

SPECTRAL PROPERTIES BETWEEN OPERATOR MATRICES AND HELTON CLASS

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Abstract. In this paper, we study properties of Helton class of an operator matrix. In particular, we show that some upper operator matrix belongs to Helton class of an operator matrix have the property $(\beta)_\varepsilon$. As an application, we get that such operators have nontrivial invariant subspaces. Finally, we prove that Helton class preserves the generalized Weyl's theorem under some conditions.

1. Introduction

Let \mathcal{H} be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on \mathcal{H} . If $T \in \mathcal{L}(\mathcal{H})$, then we shall use the notations $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_{su}(T)$, and $\sigma_e(T)$ for the spectrum, the point spectrum, the approximate point spectrum, the surjective spectrum, and the essential spectrum of T , respectively.

The following concept which is a generalization of the ordinary intertwining condition $RA = AS$ where R and S are in $\mathcal{L}(\mathcal{H})$ stems from [5] and [17]. Let $C(R, S) : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ be defined by $C(R, S)(A) = RA - AS$. The higher order intertwining condition

$$C(R, S)^n(A) = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} R^j A S^{n-j}. \quad (1)$$

In particular, if $A = I$ in (1), then

$$C(R, S)^n(I) = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} R^j S^{n-j}. \quad (2)$$

For an operator $R \in \mathcal{L}(\mathcal{H})$ if there is an integer $n \geq 1$ such that an operator S satisfies $C(R, S)^n(I) = 0$, then we say that S belongs to *Helton class* of R with order n . We denote this by $S \in \text{Helton}_n(R)$. Such an operator S in Helton class of R with order n has been called an intertwining of R and S by the identity I .

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We remark that $C(R, S)^n(I) = 0$ does not imply that $C(S, R)^n(I) \neq 0$ in general. For example, define two operator matrices S and R acting on \mathbb{C}^3 by

$$S = \begin{pmatrix} 0 & 1 & 2011 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to verify that $C(R, S)^n(I) = 0$, but $C(S, R)^n(I) \neq 0$ for some integer $n \geq 2$. In 2005, Helton class was defined by Y. Kim, E. Ko, and J. Lee (see [12]). In [12]–[16], and [18], the authors have been studied spectral and local spectral properties of this class of an operator. Moreover, Y. M. Han and J. Lee (see [8]) gave the result which is related to a -Browder’s theorem under Helton class condition.

In this paper, we study spectral properties of Helton class of an operator matrix. In particular, we show that some upper operator matrix belongs to Helton class of an operator matrix have the property $(\beta)_\epsilon$. As a corollary, we get that such operators have nontrivial invariant subspaces. Finally, we prove that Helton class preserves the generalized Weyl’s theorem under some conditions.

2. Spectral properties

Now we focus on local spectral theory, one of the most important topics in operator theory. This theory is related to the invariant subspace problem, i.e., does every bounded operator T on a separable Hilbert space \mathcal{H} over \mathbb{C} have a non-trivial invariant subspace? Whether every operator in $\mathcal{L}(\mathcal{H})$ has a nontrivial invariant subspace is an unsolved problem, but some special classes of operators are known to have nontrivial invariant subspaces. Another achievement of local spectral theory gives the concept of the property (β) or $(\beta)_\epsilon$. It is known that if an operator $T \in \mathcal{L}(\mathcal{H})$ has the property (β) with thick spectra, then T has a nontrivial invariant subspace (see [10]). Hence the property (β) or $(\beta)_\epsilon$ is closely connected to the invariant subspace problem which is still unsolved. So we focus our research to operators which have the property $(\beta)_\epsilon$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the single-valued extension property at z_0 if for every neighborhood D of z_0 in \mathbb{C} and any analytic function $f : D \rightarrow \mathcal{H}$, with $(T - z)f(z) \equiv 0$, it results $f(z) \equiv 0$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *single-valued extension property* (or SVEP) if it has the single-valued extension property at every z in \mathbb{C} .

Let $\mathcal{R}(T) = \{\lambda \in \mathbb{C} \mid T \text{ fails to SVEP at } \lambda\}$ be an analytic residuum of T . It is a open subset of \mathbb{C} contained in the point spectrum $\sigma_p(T)$ of T . If T has the single-valued extension property, then $\mathcal{R}(T) = \emptyset$. For an operator $T \in \mathcal{L}(\mathcal{H})$ and for $x \in \mathcal{H}$ we can consider the set $\rho_T(x)$ of elements z_0 in \mathbb{C} such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0 , with values in \mathcal{H} , which verifies $(T - z)f(z) \equiv x$. We let $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$ and $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subseteq F\}$, where F is a subset of \mathbb{C} . An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *property (β)* if for every open subset G of \mathbb{C} and every sequence $f_m : G \rightarrow \mathcal{H}$ of \mathcal{H} -valued analytic functions such that $(T - z)f_m(z)$ converges uniformly to 0 in norm on compact subsets of G , then $f_m(z)$ converges uniformly to 0 in norm on compact subsets of G .

