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# $C^*$–ALGEBRAS GENERATED BY THREE PROJECTIONS

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*Abstract.* In this short note, we prove that for a $C^*$–algebra $\mathscr{A}$ generated by $n$ elements, $\mathrm{M}_k(\tilde{\mathscr{A}})$ is generated by $k$ mutually unitarily equivalent and almost mutually orthogonal projections for any $k \geqslant \delta(n) = \min\{k \in \mathbb{N} \mid (k-1)(k-2) \geqslant 2n\}$. Then combining this result with recent works of Nagisa, Thiel and Winter on the generators of $C^*$–algebras, we show that for a $C^*$–algebra $\mathscr{A}$ generated by finite number of elements, there is $d \geqslant 3$ such that $\mathrm{M}_d(\tilde{A})$ is generated by three mutually unitarily equivalent and almost mutually orthogonal projections. Furthermore, for certain separable purely infinite simple unital $C^*$–algebras and $AF$–algebras, we give some conditions that make them be generated by three mutually unitarily equivalent and almost mutually orthogonal projections.

## 1. Introduction

Let $H$ be a separable complex Hilbert space with $\dim H = \infty$. Let $P$ and $Q$ be two (orthogonal) projections on $H$. Put $M = PH$ and $N = QH$. Due to Halmos [5], $P$ and $Q$ are in generic position if

$$M \cap N = \{0\},\ M \cap N^{\perp} = \{0\},\ M^{\perp} \cap N = \{0\},\ M^{\perp} \cap N^{\perp} = \{0\}.$$

Then the unital $C^*$–algebra generated by two projections $P$ and $Q$, which are in generic position, is $*$–isomorphic to $\{f \in \mathrm{M}_2(C(\sigma((P-Q)^2)) \mid f(0), f(1) \text{ are diagonal}\}$ (cf. [18, Theorem 1.1]). Furthermore, by [13, Theorem 1.3], the the universal $C^*$–algebra $C^*(p,q)$ generated by two projections $p$ and $q$ is $*$–isomorphic to the $C^*$–algebra

$$\{f \in M_2(C([0,1])) \mid f(0),\ f(1) \text{ are diagonal}\}$$

which is of Type I. But in the general case of the $C^*$–algebra generated by a finite set of orthogonal projections (at least three projections), the situation becomes unpredictable. For example, Davis showed in [4] that there exist three projections $P_1$, $P_2$ and $P_3$ on $H$ such that the von Neumann algebra $W^*(P_1,P_2,P_3)$ generated by $P_1$, $P_2$ and $P_3$ coincides with $B(H)$ of all bounded linear operators acting on $H$. Furthermore, Sunder proved in [16] that for each $n \geqslant 3$, there exist $n$ projections $P_1,\cdots,P_n$ on $H$ such that the von Neumann algebra $W^*(P_1,\cdots,P_n)$ generated by $P_1,\cdots,P_n$ is $B(H)$ and $W^*(\mathscr{M}) \subsetneqq B(H)$, whenever $\mathscr{M} \subsetneqq \{P_1,\cdots,P_n\}$, where $W^*(\mathscr{M})$ is the von Neumann algebra generated by all elements in $\mathscr{M}$.

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Therefore investigating the $C^*$–algebra generated by $n$ $(n \geqslant 3)$ projections is an interesting topic. Shulman studied the universal $C^*$–algebras generated by $n$ projections $p_1, \cdots, p_n$ subject to the relation $p_1 + \cdots + p_n = \lambda 1$, $\lambda \in \mathbb{R}$ in [15]. She gave some conditions to make these $C^*$–algebras type I, nuclear or exact and proved that among these $C^*$–algebras, there is a continuum of mutually non–isomorphic ones. Meanwhile, Vasilevski considered the problem in [18] that given finite set of (orthogonal) projections $P, Q_1, \cdots, Q_n$ on $H$ with the conditions

$$Q_j Q_k = \delta_{j,k} Q_k, \qquad j,k = 1, \cdots, n, \quad Q_1 + \cdots + Q_n = I, \tag{1}$$

$$PH \cap (Q_k H)^{\perp} = \{0\}, \quad Q_k H \cap (PH)^{\perp} = \{0\}, \quad k = 1, \cdots, n. \tag{2}$$

Then what is the $C^*$–algebra $C^*(Q, P_1, \cdots, P_n)$ generated $Q, P_1, \cdots, P_n$? One of interesting results concerning this problem is Corollary 4.5 of [18], which can be described as follows.

Let $\mathscr{A}$ be a finitely generated $C*$–algebra with identity in $B(H)$ and let $n_0$ be a minimal number of self–adjoint elements generating $\mathscr{A}$. Then for each $n > n_0$, there exist projections $P, Q_1, \cdots, Q_n$ on $H$ satisfying (1) and (2) such that $M_n(\mathscr{A})$ is $*$–isomorphic to $C^*(P, Q_1, \cdots, Q_n)$.

Inspired by above works, we study the problem: find least number of projections in the matrix algebra of a given finitely generated $C^*$–algebra such that these projections generates this $C^*$–algebra in this short note. The main results of the paper are the following:

Let $\mathscr{A} = C^*(a_1, \cdots, a_n)$ be the $C^*$–algebra generated by elements $a_1, \cdots, a_n$. Let $\tilde{\mathscr{A}}$ denote the $C^*$–algebra obtained by adding the unit $1$ to $\mathscr{A}$ (if $\mathscr{A}$ is non–unital) and let $\mathrm{M}_k(\tilde{\mathscr{A}})$ denote the algebra of all $n \times n$ matrices with entries in $\tilde{\mathscr{A}}$. Then

(1) for any $k \geqslant \delta(n) = \min\{k \in \mathbb{N} \,|\, (k-1)(k-2) \geqslant 2n\}$, $\mathrm{M}_k(\tilde{\mathscr{A}})$ is generated by $k$ mutually unitarily equivalent and almost mutually orthogonal projections (see Theorem 2.3).

