLE 4.19.** By Remark 3.14 there exists a unitary operator  $U$  in an (infinite-dimensional) Kreĭn space  $\{\mathfrak{K}, [\cdot, \cdot]\}$  which maps a neutral subspace  $\mathfrak{L}$  with unequal defect numbers onto a hyper-maximal neutral subspace. Now let  $U_b$  be the unitary operator from  $\{\mathfrak{K}, [\cdot, \cdot]\}$  to  $\{0\}$  whose graph is  $U(\mathfrak{L}) \times \{0\}$ . Then  $U_bU$  is an isometric operator from  $\{\mathfrak{K}, [\cdot, \cdot]\}$  to  $\{0\}$  whose graph is given by  $\mathfrak{L} \times \{0\}$ . Clearly,  $U_bU$  cannot be extended to a unitary operator, because  $\mathfrak{L}$  can not be extended to a hyper-maximal neutral subspace.

Finally, Proposition 4.18 is applied to the abstract equivalent of generalized boundary triplets. Note that the following result will be used in Section 5.2 below to obtain results on the boundary relations for intermediate extensions.

**COROLLARY 4.20.** *Let  $U$  be a unitary operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$ , let  $j_2$  be a fundamental symmetry of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and let  $\mathfrak{M}$  be a hyper-maximal neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  such that  $\mathcal{P}_{\mathfrak{M}}\text{ran}U = \mathfrak{M}$  and that  $\ker(\mathcal{P}_{\mathfrak{M}}U)$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$ . Moreover, let  $U_b$  be a bounded unitary operator from  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  onto  $\{\mathfrak{K}_3, [\cdot, \cdot]_3\}$  such that  $j_2\mathfrak{M} \subseteq \text{dom}U_b$  or, equivalently,  $\ker U_b \subseteq j_2\mathfrak{M}$ . Then  $U_bU$  is a unitary operator from  $\{\mathfrak{K}_1, [\cdot, \cdot]_1\}$  to  $\{\mathfrak{K}_3, [\cdot, \cdot]_3\}$  and  $\mathfrak{N} := U_b(\mathfrak{M} \cap \text{dom}U_b)$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_3, [\cdot, \cdot]_3\}$  such that*

$$\mathcal{P}_{\mathfrak{N}}(\text{ran}(U_bU)) = \mathfrak{N} \quad \text{and} \quad \ker(\mathcal{P}_{\mathfrak{N}}(U_bU)) = \ker(\mathcal{P}_{\mathfrak{M}}U),$$

where  $\mathcal{P}_{\mathfrak{N}}$  is the orthogonal projection onto  $\mathfrak{N}$  w.r.t.  $[U_bj_2U_b^{-1}\cdot, \cdot]_3$ .

*Proof.* Theorem 4.12 shows that to prove the statement it suffices to show that the last two equalities hold. Note therefore first that the assumption  $\ker U_b \subseteq j_2\mathfrak{M}$  implies that  $\mathfrak{M}_r := \mathfrak{M} \cap \text{dom}U_b$  is a closed subspace such that

$$\text{dom}U_b = \mathfrak{M}_r \oplus j_2\mathfrak{M} = \mathfrak{M}_r \oplus j_2\mathfrak{M}_r \oplus \ker U_b.$$

Since  $(\text{dom}U_b)^{\perp_2} = j_2\ker U_b$ , the above formula line shows that  $\mathfrak{M}_r + \ker U_b \subseteq \text{dom}U_b$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_2, [\cdot, \cdot]_2\}$  and, hence,  $\mathfrak{N} := U_b(\mathfrak{M}_r)$  is a hyper-maximal neutral subspace of  $\{\mathfrak{K}_3, [\cdot, \cdot]_3\}$ .

Next note that the assumption  $\mathcal{P}_{\mathfrak{M}}\text{ran}U = \mathfrak{M}$  together with  $j_2\mathfrak{M} \subseteq \text{dom}U_b$  implies that  $\mathcal{P}_{\mathfrak{M}_r}(\text{ran}U \cap \text{dom}U_b) = \mathfrak{M}_r$ . Since  $j_3\mathfrak{N} = U_b(j_2\mathfrak{M})$ , where  $j_3 = U_bj_2U_b^{-1}$ , the preceding observations imply that  $\mathcal{P}_{\mathfrak{N}}(\text{ran}(U_bU)) = \mathfrak{N}$ . Moreover,  $j_3\mathfrak{N} = U_b(j_2\mathfrak{M})$  together with the assumption  $j_2\mathfrak{M} \subseteq \text{dom}U_b$  yields

$$\ker(\mathcal{P}_{\mathfrak{M}}U) = U^{-1}(j_2\mathfrak{M} \cap \text{ran}U) = (U_bU)^{-1}(j_3\mathfrak{N} \cap \text{ran}(U_bU)) = \ker(\mathcal{P}_{\mathfrak{N}}(U_bU)).$$