Let $\mathcal{E}(\mathcal{U}, \mathcal{H})$ be the Fréchet space of all \mathcal{H} -valued \mathcal{C}^∞ -functions on $\mathcal{U} \subset \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of \mathcal{U} of all derivatives. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the property $(\beta)_\varepsilon$ if for each open set \mathcal{U} in \mathbb{C} , the operator

$$\lambda - T : \mathcal{E}(\mathcal{U}, \mathcal{H}) \rightarrow \mathcal{E}(\mathcal{U}, \mathcal{H}), f \rightarrow (\lambda - T)f$$

is a topological monomorphism, i.e., $(\lambda - T)f_m \rightarrow 0$ in $\mathcal{E}(\mathcal{U}, \mathcal{H})$ implies $f_m \rightarrow 0$ in $\mathcal{E}(\mathcal{U}, \mathcal{H})$ for any $f_m \in \mathcal{E}(\mathcal{U}, \mathcal{H})$. It is well known from [17] that

$$\text{property } (\beta)_\varepsilon \Rightarrow \text{Property } (\beta) \Rightarrow \text{SVEP.}$$

It can be shown that the converse implications do not hold in general as can be seen from [5] and [17].

For given operators $R_1, S_1 \in \mathcal{L}(\mathcal{H}_1)$ and $R_3, S_3 \in \mathcal{L}(\mathcal{H}_2)$, we denote by M_{R_2} and M_{S_2} the operator acting on $\mathcal{H}_1 \oplus \mathcal{H}_2$ of the form

$$M_{R_2} = \begin{pmatrix} R_1 & R_2 \\ 0 & R_3 \end{pmatrix} \text{ and } M_{S_2} = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$$

where two operators R_2, S_2 are in $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$.

Generally, if $M_{S_2} \in \text{Helton}_n(M_{R_2})$, then $S_i \in \text{Helton}_n(R_i)$ for $i = 1, 3$. But the converse statement does not hold even if $n = 2$ (See [14] or [18]). So we consider spectral and local spectral properties between operator matrices M_{R_2} , M_{S_2} , and Helton class (i.e., $S_i \in \text{Helton}_n(R_i)$ for $i = 1, 3$). We begin with the following lemma.

LEMMA 2.1. If $S \in \text{Helton}_n(R)$, then $\mathcal{R}(S) \subset \mathcal{R}(R)$.

Proof. Assume that $\lambda_0 \notin \mathcal{R}(R)$. Let $f : D_{\lambda_0} \rightarrow \mathcal{H}$ be an analytic function such that $(\lambda - S)f(\lambda) \equiv 0$ for all $\lambda \in D_{\lambda_0}$. Since the terms of the below equation are equal to zero when $j + s \neq r$, it suffices to consider only the case of $j + s = r$. This ensures the following equations:

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} (R - \lambda)^j (\lambda - S)^{n-j} \\ &= \sum_{j=0}^n \sum_{r=0}^j \sum_{s=0}^{n-j} (-1)^{n-(s+r)} \binom{n}{j} \binom{j}{r} \binom{n-j}{s} R^r \lambda^{j+s-r} S^{n-(j+s)} \\ &= \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} R^j S^{n-j}. \end{aligned}$$

Hence we have

$$\begin{aligned} & \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} R^j S^{n-j} f(\lambda) - (R - \lambda)^n f(\lambda) \\ &= \sum_{j=0}^n \binom{n}{j} (R - \lambda)^j (\lambda - S)^{n-j} f(\lambda) - (R - \lambda)^n f(\lambda) \\ &= \sum_{j=0}^{n-1} \binom{n}{j} (R - \lambda)^j (\lambda - S)^{n-j} f(\lambda) \\ &= \left[\sum_{j=0}^{n-1} \binom{n}{j} (R - \lambda)^j (\lambda - S)^{n-j-1} \right] (\lambda - S) f(\lambda) = 0. \end{aligned}$$

Since $\sum_{j=0}^n (-1)^{n-j} \binom{n}{j} R^j S^{n-j} = 0$, we get that $(R - \lambda)^n f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$.