(2) for every $l \geqslant \{\sqrt{n-1}\}$ and $k \geqslant 3$, $\mathrm{M}_{kl}(\tilde{\mathscr{A}})$ is generated by $k$ mutually unitarily equivalent and almost mutually orthogonal projections (see Proposition 3.4), where $\{x\}$ stands for the least natural number that is greater than or equal to the positive number $x$.

## 2. The main result

In this section, we will give our main result (1) mentioned in §1. Firstly, we have

LEMMA 2.1. *Let $\mathscr{A}$ be a $C^*$–algebra with unit $1$ and $B_{ij} \in \mathscr{A}$, for any $1 \leqslant i < j \leqslant k$. Suppose that $\eta = \max\{\|B_{ij}\| \,|\, 1 \leqslant i < j \leqslant k\} < \frac{1}{2(k-1)}$, then*

$$T = \begin{bmatrix} 1 & B_{12} & \cdots & B_{1k} \\ B_{12}^* & 1 & \cdots & B_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ B_{1k}^* & B_{2k}^* & \cdots & 1 \end{bmatrix}$$

*is invertible and positive, and*

$$\|T-1_k\| \leqslant (k-1)\eta, \quad \|T^{-1/2}-1_k\| \leqslant 2(k-1)\eta,$$

*where* $1_k$ *is the unit of* $\mathrm{M}_k(\mathscr{A})$.

*Proof.* By the definition of the norm of $\mathrm{M}_k(\tilde{\mathscr{A}})$, $\|A\| = \|[\pi(A_{ij})]_{k\times k}\|$, for $A = [A_{ij}]_{k\times k} \in \mathrm{M}_k(\tilde{\mathscr{A}})$, where $\pi$ is any faithful representation of $\tilde{\mathscr{A}}$ on a Hilbert space $K$ (see [10]), we may assume that $\tilde{\mathscr{A}} \subset B(K)$ and the identity operator on $K$ is the unit of $\tilde{\mathscr{A}}$. So $T \in B(K_k)$, where $K_k = \underbrace{K\oplus\cdots\oplus K}_{k}$.

For any $\lambda < 1-(k-1)\eta$, set

$$A = \begin{bmatrix} 1-\lambda & -\|B_{12}\| & \cdots & -\|B_{1k}\| \\ -\|B_{12}\| & 1-\lambda & \cdots & -\|B_{2k}\| \\ \vdots & & & \\ -\|B_{1k}\| & -\|B_{2k}\| & \cdots & 1-\lambda \end{bmatrix}.$$

Since for any $i$, $\sum\limits_{i\neq j}\|B_{ij}\| < 1-\lambda$, it follows from Levy–Dedplanques Theorem in Matrix Analysis (see [7]) that $A$ is positive and invertible. So the quadratic form

$$f(x_1,x_2,\cdots,x_k) = \sum_{i=1}^{k} x_i^2 - 2\sum_{1\leqslant i<j\leqslant k}\|B_{ij}\|x_ix_j$$

is positive definite and consequently, there exits $\delta > 0$ such that for any $(x_1,\cdots,x_k) \in \mathbb{R}^n$, $f(x_1,\cdots,x_k) \geqslant \delta\big(\sum\limits_{i=1}^{k} x_i^2\big)$.

Now for any $\xi = (\xi_1,\cdots,\xi_n) \in K_k$, we have

$$\begin{aligned} \langle (T-\lambda 1_k)\xi,\xi\rangle &= \sum_{i=1}^{k}\|\xi_i\|^2 + \sum_{1\leqslant i<j\leqslant k}\Big(\langle B_{ij}\xi_i,\xi_j\rangle + \langle B_{ij}^*\xi_j,\xi_i\rangle\Big) \\ &\geqslant \sum_{i=1}^{k}\|\xi_i\|^2 - 2\sum_{1\leqslant i<j\leqslant k}\|B_{ij}\|\|\xi_i\|\|\xi_j\| \\ &= f(\|\xi_1\|,\cdots,\|\xi_k\|) \geqslant \delta(\sum_{i=1}^{k}\|\xi_i\|^2) \end{aligned}$$

by above argument. Thus, $T-\lambda 1_k$ is invertible. Similarly, for any $\lambda > 1+(k-1)\eta$, $T-\lambda 1_k$ is also invertible.

Let $\sigma(T)$ denote the spectrum of $T$. Then we have

$$\sigma(T) \subset [1-(k-1)\eta, 1+(k-1)\eta] \subset (0,2),$$

This indicates that $T$ is positive and invertible. Finally, by the Spectrum Mapping Theorem, $\sigma(1_k - T) \subset [-(k-1)\eta, (k-1)\eta]$ and

$$\begin{aligned}\sigma(1_k - T^{-1/2}) &\subset [1-(1-(k-1)\eta)^{-1/2}, 1-(1+(k-1)\eta)^{-1/2}]\\ &\subset [-2(k-1)\eta, 2(k-1)\eta].\end{aligned}$$

So $\|T - 1_k\| \leqslant (k-1)\eta$ and $\|T^{-1/2} - 1_k\| \leqslant 2(k-1)\eta$.