This completes the proof.  $\square$

### 5. Boundary relations in Kreĭn spaces

The results from Section 4 are now used to study quasi-boundary relations and to generalize some results concerning boundary relations for intermediate extensions from [12, Section 4] to the Kreĭn space setting. Therefore it is first shown which form the archetypical isometric (unitary) operators  $\Upsilon_1(S)$  and  $\Upsilon_2(B)$  take in the Kreĭn spaces used in the definition of (quasi-) boundary relations.

Let  $\{\mathcal{H}, (\cdot, \cdot)\}$  be a Hilbert space and let  $\{\mathcal{H}^2, \langle \cdot, \cdot \rangle\}$  be its associated Kreĭn space as in Section 2.4, see (2.7). Then note that  $\mathfrak{M} := \mathfrak{H} \times \{0\}$  is hyper-maximal neutral in  $\{\mathcal{H}^2, \langle \cdot, \cdot \rangle\}$  and  $j_{\mathcal{H}}$  defined as  $j_{\mathcal{H}}\{f, f'\} = \{-if', if\}$ , where  $f, f' \in \mathcal{H}$ , is a fundamental symmetry for this Kreĭn space. Now for a symmetric relation  $S$

and a relation  $B$  in the Hilbert space  $\{\mathfrak{M}, \langle j_{\mathcal{H}}, \cdot \rangle\}$ , which can and will be identified with  $\{\mathcal{H}, (\cdot, \cdot)\}$ ,  $\Upsilon_1(S)$  and  $\Upsilon_2(B)$  take the form

$$\begin{aligned} \Upsilon_1(S)\{f, g\} &= \{f, Sf + g\}, \quad f \in \text{dom}S, g \in \mathcal{H}; \\ \Upsilon_2(B)\{f, g\} &= \{Bf, B^{-*}g\}, \quad f \in \text{dom}B, g \in \text{ran}B^*. \end{aligned} \tag{5.1}$$

In particular, if  $\Upsilon_1(S)$  and  $\Upsilon_2(B)$  are operators, then w.r.t. the decomposition  $\mathcal{H} \times \mathcal{H}$  of  $\mathcal{H}^2$ , they have the following block representation:

$$\Upsilon_1(S) = \begin{pmatrix} I & 0 \\ S & I \end{pmatrix} \quad \text{and} \quad \Upsilon_2(B) = \begin{pmatrix} B & 0 \\ 0 & B^{-*} \end{pmatrix}, \tag{5.2}$$

Henceforth,  $\Upsilon_1(S)$  and  $\Upsilon_2(B)$  are used to denote the relations in (5.1).

### 5.1. Quasi-boundary triplets

Proposition 5.1 implies that there exists a strong connection between quasi-boundary triplets and (multi-valued) generalized boundary triplets. Recall that  $\Gamma_0$  is as in (2.8).

**PROPOSITION 5.1.** *Let  $\{\mathcal{H}, \Gamma_q\}$  be a quasi-boundary triplet for the adjoint of the closed symmetric relation  $S$  in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ . Then there exists a boundary relation  $\{\mathcal{H}, \Gamma\}$  for  $S^{[*]}$  with  $\mathcal{H} \times \{0\} \subseteq \text{ran}\Gamma$  and  $\ker\Gamma_0 = (\ker\Gamma_0)^{[*]}$ , and a symmetric operator  $T$  in  $\{\mathcal{H}, (\cdot, \cdot)\}$  with  $\overline{\text{dom}T} = \mathcal{H}$  and  $\text{dom}T^* \cap \text{mul}\Gamma_0 = \{0\}$  such that  $\Gamma_q = \Upsilon_1(T)\Gamma$ . Conversely, if  $T$  and  $\Gamma$  are as above, then  $\{\mathcal{H}, \Upsilon_1(T)\Gamma\}$  is a quasi-boundary triplet for  $S^{[*]}$ .*