Since R has the single-valued extension property at λ_0 , it follows that $(R - \lambda)^{n-1} f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$. By induction, we have that $f(\lambda) \equiv 0$ for all $\lambda \in D_{\lambda_0}$. So we conclude that S has the single-valued extension property at λ_0 . Hence $\lambda_0 \notin \mathcal{R}(S)$, and the proof is finished. \square

THEOREM 2.2. Let $R = M_{R_2}$ and let $S = M_{S_2}$ be above defined where $S_i \in \text{Helton}_n(R_i)$ for $i = 1, 3$ and for $n \geq 2$. Then the following relations hold.

(i) If S_1^* has the single-valued extension property, then

$$\sigma_p(S) \subset \sigma_p(R), \text{ and } \sigma_{ap}(S) \subset \sigma_{ap}(R).$$

(ii) If R_3 has the single-valued extension property, then

$$\sigma_{su}(R) \subset \sigma_{su}(S), \text{ and } \sigma(R) \subset \sigma(S).$$

(iii) If $\mathcal{R}(S_1) \cup \mathcal{R}(S_3) \subset \mathcal{R}(S)$, then $\mathcal{R}(S) \subset \mathcal{R}(R)$.

Proof. (i) Assume that $\lambda \in \sigma_{ap}(S)$. Then there is a sequence $\{x_n \oplus y_n\}$ of unit vectors in $\mathcal{H}_1 \oplus \mathcal{H}_2$ such that

$$(S - \lambda)(x_n \oplus y_n) \rightarrow 0.$$

Then we obtain that $\lim_{n \rightarrow \infty} [(S_1 - \lambda)x_n + S_2 y_n] = 0$ and $\lim_{n \rightarrow \infty} [(S_3 - \lambda)y_n] = 0$. If $y_n \rightarrow 0$, then $(S_1 - \lambda)x_n \rightarrow 0$. We note that $\|x_n \oplus y_n\|^2 = \|x_n\|^2 + \|y_n\|^2 = 1$ for all $n \in \mathbb{N}$. It means that x_n does not converge to 0. Thus $\lambda \in \sigma_{ap}(S_1)$. If y_n does not converge to 0, then it is clear that $\lambda \in \sigma_{ap}(S_3)$. From Theorem 3.6.1 in [18], if $S_i \in \text{Helton}_n(R_i)$ with $i = 1, 2$, then $\sigma_{ap}(S_1) \subset \sigma_{ap}(R_1)$ and $\sigma_{ap}(S_3) \subset \sigma_{ap}(R_3)$. Hence we have $\sigma_{ap}(S) \subset \sigma_{ap}(R_1) \cup \sigma_{ap}(R_3)$. It is well known from [22] that

$$\sigma_{ap}(R) \cup \mathcal{R}(R_1^*) = \sigma_{ap}(R_1) \cup \sigma_{ap}(R_3) \cup \mathcal{R}(R_1^*).$$

Moreover, note from [18] that if S_1^* has the single-valued extension property and $S_1 \in \text{Helton}_n(R_1)$, then $R_1^* \in \text{Helton}_n(S_1^*)$ and so R_1^* has the single-valued extension property. Thus we get that $\sigma_{ap}(R) = \sigma_{ap}(R_1) \cup \sigma_{ap}(R_3)$. Therefore we have

$\sigma_{ap}(S) \subset \sigma_{ap}(R)$. By using a similar way, we have $\sigma_p(S) \subset \sigma_p(R)$.

(ii) Let $\lambda \in \sigma_{su}(R)$. Then we know that $\lambda \in \sigma_{su}(R_1)$ or $\lambda \in \sigma_{su}(R_3)$. If $S_1 \in \text{Helton}_\eta(R_1)$ and $S_3 \in \text{Helton}_\eta(R_3)$, then this relations imply from [18] that $\sigma_{su}(R_1) \subset \sigma_{su}(S_1)$ and $\sigma_{su}(R_3) \subset \sigma_{su}(S_3)$. This ensures that $\lambda \in \sigma_{su}(S_1) \cup \sigma_{su}(S_3)$. As you see from Theorem 2 in [12] that S_3 has the single-valued extension property when R_3 has the single-valued extension property. Since we notice from [22] that $\sigma_{su}(S) \cup \mathcal{R}(S_3) = \sigma_{su}(S_1) \cup \sigma_{su}(S_3) \cup \mathcal{R}(S_3)$, it follows that $\sigma_{su}(S) = \sigma_{su}(S_1) \cup \sigma_{su}(S_3)$. Hence $\lambda \in \sigma_{su}(S)$. The second inclusion holds by a similar method.