DEFINITION 2.2. We say that a unital $C^*$–algebra $\mathscr{E}$ is generated by $n$ $(n \geqslant 2)$ mutually unitarily equivalent and almost mutually orthogonal projections if for any given $\varepsilon > 0$, there exist projections $p_1, \cdots, p_n$ in $\mathscr{E}$ satisfying following conditions:

(1) $p_1 + \cdots + p_n$ is invertible in $\mathscr{E}$,

(2) $C^*(p_1, \cdots, p_n) = \mathscr{E}$ and

(3) for any $i \neq j$, $p_i$ is unitarily equivalent to $p_j$ in $\mathscr{E}$ and $\|p_i p_j\| < \varepsilon$.

Now we present one of our main results as follows.

THEOREM 2.3. *Suppose that the $C^*$–algebra $\mathscr{A}$ is generated $n$ elements $a_1, \cdots,$ $a_n$. Then for each $k \geqslant \delta(n) = \min\{k \in \mathbb{N} \,|\, (k-1)(k-2) \geqslant 2n\}$, $\mathrm{M}_k(\tilde{\mathscr{A}})$ is generated by $k$ mutually unitarily equivalent and almost mutually orthogonal projections.*

*Proof.* We assume that $\mathscr{A}$ is non–unital. If $\mathscr{A}$ is unital, $\tilde{\mathscr{A}} = \mathscr{A}$. Without loss generality, we may assume that $\|a_i\| = 1$, $i = 1, \cdots, n$. Furthermore, we can assume $n = \frac{(k-1)(k-2)}{2}$. Otherwise, for any $n < i \leqslant \frac{(k-1)(k-2)}{2}$, put $a_i = 1$, where $1$ is the unit of $\tilde{\mathscr{A}}$.

Rewrite $\{a_1, \cdots, a_n\} = \{B_{ij} : 1 \leqslant i < j \leqslant k-2\}$ (for $\delta(n) \geqslant 3$) and define

$$T_\varepsilon = \begin{bmatrix} 1 & \varepsilon B_{12} & \cdots & \varepsilon B_{1,k-1} & \varepsilon 1 \\ \varepsilon B_{12}^* & 1 & \cdots & \varepsilon B_{2,k-1} & \varepsilon 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varepsilon B_{1,k-1}^* & \varepsilon B_{2,k-1}^* & \cdots & 1 & \varepsilon 1 \\ \varepsilon 1 & \varepsilon 1 & \cdots & \varepsilon 1 & 1 \end{bmatrix}, \quad \forall \varepsilon \in (0, 1/8(k-1)).$$

Using the canonical matrix units $\{e_{ij}\}$ for $\mathrm{M}_k(\mathbb{C})$, we have

$$T_\varepsilon = \sum_{i=1}^{k} \big(1 \otimes e_{ii}\big) + \sum_{i=1}^{k-1} \big(\varepsilon 1 \otimes e_{i,k} + \varepsilon 1 \otimes e_{k,i}\big) + \sum_{1 \leqslant i < j \leqslant k-1} \big(\varepsilon B_{ij} \otimes e_{ij} + \varepsilon B_{ij}^* \otimes e_{ji}\big).$$

By Lemma 2.1, $T_\varepsilon$ is positive and invertible with $\|1_k - T_\varepsilon\| \leqslant (k-1)\varepsilon$ and $\|1_k - T_\varepsilon^{-1/2}\| \leqslant 2(k-1)\varepsilon$.

Define $p_i(\varepsilon) = T_\varepsilon^{1/2}(1 \otimes e_{ii}) T_\varepsilon^{1/2}$, $i = 1, \cdots, k$. It is easy to verify that $p_i(\varepsilon)$ is a projection and $C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon)) \subset \mathrm{M}_k(\tilde{\mathscr{A}})$. In the following, we will show $\mathrm{M}_k(\tilde{\mathscr{A}}) \subset C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon))$.

For all $1 \leqslant i \leqslant k$, $p_i(\varepsilon) \in C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon))$ implies $T_\varepsilon = \sum_{i=1}^{k} p_i(\varepsilon)$ is contained in $C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon))$. Then $T_\varepsilon^{-1/2} \in C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon))$ by Gelfand's Theorem (cf. [19, Theorem 1.5.10]), which implies that for any $1 \leqslant i \leqslant k$,

$$1 \otimes e_{ii} = T_\varepsilon^{-1/2} p_i(\varepsilon) T_\varepsilon^{-1/2} \in C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon)).$$

It follows that for any $1 \leqslant i < j \leqslant k-1$,

$$B_{ij} \otimes e_{ij} = (1 \otimes e_{ii}) T_\varepsilon (1 \otimes e_{jj}) \in C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon))$$

and for any $1 \leqslant i \leqslant k-1$,

$$1 \otimes e_{ik} = (1 \otimes e_{ii}) T_\varepsilon (1 \otimes e_{kk}) \in C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon)).$$

So $1 \otimes e_{ki} = (1 \otimes e_{ik})^* \in C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon))$ and hence, for any $1 \leqslant i < j \leqslant k-1$,

$$1 \otimes e_{ij} = (1 \otimes e_{ii})(1 \otimes e_{ik})(1 \otimes e_{kj}) \in C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon))$$

and $1 \otimes e_{ji} = (1 \otimes e_{ij})^* \in C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon))$. Consequently, for any $1 \leqslant i < j \leqslant k$ and $1 \leqslant m \leqslant k$,

$$B_{ij} \otimes e_{mm} = (1 \otimes e_{mi})(B_{ij} \otimes e_{ij})(1 \otimes e_{jm}) \in C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon)).$$

Since for $i = 1, \cdots, k$, $\tilde{\mathscr{A}} \otimes e_{ii}$ is a $C^*$–algebra, we get for $1 \leqslant i \leqslant k$, $\tilde{\mathscr{A}} \otimes e_{ii} \subset C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon))$ and for $1 \leqslant i, j \leqslant k$,

$$\tilde{\mathscr{A}} \otimes e_{ij} = (\tilde{\mathscr{A}} \otimes e_{ii})(1 \otimes e_{ij}) \subset C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon)).$$

At last, we obtain that $\mathrm{M}_k(\tilde{\mathscr{A}}) \subset C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon))$.