*Proof.* The converse part follows directly by checking the criteria for quasi-boundary triplets. For the direct part recall that by Theorem 4.2 there exists an operator  $B$  in  $\{\mathcal{H}, (\cdot, \cdot)\}$  with  $\overline{\text{dom}B} = \mathcal{H} = \text{ran}\text{clos}(B)$  and  $\text{kerclos}(B) = \{0\}$ , a symmetric operator  $T$  in  $\{\mathcal{H}, (\cdot, \cdot)\}$  with  $\text{dom}T = \text{ran}B$  and  $\text{dom}T^* \cap \text{mul}\text{clos}(B) = \{0\}$ , and a bounded unitary operator  $U_t$  from  $\{\mathfrak{K}^2, \langle\langle \cdot, \cdot \rangle\rangle\}$  onto  $\{\mathcal{H}^2, \langle \cdot, \cdot \rangle\}$  with  $\text{dom}\Gamma_q \subseteq \text{dom}U_t$  such that  $\Gamma_q U_t^{-1} = \Upsilon_1(T)\Upsilon_2(B)$ , where  $\Upsilon_1(T)$  and  $\Upsilon_2(B)$  are as in (5.2). Since  $\text{dom}T = \text{ran}B$ , it is now clear that  $\Gamma := \Upsilon_2(\text{clos}(B))U_t$  satisfies the stated conditions.  $\square$

Note that the condition  $\text{dom}T^* \cap \text{mul}\Gamma_0$  in Proposition 5.1 guaranties that  $\overline{\text{ran}}(\Upsilon_1(T)\Gamma) = \mathfrak{K}_2$ , see step 3 of the proof of Theorem 4.2. Hence, in particular, that condition guaranties that  $\Upsilon_1(T)\Gamma$  is an (isometric) operator, see (2.5).

As a consequence of Theorem 4.15, the following necessary and sufficient conditions hold for a quasi-boundary triplet to be (extendable to) a boundary relation.

**THEOREM 5.2.** *Let  $S$  be a closed symmetric relation in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and let  $\{\mathcal{H}, \Gamma\}$  be a quasi-boundary triplet for  $S^{[*]}$ . Then  $\{\mathcal{H}, \Gamma\}$  is a boundary relation with  $\text{mul}\Gamma = \{0\}$  if and only if  $\Gamma(\text{jker}\Gamma_0 \cap \text{dom}\Gamma)$  is a hyper-maximal neutral subspace of  $\{\mathcal{H}^2, \langle \cdot, \cdot \rangle\}$  for some (and hence for every) fundamental symmetry  $j$  of  $\{\mathfrak{K}^2, \langle\langle \cdot, \cdot \rangle\rangle\}$ .*

A further sufficient condition for a quasi-boundary triplet to be a boundary relation follows from Theorem 4.12: a quasi-boundary triplet  $\{\mathcal{H}, \Gamma\}$  is a boundary relation with  $\text{mul}\Gamma = \{0\}$  if  $\text{ran}\Gamma_0 = \mathcal{H}$ . Quasi-boundary triplets can also be characterized by their associated Weyl functions, cf. [7, Proposition 2.6] and [1, Proposition 2.6].

PROPOSITION 5.3. *Let  $\{\mathcal{H}, (\cdot, \cdot)\}$  be a Hilbert space and let  $M(\cdot)$  be a  $\mathcal{H}$ -valued operator function. Then  $M(\cdot)$  is the Weyl family of a quasi-boundary triplet (for the adjoint of a closed symmetric relation in the Hilbert space  $\{\mathfrak{H}, (\cdot, \cdot)\}$ ) if and only if there exists a symmetric operator  $T$  in  $\{\mathcal{H}, (\cdot, \cdot)\}$  such that  $\text{dom}M(\cdot) \subseteq \text{dom}T$  and that  $M_{\Gamma'}(\cdot) := \text{clos}(M(\cdot) + T)$  is a Nevanlinna family<sup>2</sup> which satisfies  $\text{dom}M_{\Gamma'}(\lambda) = \mathcal{H}$  and  $\text{ker}M_{\Gamma'}(\lambda) \cap \text{dom}T^* = \{0\}$  for all  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .*