(iii) If you apply to Lemma 2.1 in the statement (iii) of Proposition 2.2, then we get this result. \square

LEMMA 2.3. ([22]) For an injective operator B with $CB^m = 0$ for some $m \geq 1$, the following equation holds: for each $x \oplus y$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$,

$$\sigma_{M_C}(x \oplus B^m y) = \sigma_A(x) \cup \sigma_B(y).$$

LEMMA 2.4. Let $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, and $Y = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$ be operator matrices. If $\sigma(A) \cap \sigma(B) = \emptyset$, then for every $x \oplus y$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$,

$$\sigma_{M_C}(x \oplus y) = \sigma_{M_0}(z \oplus y)$$

where $z = x + Xy$.

Proof. Let $\lambda_0 \notin \sigma_{M_C}(x \oplus y)$. There is an analytic function $f : \rho_{M_C(x \oplus y)} \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $(M_C - \lambda)f(\lambda) = (x \oplus y)$. Since $\sigma(A) \cap \sigma(B) = \emptyset$, it ensures that

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}.$$

Since $(M_0 - \lambda)Y = Y(M_C - \lambda)$ for all $\lambda \in \mathbb{C}$, we deduce that $Y(M_C - \lambda)f(\lambda) = (M_0 - \lambda)Yf(\lambda)$. Take a function $h(\lambda) := Yf(\lambda)$. Then h is an analytic function from $\rho_{M_C(x \oplus y)}$ onto $\mathcal{H}_1 \oplus \mathcal{H}_2$ such that $(M_0 - \lambda)h(\lambda) = Y(x \oplus y) = (z \oplus y)$ where $z = x + Xy$. Hence $\lambda_0 \notin \sigma_{M_0}(z \oplus y)$. We obtain the reverse inclusion in a similar fashion. \square

PROPOSITION 2.5. Let $R = M_{R_2}$ and let $S = M_{S_2}$ be above defined in Proposition 2.2. Then the following assertions hold.

(i) If R_1 and R_3 have the single-valued extension property, then

$$\sigma_R(x \oplus 0) \subset \sigma_S(x \oplus 0) \text{ for all } x \in \mathcal{H}_1.$$

(ii) If S_3 is invertible where $R_2 R_3 = 0$ and $R_2 = S_2$, then

$$\sigma_R(x \oplus R_3 y) \subset \sigma_S(x \oplus S_3^m y)$$

for every $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$ and for some integer $m \geq 1$.

(iii) If R_1 and R_3 have the single-valued extension property, and $\sigma(S_1) \cap \sigma(S_3) = \emptyset$, then $\sigma_R(x \oplus y) = \sigma_S(x \oplus y)$ for every $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$.

Proof. (i) Suppose that $\lambda_0 \notin \sigma_S(x \oplus 0)$. Then there exists a neighborhood D of λ_0 and an analytic function $f = f_1 \oplus f_2 : D \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$ where f_1 and f_2 are analytic functions such that

$$(S - \lambda)(f_1(\lambda) \oplus f_2(\lambda)) \equiv (x \oplus 0).$$

Thus $(S_1 - \lambda)f_1(\lambda) + S_2f_2(\lambda) \equiv x$ and $(S_3 - \lambda)f_2(\lambda) \equiv 0$. Since S_3 has the single-valued extension property by hypothesis and Theorem 2 in [12], it follows that $f_2(\lambda) \equiv 0$. Thus we have $(S_1 - \lambda)f_1(\lambda) \equiv x$. Hence $\lambda_0 \notin \sigma_{S_1}(x)$. From Theorem 3.6.4 in [18], if R has the single-valued extension property and $S \in \text{Helton}_n(R)$, then $\sigma_R(x) \subset \sigma_S(x)$ for all $x \in \mathcal{H}$. Thus $\lambda_0 \notin \sigma_{R_1}(x)$ and it means that there exists a neighborhood D of λ_0 and an analytic function $f_1 : D \rightarrow \mathcal{H}$ such that $(R_1 - \lambda)f_1(\lambda) \equiv x$. Then

$$(R - \lambda)(f_1(\lambda) \oplus 0) = ((R_1 - \lambda)f_1(\lambda) \oplus 0) = (x \oplus 0).$$

Hence $\lambda_0 \notin \sigma_R(x \oplus 0)$. So we complete the proof.

