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ON THE BEHAVIOR AT INFINITY OF SOLUTIONS TO DIFFERENCE EQUATIONS IN SCHRÖDINGER FORM

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Abstract. We study the behavior at infinity of solutions of discrete Schrödinger equations. First we study pairs of discrete Schrödinger equations whose potential functions differ by a quantity that can be considered small in a suitable sense as the index $n \to \infty$. With simple assumptions on the growth rate of the solutions of the original system, we show that the perturbed system has a fundamental set of solutions with the same behavior at infinity, employing a variation-of-constants scheme to produce a convergent iteration for the solutions of the second equation in terms of those of the original one.

Next, we present a sharp discrete analogue of the Liouville-Green (JWKB) transformation, making it possible to derive exponential behavior at infinity of a single difference equation, by explicitly constructing a comparison equation to which the perturbation results apply. After that we use the relations between the solution sets of two discrete Schrödinger equations differing by a perturbation to derive exponential dichotomy of solutions and to elucidate the structure of transfer matrices.

A final section contains illustrative examples, including some with large, oscillatory potentials, and an appendix discusses the connection between the discrete Schrördinger problem and orthogonal polynomials on the real line.

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