*Proof.* If  $\{\mathcal{H}, \Gamma\}$  is a quasi-boundary triplet for the adjoint of a symmetric relation  $S$  in a Hilbert space  $\{\mathfrak{H}, (\cdot, \cdot)\}$ , then by Proposition 5.1 there exists a symmetric operator  $T$  in  $\{\mathcal{H}, (\cdot, \cdot)\}$  with  $\text{dom}T = \mathcal{H}$  and a boundary relation  $\{\mathcal{H}, \Gamma'\}$  for  $S^*$  with  $(\text{ker}\Gamma'_0)^* = \text{ker}\Gamma'_0$ ,  $\text{ran}\Gamma'_0 = \mathcal{H}$  and  $\text{dom}T^* \cap \text{mul}\Gamma_0 = \{0\}$  such that  $\Gamma = \Upsilon_1(T)\Gamma'$ . The Weyl family  $M_{\Gamma'}(\cdot)$  associated to  $\Gamma'$  is a Nevanlinna family of bounded operators, i.e.  $\text{dom}M_{\Gamma'}(\lambda) = \mathcal{H}$  for all  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , see [12, Proposition 3.15]. Note also that the condition  $\text{dom}T^* \cap \text{mul}\Gamma_0 = \{0\}$  implies that  $\text{ker}M_{\Gamma'}(\lambda) \cap \text{dom}T^* = \{0\}$  for all  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . Now a calculation shows that the Weyl family  $M(\lambda)$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , associated to  $\Gamma$  is

$$M(\lambda) = T + M_{\Gamma'}(\lambda), \quad \text{dom}M = \text{dom}T. \tag{5.3}$$

Since  $\overline{\text{dom}T} = \mathcal{H}$  and  $\text{dom}M_{\Gamma'}(\cdot) = \mathcal{H}$ , (5.3) gives that  $M_{\Gamma'}(\cdot) = \text{clos}(M(\cdot) - T)$  and that  $\text{dom}M(\cdot) \subseteq \text{dom}T$ .

Conversely, if  $M_{\Gamma'} := \text{clos}(M(\cdot) + T)$  is a Nevanlinna family which satisfies  $\text{dom}M_{\Gamma'}(\lambda) = \mathcal{H}$  for all  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , then, see [12, Proposition 3.15], there exists a symmetric operator  $S$  in a Hilbert space  $\{\mathfrak{H}, (\cdot, \cdot)\}$  and a boundary relation  $\{\mathcal{H}, \Gamma'\}$  for  $S^*$  satisfying  $(\text{ker}\Gamma'_0)^* = \text{ker}\Gamma'_0$  and  $\text{ran}\Gamma'_0 = \mathcal{H}$  such that its associated Weyl family is  $M_{\Gamma'}$ . Then, since  $\overline{\text{dom}T} = \mathcal{H}$  and  $\text{mul}\Gamma_0 \cap \text{dom}T^* = \text{ker}M_{\Gamma'}(\lambda) \cap \text{dom}T^* = \{0\}$ , Proposition 5.1 implies that  $\{\mathcal{H}, \Upsilon_1(-T)\Gamma'\}$  is a quasi-boundary triplet for  $S^*$  and a calculation shows that its Weyl family is  $M_{\Gamma'}(\cdot) - T = M(\cdot)$ .  $\square$

Note that if  $T$  has equal defect numbers in the above statement, then the quasi-boundary triplet for  $M(\cdot)$  can be extended to a boundary relation.

### 5.2. Boundary relations for intermediate extensions

The results in [12, Section 4] for boundary relations in the Hilbert space setting are here shown to remain valid in the Kreĭn space setting. Therefore recall that for a boundary relation  $\{\mathcal{H}, \Gamma\}$ ,  $\Gamma_0$  and  $\Gamma_1$  are defined as in (2.8). Using those definitions a boundary relation  $\{\mathcal{H}, \Gamma\}$  for the adjoint of a closed symmetric relation in the Kreĭn space  $\{\mathfrak{K}, [\cdot, \cdot]\}$  is, analogous to the Hilbert space case, called a *generalized boundary triplet* if  $\text{mul}\Gamma = \{0\}$ ,  $\text{ran}\Gamma_0 = \mathcal{H}$  and  $A_0$ , defined via  $\text{gr}A_0 = \text{ker}\Gamma_0$ , is a selfadjoint relation in  $\{\mathfrak{K}, [\cdot, \cdot]\}$ , cf. [13, Definition 6.1].

<sup>2</sup>For the definition of a Nevanlinna family see for example [11, Section 2.6].

For later use first observe the following statement about the transformation of a boundary relation by certain standard unitary operators, see [12, Proposition 3.11].