(ii) Since S_3 is invertible, it follows from [18] that $\sigma(R_3) \subset \sigma(S_3)$. This means that R_3 also is invertible. The assertion $R_2R_3 = 0$, $R_2 = S_2$, and $S_3 \in \text{Helton}_n(R_3)$ imply $S_2S_3^m = 0$ for a fixed integer m . From Theorem 3.6.4 in [18], we obtain that $\sigma_{R_1}(x) \subset \sigma_{S_1}(x)$ for all $x \in \mathcal{H}_1$ and $\sigma_{R_3}(y) \subset \sigma_{S_3}(y)$ for all $y \in \mathcal{H}_2$. By Lemma 2.3, we obtain that for a fixed integer m

$$\sigma_R(x \oplus R_3y) = \sigma_{R_1}(x) \cup \sigma_{R_3}(y) \subset \sigma_{S_1}(x) \cup \sigma_{S_3}(y) = \sigma_S(x \oplus S_3^m y).$$

Hence the second statement holds.

(iii) By Theorem 2 in [12], the inclusion $\sigma(R) \subset \sigma(S)$ holds if $S \in \text{Helton}_n(R)$. So it is clear that $\sigma(R_1) \cap \sigma(R_3) = \emptyset$ by hypothesis. If you apply to Lemma 2.4 in the assertion (iii) of Proposition 2.5, then we complete the proof. \square

THEOREM 2.6. Let $R = M_{R_2}$ and let $S = M_{S_2}$ be above defined in Proposition 2.2. If R has Bishop’s property (β) with $R_1R_2 = R_2R_3$, then S has Bishop’s property (β) .

Proof. Let $f_n^{(j)} : D \rightarrow \mathcal{H}$ be analytic functions such that $(R_1 - \lambda)f_n^{(1)}(\lambda)$ and $(R_3 - \lambda)f_n^{(2)}(\lambda)$ converge uniformly to zero on compact subset G_j of D for each $j = 1, 2$ where D is an open set in \mathbb{C} . Assume that R has Bishop’s property (β) with $R_1R_2 = R_2R_3$. Then it holds that

$$\begin{aligned} (R - \lambda)(R_2f_n^{(2)}(\lambda) \oplus 0) &= ((R_1 - \lambda)R_2f_n^{(2)}(\lambda)) \oplus 0 \\ &= (R_2(R_3 - \lambda)f_n^{(2)}(\lambda)) \oplus 0 \end{aligned}$$

converges uniformly to zero on compact subset G_j of D . Since R has Bishop’s property (β) , we get that $R_2f_n^{(2)}(\lambda)$ converges uniformly to on compact subset G_j of D . Hence

$$(R - \lambda)(f_n^{(1)}(\lambda) \oplus 0) = ((R_1 - \lambda)f_n^{(1)}(\lambda)) \oplus 0 \rightarrow 0,$$

and

$$(R - \lambda)(0 \oplus f_n^{(2)}(\lambda)) = (R_2f_n^{(2)}(\lambda)) \oplus ((R_3 - \lambda)f_n^{(2)}(\lambda)) \rightarrow 0$$

uniformly on compact subset G_j of D . Since R has Bishop's property (β) , we conclude that $f_n^{(2)}(\lambda) \rightarrow 0$ uniformly on compact subset G_j of D . Hence both R_1 and R_3 have this property. By Theorem 3.7.1 in [18], both S_1 and S_3 has Bishop's property (β) . So we easily show that S has Bishop's property (β) . \square

By a similar method in [10] and Theorem 2.6, we obtain the following corollary.

COROLLARY 2.7. Let $R = M_{R_2}$ and let $S = M_{S_2}$ be above defined in Proposition 2.2. Then the following statements hold.

- (i) If R has the single-valued extension property with $R_1R_2 = R_2R_3$, then S has the single-valued extension property.
- (ii) Suppose that R has Bishop's property (β) where $S_1 - \lambda$ is subjectivity or $R_1R_2 = R_2R_3$. If $\sigma(S)$ has nonempty interior in \mathbb{C} , then S has a nontrivial invariant subspace.

Recall that a closed linear subspace \mathcal{Y} of \mathcal{H} is called a *spectral maximal space* of T if \mathcal{Y} is invariant to T and if \mathcal{Z} is another closed linear subspace of \mathcal{H} , invariant to T , such that $\sigma(T|_{\mathcal{Z}}) \subset \sigma(T|_{\mathcal{Y}})$, then $\mathcal{Z} \subset \mathcal{Y}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *decomposable* if for every finite open covering $\{G_i\}_{i=1}^m$ of $\sigma(T)$ there exists a system $\{\mathcal{M}_i\}_{i=1}^m$ of spectral maximal spaces of T such that $\sigma(T|_{\mathcal{M}_i}) \subset G_i$ for every $i = 1, 2, \dots, m$ and $\mathcal{H} = \sum_{i=1}^m \mathcal{M}_i$, and an operator $R \in \mathcal{L}(\mathcal{H})$ is said to be *algebraic* if $p(R) = 0$ for some nonzero polynomial p .