Put $I_i = 1 \otimes e_{ii} = T_\varepsilon^{-1/2} p_i(\varepsilon) T_\varepsilon^{-1/2}$, $i = 1, \cdots, k$. Then $\{I_1, \cdots, I_k\}$ is a family of mutually equivalent and mutually orthogonal projections in $C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon))$. Now for $1 \leqslant i, j \leqslant k$, $i \neq j$,

$$\begin{aligned} &\|p_j(\varepsilon) - I_j\| \leqslant \|(1_k - T_\varepsilon^{-1/2}) p_j(\varepsilon)\| + \|p_j(\varepsilon) T_\varepsilon^{-1/2}(1_k - T_\varepsilon^{-1/2})\| < 8(k-1)\varepsilon < 1 \\ &\|p_i(\varepsilon) p_j(\varepsilon)\| \leqslant \|p_i(\varepsilon)(p_j(\varepsilon) - I_j)\| + \|(p_i(\varepsilon) - I_i) I_j\| < 16(k-1)\varepsilon. \end{aligned}$$

So $p_j(\varepsilon)$ is unitarily equivalent to $I_j$ by Lemma 6.5.9 of [19], then to $p_i(\varepsilon)$ and $p_1(\varepsilon), \cdots, p_k(\varepsilon)$ are almost mutually orthogonal in $C^*(p_1(\varepsilon), \cdots, p_k(\varepsilon))$. $\square$

EXAMPLE 2.4. (1) Since $\mathbb{C}$ is generated by $\{1\}$, it follows from Theorem 2.3 that for any $k \geqslant 3$, $\mathrm{M}_k(\mathbb{C})$ is generated by $k$ mutually unitarily equivalent and almost mutually orthogonal projections.

(2) Let $\mathscr{B}$ be a separable unital $C^*$–algebra and $\mathscr{K}$ be the $C^*$–algebra of compact operators on the separable complex Hilbert space $H$. Then $\mathscr{B} \otimes \mathscr{K}$ is generated by a single element (cf. [12, Theorem 8]). So $\mathrm{M}_3(\widetilde{\mathscr{B} \otimes \mathscr{K}})$ is generated by 3 mutually unitarily equivalent and almost mutually orthogonal projections.

REMARK 2.5. Suppose that the $C^*$–algebra $\mathscr{E}$ with the unit $1_{\mathscr{E}}$ is generated by $k$ mutually unitarily equivalent and almost mutually orthogonal projections. Then by Definition 2.2, there are projections $p_1,\cdots,p_k$ such that $\sum_{i=1}^{k} p_i$ is invertible in $\mathscr{E}$, $p_1,\cdots,p_k$ are mutually unitarily equivalent in $\mathscr{E}$ and $\|p_ip_j\| < 1/2(k-1)$. Then by Corollary 3.8 of [6] and its proof, there exist mutually orthogonal projections $p'_1,\cdots,p'_k$ in $\mathscr{E}$ such that $\|p_i - p'_i\| < 1$ and $\sum_{i=1}^{k} p'_i = 1_{\mathscr{E}}$. Consequently, $p_i$ is unitarily equivalent to $p'_i$ in $\mathscr{E}$ by [19, Lemma 6.5.9 (2)] and so that $p'_i$ is unitarily equivalent to $p'_j$ in $\mathscr{E}$, $i,j = 1,\cdots,k$.

Now we use the $K$–Theory of $\mathscr{E}$ to describe above situations. The notations and properties of $K$–Theory of $C^*$–algebras can be found in references [10] and [19]. Let $[p_i]$ (resp. $[p'_i]$) be the class of $p_i$ (resp. $[p'_i]$) in $K_0(\mathscr{E})$, $i = 1,\cdots,k$. Then we have $[1_{\mathscr{E}}] = [\sum_{i=1}^{k} p'_i] = \sum_{i=1}^{k} [p'_i] = k[p_1]$.

## 3. Some applications

Let $\mathscr{A}$ be a $C^*$–algebra and let $M$ be a subset of $\mathscr{A}_{sa}$. We call $M$ a generator of $\mathscr{A}$ if $\mathscr{A}$ is equal to the $C^*$–algebra $C^*(M)$ generated by elements in $M$. If $M$ is finite, then we call $\mathscr{A}$ finitely generated and we define the number of generators $gen(A)$ by the minimum cardinality of $M$ which generates $\mathscr{A}$. We denote $gen(\mathscr{A}) = \infty$ unless $\mathscr{A}$ is finitely generated (cf. [11]). We call a $C^*$–algebra $\mathscr{A}$ singly generated if $gen(\mathscr{A}) \leqslant 2$. Indeed, if $\mathscr{A} = C^*(\{x,y\})$ for $x,y \in \mathscr{A}_{sa}$, then $C^*(x+iy) = \mathscr{A}$.

LEMMA 3.1. [11, Theorem 3] *Let $\mathscr{A}$ be a unital $C^*$–algebra with $gen(\mathscr{A}) \leqslant n^2+1$ $(n \in \mathbb{N})$. Then we have $gen(\mathrm{M}_n(\mathscr{A})) \leqslant 2$.*

Similar to the definition of $gen(\mathscr{A})$, we have following definition:

DEFINITION 3.2. Let $\mathscr{A}$ be a finitely generated unital $C^*$–algebra. We define the number $\mathrm{Pgen}(\mathscr{A})$ to be least integer $k \geqslant 2$ such that $\mathscr{A}$ is generated by $k$ mutually unitarily equivalent and almost mutually orthogonal projections.