LEMMA 5.4. *Let  $S$  be a closed symmetric relation in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and let  $\{\mathcal{H}, \Gamma\}$  be a boundary relation for  $S^{[*]}$  with  $\text{mul}\Gamma = \{0\}$  and with associated Weyl family  $M_\Gamma(\cdot)$ . Moreover, let  $B$  be a closed operator in  $\{\mathcal{H}, (\cdot, \cdot)\}$  with  $\text{dom}B = \mathcal{H} = \text{ran}B$  and  $\text{ker}B = \{0\}$ , and let  $K$  be a selfadjoint operator in  $\{\mathcal{H}, (\cdot, \cdot)\}$  with  $\text{dom}K = \mathcal{H}$ . Then, with  $\Gamma' := Y_1(K)Y_2(B)\Gamma$ , also  $\{\mathcal{H}, \Gamma'\}$  is a boundary relation for  $S^{[*]}$  with  $\text{mul}\Gamma' = \{0\}$ . Its Weyl family  $M_{\Gamma'}(\lambda)$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , is*

$$M_{\Gamma'}(\lambda) = K + B^{-*}M(\lambda)B^{-1}, \quad \text{dom}M_{\Gamma'}(\lambda) = \text{dom}(M(\lambda)B^{-1}).$$

*Proof.* Since  $Y_1(K)$  and  $Y_2(B)$  are standard unitary operators in the Kreĭn space  $\{\mathcal{H}^2, \langle \cdot, \cdot \rangle\}$ , it is evident that  $\{\mathcal{H}, \Gamma'\}$  is a boundary relation for  $S^{[*]}$ , see Definition 2.2. The expression for  $M_{\Gamma'}$  follows from a direct calculation after the observation that  $\text{dom}\Gamma = \text{dom}\Gamma'$  and, hence,  $\widehat{\mathfrak{N}}_\lambda(T) = \widehat{\mathfrak{N}}_\lambda(T')$ , where  $T$  and  $T'$  are the relations in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  such that  $\text{gr}T = \text{dom}\Gamma$  and  $\text{gr}T' = \text{dom}\Gamma'$ .  $\square$

To obtain results on boundary relations for intermediate extensions, the above lemma is combined with Proposition 5.5 below. Note that the following statement is a generalization of a similar statement for generalized boundary triplets to the Kreĭn space setting, cf. [12, Proposition 4.1].

PROPOSITION 5.5. *Let  $S$  be a closed symmetric relation in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  and let  $\{\mathcal{H}, \Gamma\}$  be a generalized boundary relation for  $S^{[*]}$  with associated Weyl family  $M_\Gamma(\cdot)$ . Moreover, let  $\mathcal{H}'$  be a closed subspace of  $\mathcal{H}$  and define  $\Gamma'$  from  $\mathfrak{K}^2$  to  $\mathcal{H}'^2$  by*

$$\Gamma'\{f, f'\} = \{\Gamma_0\{f, f'\}, \mathcal{P}_{\mathcal{H}'}\Gamma_1\{f, f'\}\}$$

*for all  $\{f, f'\} \in \text{dom}\Gamma$  such that  $\Gamma_0\{f, f'\} \in \mathcal{H}'$ . Then  $\{\mathcal{H}', \Gamma'\}$  is a generalized boundary triplet for  $S_r^{[*]} \subseteq S^{[*]}$ , where  $\text{gr}S_r = \text{ker}\Gamma'$ , and  $\text{ker}\Gamma_0 = \text{ker}\Gamma'_0$ . Its associated Weyl family  $M_{\Gamma'}(\lambda)$ ,  $\lambda \in \mathbb{C}$ , is*

$$M_{\Gamma'}(\lambda) = P_{\mathcal{H}'}M_\Gamma(\lambda), \quad \text{dom}M_{\Gamma'}(\lambda) = \text{dom}M_\Gamma(\lambda) \cap \mathcal{H}' = \mathcal{H}'.$$

*Proof.* The first part is a direct consequence of Corollary 4.20 with  $U_b$  defined as  $U_b\{f, f'\} = \{f, \mathcal{P}_{\mathcal{H}'}f'\}$ ,  $f \in \mathcal{H}'$  and  $f' \in \mathcal{H}$ . The formula for the Weyl family is a direct consequence of the definition of  $\Gamma'$  together with the observation that  $\text{dom}\Gamma' \subseteq \text{dom}\Gamma$  and, hence,  $\widehat{\mathfrak{N}}_\lambda(T') \subseteq \widehat{\mathfrak{N}}_\lambda(T)$ , where  $T$  and  $T'$  are the relations in  $\{\mathfrak{K}, [\cdot, \cdot]\}$  such that  $\text{gr}T = \text{dom}\Gamma$  and  $\text{gr}T' = \text{dom}\Gamma'$ , see Definition 2.4.  $\square$