PROPOSITION 2.8. Let $R = M_{R_2}$ and let $S = M_{S_2}$ be above defined in Proposition 2.2 where $R_3 \in \mathcal{L}(\mathcal{H})$ has Bishop's property (β) and $S_1 \in \text{Helton}_n(R_1)$. If S is an algebraic operator, then R is decomposable.

Proof. If S is an algebraic operator, then $p(S) = 0$ for some nonzero polynomial p . Since $p(\sigma(S)) = \sigma(p(S)) = \{0\}$ by the spectral mapping theorem, it holds that $\sigma(S)$ is contained in the set of zeros of p . It follows from the assertion (ii) in Proposition 2.2 that

$$\sigma(R_1) \cup \sigma(R_2) \subset \sigma(R) \subset \sigma(S)$$

imply $\sigma(R)$ is contained in the set of zeros of p . Hence $\sigma(R)$ is a finite set. Hence R is decomposable by [17]. \square

THEOREM 2.9. Let S_1, S_2 , and S_3 be in $\mathcal{L}(\mathcal{H})$. Let R_1 (or R_3) $\in \mathcal{L}(\mathcal{H})$ have the property $(\beta)_\varepsilon$, $S_1 \in \text{Helton}_n(R_1)$, and $S_3 \in \text{Helton}_n(R_3)$ where $R_1R_2 = R_2R_3$, and R_2 is a bounded below. Then $S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$ has the property $(\beta)_\varepsilon$.

Proof. Suppose that R_1 has the property $(\beta)_\varepsilon$. We will prove that S has the property $(\beta)_\varepsilon$. First, we claim that R_3 has the property $(\beta)_\varepsilon$. Let $R_1 \in \mathcal{L}(\mathcal{H})$ have the property $(\beta)_\varepsilon$ where $R_1R_2 = R_2R_3$, and R_2 is a bounded below. If $\{f_m\}$ is a sequence in $\mathcal{E}(U, \mathcal{H})$ such that $(R_3 - \lambda)f_m(\lambda) \rightarrow 0$ in $\mathcal{E}(U, \mathcal{H})$, then $(R_1 - \lambda)R_2f_m(\lambda) \rightarrow 0$ in $\mathcal{E}(U, \mathcal{H})$. Since R_1 has the property $(\beta)_\varepsilon$, it follows that $R_2f_m(\lambda) \rightarrow 0$ in $\mathcal{E}(U, \mathcal{H})$. Since R_2 is a bounded below, it ensures that $f_m(\lambda) \rightarrow 0$ in $\mathcal{E}(U, \mathcal{H})$. So we complete the claim. Let $f_j = \bigoplus_{i=1}^2 f_j^i$ be an analytic $\bigoplus_{i=1}^2 \mathcal{H}$ -valued function defined

on an open set \mathcal{U} , where $f_j^i : \mathcal{U} \rightarrow \mathcal{H}$ are analytic functions for $i = 1, 2$. Now if $(\lambda - S)f_j(\lambda) \rightarrow 0$ in $\mathcal{E}(\mathcal{U}, \mathcal{H})$, then we get the following equations:

$$\begin{cases} \lim_{j \rightarrow \infty} [(\lambda - S_1)f_j^1(\lambda) - S_2f_j^2(\lambda)] = 0 \\ \lim_{j \rightarrow \infty} [(\lambda - S_3)f_j^2(\lambda)] = 0 \end{cases}$$

in $\mathcal{E}(\mathcal{U}, \mathcal{H})$. Then we get that

$$\begin{aligned} & \lim_{m \rightarrow \infty} \left\| \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} R_3^j S_3^{n-j} f_m^2(\lambda) - (R_3 - \lambda)^n f_m^2(\lambda) \right\| \\ &= \lim_{m \rightarrow \infty} \left\| \sum_{j=0}^n \binom{n}{j} (R_3 - \lambda)^j (\lambda - S_3)^{n-j} f_m^2(\lambda) - (R_3 - \lambda)^n f_m^2(\lambda) \right\| \\ &= \lim_{m \rightarrow \infty} \left\| \sum_{j=0}^{n-1} \binom{n}{j} (R_3 - \lambda)^j (\lambda - S_3)^{n-j} f_m^2(\lambda) \right\| \\ &\leq \lim_{m \rightarrow \infty} \left\| \sum_{j=0}^{n-1} \binom{n}{j} (R_3 - \lambda)^j (\lambda - S_3)^{n-j-1} \right\| \|(\lambda - S_3) f_m^2(\lambda)\| \\ &= 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{H}). \end{aligned}$$