If no such $k$ exists, we set $\mathrm{Pgen}(\mathscr{A}) = \infty$.

REMARK 3.3. (1) There is a finitely generated unital $C^*$–algebra $\mathscr{A}$ such that $\mathrm{Pgen}(\mathscr{A}) = 2$. For example, take $\mathscr{A} = \mathrm{M}_2(\mathbb{C})$ and projections

$$p_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} \varepsilon & \sqrt{\varepsilon(1-\varepsilon)} \\ \sqrt{\varepsilon(1-\varepsilon)} & 1-\varepsilon \end{bmatrix}, \ \forall \varepsilon \in (0,1).$$

Clearly, $p_1$ and $p_2$ are unitarily equivalent, $p_1+p_2$ is invertible and $\|p_1p_2\| \leqslant \varepsilon^{1/2}$. Moreover, it is easy to check that $C^*(p_1,p_2) = \mathscr{A}$. Thus, $\mathrm{Pgen}(\mathscr{A}) = 2$.

(2) If the unital $C^*$–algebra $\mathscr{A}$ is infinite–dimensional and simple, then $\mathrm{Pgen}(\mathscr{A}) \geqslant 3$. In fact, if $\mathscr{A}$ is generated by two mutually unitarily equivalent and almost mutually

orthogonal projections $p_1$ and $p_2$, then there is a $*$–homomorphism $\pi\colon C^*(p,q)\to\mathscr{A}$ such that $\pi(p)=p_1$ and $\pi(q)=p_2$. Thus, $\mathscr{A}=\pi(C^*(p,q))$ and hence $\mathscr{A}$ is of Type $I$. But it is impossible since $\mathscr{A}$ is infinite–dimensional and simple.

Now we present main result (2) mentioned in the end of §1.

PROPOSITION 3.4. *Assume that the unital $C^*$–algebra $\mathscr{A}$ is generated by $n$ self–adjoint elements. Then for any $l\geqslant\{\sqrt{n-1}\}$ and $k\geqslant 3$,* $\mathrm{Pgen}(\mathrm{M}_{kl}(\mathscr{A}))\leqslant k$.

*Proof.* Since $l\geqslant\sqrt{n-1}$ and $l^2+1\geqslant n\geqslant gen(\mathscr{A})$, it follow from Lemma 3.1 that $\mathrm{M}_l(\mathscr{A})$ is singly generated. In this case, $\delta(1)=3$. So for any $k\geqslant 3$, $\mathrm{M}_{kl}(\mathscr{A})=\mathrm{M}_k(\mathrm{M}_l(\mathscr{A}))$ is generated by $k$ mutually unitarily equivalent and almost mutually orthogonal projections Theorem 2.3. $\square$

Since simple $AF$ $C^*$–algebra and the irrational rotation algebra are all singly generated by [11], we have by Proposition 3.4:

COROLLARY 3.5. *If $\mathscr{A}$ is a simple unital AF $C^*$–algebra or an irrational rotation algebra, then* $\mathrm{Pgen}(\mathrm{M}_3(\mathscr{A}))\leqslant 3$.

COROLLARY 3.6. *Let $X$ be a compact metric space with* $\dim X\leqslant m$. *If $X$ can be embedded into $\mathbb{C}^m$, then* $\mathrm{Pgen}(\mathrm{M}_{3k}(C(X)))\leqslant 3$, *where $k=\{\sqrt{2m-1}\}$. In general,* $\mathrm{Pgen}(\mathrm{M}_{3s}(C(X)))\leqslant 3$, *where $s=\{\sqrt{2m}\}$.*

*Proof.* By [11, Proposition 2],

$$gen(C(X))=\min\{m\in\mathbb{N}\,|\,\text{there is an embedding of } X \text{ into } \mathbb{R}^m\}.$$

Therefore, if $X$ can be embedded into $\mathbb{C}^m$, then $gen(C(X))\leqslant 2m$ and in general, $X$ can be embedded into $\mathbb{R}^{2m+1}$ by [1, Theorem III.4.2]. In this case, $gen(C(X))\leqslant 2m+1$.

So the assertions follow from Proposition 3.4. $\square$

Recall that a projection $p$ in a $C^*$–algebra $\mathscr{A}$ is infinite if there is a projection $q$ in $\mathscr{A}$ with $q<p$ such that $p$ and $q$ are equivalent (denoted by $p\sim q$) in the sense of Murray–von Neumann. $\mathscr{A}$ is called to be purely infinite if the closure of $a\mathscr{A}a$ contains an infinite projection for every non–zero positive element $a$ in $\mathscr{A}$ (cf. [3]).

PROPOSITION 3.7. *Let $\mathscr{A}$ be a separable purely infinite simple $C^*$–algebra with the unit $1_{\mathscr{A}}$. Suppose the class $[1_{\mathscr{A}}]$ in $K_0(\mathscr{A})$ has torsion. Let $m$ be the order of $[1_{\mathscr{A}}]$. Then* $3\leqslant\mathrm{Pgen}(\mathscr{A})\leqslant\min\{k\in\mathbb{N}|\,k\geqslant 3,(k,m)=1\}$.

*In particular, when $m$ has the form $m=3n-1$ or $m=3n-2$ for some $n\in\mathbb{N}$,* $\mathrm{Pgen}(\mathscr{A})=3$.

*Proof.* According to Remark 3.3 (2), $\mathrm{Pgen}(\mathscr{A})\geqslant 3$.