Now the statements from [12, Section 4], other than [12, Proposition 4.1], can be obtained by combining Proposition 5.5 with Lemma 5.4; following is an example, cf. [12, Corollary 4.5]

COROLLARY 5.6. *Let  $S_i$  be a closed symmetric relation in  $\{\mathfrak{K}_i, [\cdot, \cdot]_i\}$  and let  $\{\mathcal{H}, \Gamma^i\}$  be a generalized boundary triplet for  $S_i^{[*]}$  with associated Weyl family  $M_i$ , for  $i = 1, 2$ . Moreover, with  $\mathfrak{K} := \mathfrak{K}_1 \oplus \mathfrak{K}_2$ , define the operator  $\Gamma$  from  $\mathfrak{K}^2$  to  $\mathcal{H}$  as*

$$\Gamma\{f_1 \oplus f_2, f'_1 \oplus f'_2\} = \{\Gamma_0^1\{f_1, f'_1\}, \Gamma_1^1\{f_1, f'_1\} + \Gamma_1^2\{f_2, f'_2\}\},$$

see (2.8), where

$$\begin{aligned} \text{dom } \Gamma &= \{\{f_1 \oplus f_2, f'_1 \oplus f'_2\} \in \mathfrak{K}^2 : \{f_1, f'_1\} \in \text{dom } \Gamma^1, \{f_2, f'_2\} \in \text{dom } \Gamma^2 \\ &\text{and } \Gamma_0^1\{f_1, f'_1\} = \Gamma_0^2\{f_2, f'_2\}\}. \end{aligned}$$

Then  $\{\mathcal{H}, \Gamma\}$  is a generalized boundary triplet for  $S_r^{[*]} \subseteq S_1^{[*]} \oplus S_2^{[*]}$ , where  $\text{gr } S_r = \ker \Gamma$ , and its associated Weyl family is  $M_1 + M_2$ .

*Proof.* Define  $\Gamma'$  as

$$\Gamma'\{f_1 \oplus f_2, f'_1 \oplus f'_2\} = \left\{ \begin{pmatrix} \Gamma_0^1\{f_1, f'_1\} \\ \Gamma_0^2\{f_2, f'_2\} \end{pmatrix}, \begin{pmatrix} \Gamma_1^1\{f_1, f'_1\} \\ \Gamma_1^2\{f_2, f'_2\} \end{pmatrix} \right\},$$

where  $\{f_1, f'_1\} \in \text{dom } \Gamma^1$  and  $\{f_2, f'_2\} \in \text{dom } \Gamma^2$ . Then  $\{\mathcal{H}^2, \Gamma'\}$  is a generalized boundary triplet for  $S_1^{[*]} \oplus S_2^{[*]}$  with associated Weyl family  $M_1(\cdot) \oplus M_2(\cdot)$ . Next define the operator  $B$  on  $\mathcal{H}^2$  by  $B\{f, f'\} = \{f', f - f'\}$ ,  $f, f' \in \mathcal{H}$ . Then, see Lemma 5.4,  $\{\mathcal{H}^2, \Gamma^B\}$ , where  $\Gamma^B := \Upsilon_2(B)\Gamma'$ , is a generalized boundary triplet for  $S_1^{[*]} \oplus S_2^{[*]}$ . Here

$$\Gamma^B\{f_1 \oplus f_2, f'_1 \oplus f'_2\} = \left\{ \begin{pmatrix} \Gamma_0^2\{f_2, f'_2\} \\ \Gamma_0^1\{f_2, f'_2\} - \Gamma_0^2\{f_1, f'_1\} \end{pmatrix}, \begin{pmatrix} \Gamma_1^1\{f_2, f'_2\} + \Gamma_1^2\{f_1, f'_1\} \\ \Gamma_1^1\{f_2, f'_2\} \end{pmatrix} \right\},$$

for  $\{f_1, f'_1\} \in \text{dom } \Gamma^1$  and  $\{f_2, f'_2\} \in \text{dom } \Gamma^2$ . Its associated Weyl family is

$$M_B(\lambda) = \begin{pmatrix} M_1(\lambda) + M_2(\lambda) & M_1(\lambda) \\ M_1(\lambda) & M_1(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}.$$

After these observations the statement follows directly from Proposition 5.5.  $\square$

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