Since $\sum_{j=0}^n (-1)^{n-j} \binom{n}{j} R_3^j S_3^{n-j} = 0$, it follows that

$$\lim_{m \rightarrow \infty} \left\| (R_3 - \lambda)^n f_m^2(\lambda) \right\| = 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{H}).$$

Since R_3 has the property $(\beta)_\varepsilon$, it ensures that $f_j^2(\lambda) \rightarrow 0$ in $\mathcal{E}(\mathcal{U}, \mathcal{H})$. Thus $S_2f_j^2(\lambda) \rightarrow 0$ in $\mathcal{E}(\mathcal{U}, \mathcal{H})$. Hence $(\lambda - S_1)f_j^1(\lambda) \rightarrow 0$ in $\mathcal{E}(\mathcal{U}, \mathcal{H})$. Since R_1 has the property $(\beta)_\varepsilon$ and $S_1 \in \text{Helton}_n(R_1)$, it follows that S_1 has the property $(\beta)_\varepsilon$ by a similar method. Hence $f_j^1(\lambda) \rightarrow 0$ in $\mathcal{E}(\mathcal{U}, \mathcal{H})$. Thus $f_j(\lambda) = f_j^1(\lambda) \oplus f_j^2(\lambda) \rightarrow 0$ in $\mathcal{E}(\mathcal{U}, \mathcal{H})$. Hence S has the property $(\beta)_\varepsilon$. \square

The next corollary follows from Theorem 2.9 and [10].

COROLLARY 2.10. Under the same hypothesis in Theorem 2.9. If $\sigma(R_3)$ has interior in \mathbb{C} , then S has a nontrivial invariant subspace.

3. Weyl type theorem

We say that *Weyl's theorem holds* for T if

$$\sigma(T) \setminus \pi_{00}(T) = \sigma_w(T), \text{ or equivalently, } \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

where $\pi_{00}(T) = \{\lambda \in \text{iso}(\sigma(T)) : 0 < \dim \ker(T - \lambda) < \infty\}$ and $\text{iso}(\sigma(T))$ denotes the set of all isolated points of $\sigma(T)$. Recall from [4] that for each nonnegative integer

k define T_k to the restriction of T to $\text{ran}(T^k)$ viewed as a map from $\text{ran}(T^k)$ into $\text{ran}(T^k)$, in particular, $T_0 = T$. If for some k the space $\text{ran}(T^k)$ is closed and T_k is a Fredholm operator, then T is called a *B-Fredholm operator*. In this case, by Proposition in [4], T_m is a Fredholm operator and $\text{ind}(T_m) = \text{ind}(T_k)$ for each $m \geq k$. Thus, the *index* $\text{ind}(T)$ of T is defined as the index of the Fredholm operator T_k . Let $BF(\mathcal{H})$ be the class of all *B-Fredholm operators*. In [4], Berkani has proved that if an operator T is B-Fredholm if and only if $T = T_1 \oplus T_2$ where T_1 is a Fredholm operator and T_2 is nilpotent. Let $SBF_+^-(\mathcal{H})$ be the class of all upper semi-*B-Fredholm operators* such that $\text{ind}(T) \leq 0$, and let

$$\sigma_{SBF_+^-}(T) := \{\lambda \in \mathbb{C} \mid T - \lambda \in SBF_+^-(\mathcal{H})\}.$$

An operator T is called a *B-Weyl operator* if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is not a B-Weyl operator}\}.$$

Recall that $p_0(T)$ denotes the set of all poles of T and $\pi_0(T)$ is the set of all eigenvalues of T which is an isolated point in $\sigma(T)$.

DEFINITION 3.1. Let $T \in \mathcal{L}(\mathcal{H})$. We call that

- (i) T satisfies the generalized Browder's theorem if $\sigma_{BW}(T) = \sigma(T) \setminus p_0(T)$.
- (ii) T satisfies the generalized Weyl's theorem if $\sigma_{BW}(T) = \sigma(T) \setminus \pi_0(T)$.

THEOREM 3.2. Let R have the single-valued extension property and $S \in \text{Helton}_n(R)$. If R satisfies Weyl's theorem and $\pi_{00}(S) \subset \pi_{00}(R)$, then so does S .

Proof. Since S has the single-valued extension property by [18], it ensures that S enjoys Browder's theorem. We will establish Weyl's theorem holds for S . It suffices to prove $\pi_{00}(T) = p_{00}(T)$, or equivalently, $H(S - \lambda)$ is finite-dimensional for all $\lambda \in \pi_{00}(T)$ by Theorem 3.84 in [1]. Let $\lambda \in \pi_{00}(S)$. Then $\lambda \in \pi_{00}(R)$. Since R entails Weyl's theorem, it follows that $\pi_{00}(R) = p_{00}(R)$.