Since $(k,m)=1$, $s,t\in\mathbb{Z}$ such that $ks-mt=1$ (cf. [8]). Let $c=s+ml$ and $d=t+kl$. Then $kc-md=1$, $\forall l\in\mathbb{N}$. So we can choose $c,d\in\mathbb{N}$ such that $kc-md=1$. Set $r=kc$.

Since $r \equiv 1 \mod m$, it follows from [20, Lemma 1] that there exist isometries $s_1, \cdots, s_r$ in $\mathscr{A}$ such that

$$s_i^* s_j = 0, \ \ i \neq j, \ \ i, j = 1, \cdots, r \ \text{ and } \ \sum_{i=1}^{r} s_i s_i^* = 1_{\mathscr{A}}. \tag{1}$$

Define a linear map $\phi\colon \mathscr{A} \to \mathrm{M}_k(\mathscr{A})$ by $\phi(a) = [s_i^* a s_j]_{r\times r}$. It is easy to check that $\phi$ is a $*$–homomorphism and injective by using (1). Now let $A = [a_{ij}]_{r\times r} \in \mathrm{M}_r(\mathscr{A})$ and put $a = \sum\limits_{i,j=1}^{r} s_i a_{ij} s_j^* \in \mathscr{A}$. Then $\phi(a) = A$ in terms of (1). Therefore, $\phi$ is a $*$–isomorphism and $\mathscr{A}$ is $*$–isomorphic to $\mathrm{M}_r(\mathscr{A})$.

Now by Theorem 2.3 of [17], $gen(\mathscr{A}) \leqslant 2$. Thus, by Proposition 3.4, for above $k \geqslant \delta(1) = 3$, $c \geqslant 1$, $\mathrm{M}_{kc}(\mathscr{A})$ is generated by $k$ mutually unitarily equivalent and almost mutually orthogonal projections and consequently, $\mathrm{Pgen}(\mathscr{A}) \leqslant k$.

When $m$ has the form $m = 3n-1$ or $m = 3n-2$ for some $n \in \mathbb{N}$, $(3,m) = 1$. In this case, $\mathrm{Pgen}(\mathscr{A}) = 3$ by above argument. $\square$

EXAMPLE 3.8. Let $\mathscr{O}_n$ $(2 \leqslant n \leqslant +\infty)$ be the Cuntz algebra. $\mathscr{O}_n$ is a separable purely infinite simple unital $C^*$–algebra with $K_0(\mathscr{O}_n) \cong \begin{cases} \mathbb{Z}/(n-1)\mathbb{Z}, & 2 \leqslant n < +\infty \\ \mathbb{Z}, & n = +\infty \end{cases}$ and the generator $[1_{\mathscr{O}_n}]$ (cf. [3]). Then we have

(1) $\mathrm{Pgen}(\mathscr{O}_\infty) = +\infty$ by Remark 2.5.

(2) $\mathrm{Pgen}(\mathscr{O}_n) = 3$ if $n = 3m$ or $n = 3m-1$ for some $m \in \mathbb{N}$ by Proposition 3.7.

(3) $\mathrm{Pgen}(\mathscr{O}_n) = \min\{k \in \mathbb{N} \,|\, k \geqslant 3,\ (k, n-1) = 1\}$. In fact, Proposition 3.7 shows that $\mathrm{Pgen}(\mathscr{O}_n) \leqslant \min\{k \in \mathbb{N} \,|\, k \geqslant 3,\ (k, n-1) = 1\}$. Now, $\mathrm{Pgen}(\mathscr{O}_n) = m$ implies that there is a projection $e \in \mathscr{O}_n$ such that $m[e] = 1$ in $K_0(\mathscr{O}_n)$ by Remark 2.5. So there exists $s \in \mathbb{N}$ such that $[e] = s[1_{\mathscr{O}_n}]$. Then $ms - 1 \equiv 0 \mod (n-1)$ and hence $(m, n-1) = 1$.

For example: $\mathrm{Pgen}(\mathscr{O}_4) = 4$, $\mathrm{Pgen}(\mathscr{O}_{13}) = 5$, $\mathrm{Pgen}(\mathscr{O}_{211}) = 11$, etc..

According to [2], a unital separable $C^*$–algebra $\mathscr{A}$ with the unit $1_{\mathscr{A}}$ is approximately divisible if, for every $x_1, \cdots, x_n \in A$ and any $\varepsilon > 0$, there is a finite–dimensional $C^*$–subalgebra $\mathscr{B}$ with unit $1_{\mathscr{A}}$ of $\mathscr{A}$ such that $\mathscr{B}$ has no Abelian central projections and $\|x_i y - y x_i\| < \varepsilon \|y\|$, $\forall 1 \leqslant i \leqslant n$ and $y \in \mathscr{B}$.

PROPOSITION 3.9. *Suppose that two separable and unital $C^*$–algebras $\mathscr{A}$ and $\mathscr{B}$ satisfies following conditions:*

(1) *$\mathscr{A}$ or $\mathscr{B}$ is nuclear;*

(2) *there is an integer $k \geqslant 3$ and a unital $C^*$–algebra $\mathscr{C}$ such that $\mathscr{B} \cong \mathrm{M}_k(\mathscr{C})$;*

(3) *$\mathscr{A} \otimes \mathscr{B}$ is approximately divisible.*

*Then* $\mathrm{Pgen}(\mathscr{A} \otimes \mathscr{B}) \leqslant k$. *Furthermore, if* $k \equiv 0 \mod 3$, *then* $\mathrm{Pgen}(\mathscr{A} \otimes \mathscr{B}) \leqslant 3$.

*Proof.* If $\mathscr{B}$ is nuclear, applying [10, Proposition 2.3.8] to $\mathrm{M}_k(\mathscr{C})$ , we get that $\mathscr{C}$ is also nuclear since $\mathscr{C}$ is a hereditary $C^*$–subalgebra of $\mathrm{M}_k(\mathscr{C})$.

Now from $\mathscr{A}\otimes\mathscr{B}\cong\mathrm{M}_k(\mathscr{A}\otimes\mathscr{C})$, we get that $\mathscr{A}\otimes\mathscr{C}$ is approximately divisible by [2, Corollary 2.9]. Since every unital separable approximately divisible $C^*$–algebra is singly generated by [9, Theorem 3.1], we obtain that $\mathscr{A}\otimes\mathscr{B}$ is generated by $k$ mutually unitarily equivalent and almost mutually orthogonal projections, by applying Proposition 3.4 to $\mathscr{A}\otimes\mathscr{C}$.

If $k=3t$ for some $t\in\mathbb{N}$, then $\mathrm{Pgen}(\mathrm{M}_{3t}(\mathscr{A}\otimes\mathscr{C}))\leqslant 3$ by Proposition 3.4. Thus, $\mathrm{Pgen}(\mathscr{A}\otimes\mathscr{B})\leqslant 3$ for $\mathscr{A}\otimes\mathscr{B}\cong\mathrm{M}_k(\mathscr{A}\otimes\mathscr{C})$. $\square$

Which type of $C^*$–algebras satisfy Condition (2) and (3) of Proposition 3.9? For $AF$–algebras, we have the following:

PROPOSITION 3.10. *Let $\mathscr{A}=\overline{\bigcup_{n=1}^{\infty}\mathscr{A}_n}$ be a AF–algebra with unit $1_{\mathscr{A}}$, where $\mathscr{A}_n$ is a finite–dimensional $C^*$–algebra with the unit $1_{\mathscr{A}}$ such that $\mathscr{A}_m\subset\mathscr{A}_n$, $\forall m\leqslant n$, $m,n=1,2,\cdots$. Assume that $\mathscr{A}$ satisfies following conditions:*

(1) *no quotient of $\mathscr{A}$ has an abelian projection, especially, $\mathscr{A}$ is infinite dimensional simple;*

(2) *there is an integer $n\geqslant 3$ and an element $a$ in $K_0(\mathscr{A})$ such that $na=[1_{\mathscr{A}}]$ in $K_0(\mathscr{A})$.*

*If there is $k\geqslant 3$ such that $n\equiv 0 \mod k$, then $\mathscr{A}$ is generated by $k$ mutually unitarily equivalent and almost mutually orthogonal projections.*

*Proof.* By [10, Proposition 3.4.5], $a\in K_0(\mathscr{A})_+$ (the positive cone of $K_0(\mathscr{A})$). So we can find a projection $p$ in $\mathrm{M}_s(\mathscr{A}_m)$ for some $s,m\in\mathbb{N}$ such that $[p]=a$ in $K_0(\mathscr{A})$. Consequently, there are projections $p_1,\cdots,p_s$ in $\mathscr{A}_m$ such that $p$ is unitarily equivalent to $\mathrm{diag}(p_1,\cdots,p_s)$ in $\mathrm{M}_s(\mathscr{A}_m)$. This indicates that

$$[\mathrm{diag}(\underbrace{p_1,\cdots,p_1}_{n},\cdots,\underbrace{p_s,\cdots,p_s}_{n})]=[1_{\mathscr{A}}]\quad\text{in } K_0(\mathscr{A}). \tag{2}$$

Since $\mathrm{M}_t(\mathscr{A})$ has the cancellation property of projections for all $t\in\mathbb{N}$, we have

$$\mathrm{diag}(\underbrace{p_1,\cdots,p_1}_{n},\cdots,\underbrace{p_s,\cdots,p_s}_{n})\sim\mathrm{diag}(1_{\mathscr{A}},\underbrace{0,\cdots,0}_{ns-1})\quad\text{in } \mathrm{M}_{ns}(\mathscr{A}) \tag{3}$$

by (2). Applying [10, Lemma 3.4.2] to (3), we can find mutually orthogonal projections $q_1,\cdots,q_{ns}$ in $\mathscr{A}$ such that $q_{(i-1)s+1},\cdots,q_{is}$ are all unitarily equivalent to $p_i$, $1\leqslant i\leqslant n$ in $\mathscr{A}$.

Put $r_i=\sum_{j=1}^{s}q_{(i-1)s+j}\in\mathscr{A}$, $i=1,\cdots,n$. Then $r_ir_j=0$, $r_i\sim r_j$ and $[r_i]=[p]$ in $K_0(\mathscr{A})$, $i\neq j$, $i,j=1,\cdots,n$. So from $[r_1+\cdots+r_s]=[1_{\mathscr{A}}]$ in $K_0(\mathscr{A})$, we obtain $\sum_{i=1}^{s}r_i=1_{\mathscr{A}}$.

Let $v_i$ be partial isometries in $\mathscr{A}$ such that $v_1 = r_1$ and $r_1 = v_i^* v_i$, $r_i = v_i v_i^*$, $r_i v_i = v_i r_1$ when $2 \leqslant i \leqslant n$. Define a linear mapping $\psi\colon \mathscr{A} \to \mathrm{M}_n(r_1 \mathscr{A} r_1)$ by $\psi(a) = [v_i^* a v_j]_{n\times n}$. In terms of $v_i^* v_j = 0$, $i \neq j$, $i,j = 1,\cdots,n$ and $\sum\limits_{i=1}^{n} v_i v_i^* = 1_{\mathscr{A}}$, it is easy to check that $\psi$ is a $*$–isomorphism, that is, $\mathscr{A}$ satisfies Condition (2) of Proposition 3.9.