Let $x \in H_0(S - \lambda)$. Since $S \in \text{Helton}_n(R)$ implies $S - \lambda \in \text{Helton}_n(R - \lambda)$, and so from [18] that $(S - \lambda)^m \in \text{Helton}_n((R - \lambda)^m)$ for any positive integer n and any integer $m \geq 2$. Hence

$$\sum_{j=0}^n (-1)^{n-j} \binom{n}{j} ((R - \lambda)^m)^j ((S - \lambda)^m)^{n-j} = 0.$$

Therefore for any $x \in H_0(S - \lambda)$ we have

$$\left[\sum_{j=0}^{n-1} (-1)^{n-j} \binom{n}{j} ((R - \lambda)^m)^j ((S - \lambda)^m)^{n-j-1} \right] (S - \lambda)^m x = -(R - \lambda)^m x.$$

Since $\lim_{m \rightarrow \infty} \|(S - \lambda)^m x\|^{\frac{1}{m}} = 0$ for all $x \in H_0(S - \lambda)$, it follows that

$$\lim_{m \rightarrow \infty} \|(R - \lambda)^m x\|^{\frac{1}{m}} = 0$$

for all $x \in H_0(S - \lambda)$ and any integer $n \geq 2$. Hence $\lim_{m \rightarrow \infty} \|(R - \lambda)^{mn} x\|^{\frac{1}{mn}} = 0$ for all $x \in H_0(S - \lambda)$ and any integer $n \geq 2$. Thus $\lim_{N \rightarrow \infty} \|(R - \lambda)^N x\|^{\frac{1}{N}} = 0$ for all $x \in H_0(S - \lambda)$. Therefore $x \in H_0(R - \lambda)$, and hence $H_0(S - \lambda) \subseteq H_0(R - \lambda)$. Since Weyl's theorem holds for R , it ensures that $H_0(R - \lambda)$ is finite-dimensional. This forces that $H_0(S - \lambda)$ also is finite-dimensional. Hence S satisfies Weyl's Theorem. \square

Recall that an operator T is said to be *isoloid* if every isolated point of the spectrum $\sigma(T)$ is an eigenvalue and an operator T is said to be *polaroid* if every isolated point of $\sigma(T)$ is a pole of the resolvent operator $(\lambda - T)^{-1}$, or equivalently $0 < p(\lambda - T) = q(\lambda - T) < \infty$ for every $\lambda \in \text{iso}\sigma(T)$ (see [1] for more details).

THEOREM 3.3. Let R be a polaroid which has the single-valued extension property and $S \in \text{Helton}_n(R)$. Suppose that R entails the generalized Weyl's theorem, S^* has the single-valued extension property, and $\pi_0(S) \subset \pi_0(R)$. Then the generalized Weyl's theorem holds for S .

Proof. Since R and S^* have the single-valued extension property and $S \in \text{Helton}_n(R)$, we know that R and R^* have the single-valued extension property. Moreover, we obtain that $\sigma(R) = \sigma(S)$ from [18]. Thus the generalized Browder's theorem holds for S . It is enough to prove $\pi_0(S) = p_0(S)$. Let $\lambda \in \pi_0(R)$. Then λ is an isolated point in $\sigma(S) = \sigma(R)$. Since $R - \lambda$ is not injective, this implies that $\ker(R - \lambda) \subset H_0(R - \lambda)$. Hence, in this case, $H_0(R - \lambda) = H_0(S - \lambda)$ for all $\lambda \in \mathbb{C}$. It follows from Theorem 2.8 in [6] that $\lambda \in \sigma(S)$. Since $\sigma(S) \subset \sigma(R)$, it gives that λ is an isolated point in $\sigma(S)$. Moreover, we know that λ is an eigenvalue of S and so $\lambda \in \pi_0(S)$. Since R is polaroid, there exists a positive integer $p := p(\lambda) \in \mathbb{N}$ such that $H_0(R - \lambda) = \ker((R - \lambda)^p)$. If $(R - \lambda)$ is injective, then $(R - \lambda)^p$ should be injective. Hence $H_0(R - \lambda) = \{0\}$ which is a contradiction. Therefore we have $\lambda \in \pi_0(R)$. By symmetry we conclude that $\pi_0(S) = \pi_0(R)$. Since R enjoys the generalized Weyl's theorem, it follows that $\pi_0(R) = p_0(R)$ and so $\pi_0(S) \subset p_0(S)$. This means that the generalized Weyl's theorem holds for S . \square

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