By [2, Proposition 4.1], Condition (1) implies that $\mathscr{A}$ is approximately divisible. So the assertion follows from Proposition 3.9. $\square$

EXAMPLE 3.11. Let $\mathscr{B}$ be a *UHF*–algebra. It is in one–one correspondence with a generalized integer, formal products $q = \prod\limits_{j=1}^{\infty} p_j^{n_j}$ for some $\{n_j\}_{j=1}^{\infty} \subset \mathbb{Z}_+ \cup \{+\infty\}$, where $\{p_1, p_2, \cdots\}$ is the set of all positive prime numbers listed in increasing order. According to [14, 7.4], $K_0(\mathscr{B})$ is isomorphic to $\left\{\frac{x}{y} | x \in \mathbb{Z}, y \in \mathbb{N}, q \equiv 0 \mod y\right\} = \mathbb{Z}_{(q)}$ with $[1_{\mathscr{B}}]$ in correspondence with $1$, where $q \equiv 0 \mod y$ means that $y = \prod\limits_{j=1}^{\infty} p_j^{m_j}$ for some $m_j \in \mathbb{Z}_+$ with $m_j \leqslant n_j$, $j = 1,\cdots,\infty$ and $m_j > 0$ for only finitely many $j$.

Put $k = \min\{n \in \mathbb{N} | n \geqslant 3, q \equiv 0 \mod n\}$. Clearly, there is $a \in K_0(\mathscr{B})$ such that $ka = [1_{\mathscr{A}}]$. Thus there is a unital $C^*$–algebra $\mathscr{C}$ such that $\mathscr{B} \cong \mathrm{M}_k(\mathscr{C})$ (see the proof of Proposition 3.10). Since $\mathscr{B}$ and $\mathscr{A} \otimes \mathscr{B}$ are all approximately divisible for any unital separable $C^*$–algebra $\mathscr{A}$ by [2], it follows from Proposition 3.9 that $\mathscr{B}$ and $\mathscr{A} \otimes \mathscr{B}$ are all generated by $k$ mutually unitarily equivalent and almost mutually orthogonal projections, i.e., $\mathrm{Pgen}(\mathscr{B}) \leqslant k$ and $\mathrm{Pgen}(\mathscr{A} \otimes \mathscr{B}) \leqslant k$.

Moreover, we have $\mathrm{Pgen}(\mathscr{B}) = \min\{n \in \mathbb{N} | n \geqslant 3, q \equiv 0 \mod n\}$. In fact, since $\mathscr{B}$ is simple and infinite–dimensional, it follows from Remark 3.3 that $\mathrm{Pgen}(\mathscr{B}) \geqslant 3$. Let $m = \mathrm{Pgen}(\mathscr{B})$. Then there is a projection $e$ in $\mathscr{B}$ such that $m[e] = [1_{\mathscr{B}}]$. Thus, there are $x, y \in \mathbb{Z}_+$ with $q \equiv 0 \mod y$ such that $m\frac{x}{y} = 1$ and consequently, $q \equiv 0 \mod m$. So $\mathrm{Pgen}(\mathscr{B}) \geqslant \min\{n \in \mathbb{N} | n \geqslant 3, q \equiv 0 \mod n\}$.

For example, if $\mathscr{B}$ is a *UHF* algebra of Type $2^\infty$ or $3^\infty$, respectively, then $\mathrm{Pgen}(\mathscr{B}) = 4$ or $\mathrm{Pgen}(\mathscr{B}) = 3$.

Finally, similar to Davis' result in [4] and Sunder' work in [16], We have

PROPOSITION 3.12. *Let $H$ be a separable infinite dimensional Hilbert space. Then for any $k \geqslant 3$ there are $k$ mutually unitarily equivalent and almost mutually orthogonal projections $P_1, \cdots, P_k$ such that*

$$\mathscr{K} \subset C^*(P_1, \cdots, P_k) \subset W^*(P_1, \cdots, P_k) = B(H).$$

*Proof.* Take $H = l^2$ and let $S$ be the unilateral shift on $H$. It's well–known that $\mathscr{K} \subset C^*(S) \subset W^*(S) = B(H)$ (cf. [10]). Then there are $k$ mutually unitarily equivalent and almost mutually orthogonal projections $Q_1, \cdots, Q_k$ in $\mathrm{M}_k(C^*(S))$ such that $C^*(Q_1, \cdots, Q_k) = \mathrm{M}_k(C^*(S))$ by Theorem 2.3.

Choose isometry operators $S_1,\cdots,S_k$ on $H$ such that $S_i^*S_j=0,\ i\neq j,\ i,j=1,\cdots,k$ and $\sum_{i=1}^k S_iS_i^*=I$. Define a unitary operator $W\colon H\to\bigoplus_{i=1}^k H$ by $Wx=(S_1^*x,\cdots,S_k^*x)$, $\forall x\in H$. Then $W^*(\mathrm{M}_k(\mathscr{K}))W=\mathscr{K}$ and $W^*(\mathrm{M}_k(B(H)))W=\mathscr{B}(H)$. Put $P_i=W^*Q_iW$, $i=1,\cdots,k$. Then $P_1,\cdots,P_k$ are mutually unitarily equivalent and almost mutually orthogonal and $W^*(\mathrm{M}_k(C^*(S)))W=C^*(P_1,\cdots,P_k)$. So from

$$\mathrm{M}_k(\mathscr{K})\subset C^*(Q_1,\cdots,Q_k)\subset W^*(Q_1,\cdots,Q_k)=\mathrm{M}_k(B(H)),$$

we obtain the assertion. $\square$

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