

## COMMUTATIVITY OF UNBOUNDED NORMAL AND SELF-ADJOINT OPERATORS AND APPLICATIONS

MOHAMMED HICHEM MORTAD

(Communicated by D. Hadwin)

*Abstract.* Devinatz, Nussbaum and von Neumann established some important results on the strong commutativity of self-adjoint and normal unbounded operators. In this paper, we prove results in the same spirit.

### 1. Introduction

First, we assume that all operators operators are linear. Bounded operators are assumed to be defined on the whole Hilbert space. Unbounded operators are supposed to have dense domains, and so they will be said to be densely defined. For general references on unbounded operator theory, see [23, 24, 27, 30].

Let us, however, recall some notations that will be met below. If  $A$  and  $B$  are two unbounded operators with domains  $D(A)$  and  $D(B)$  respectively, then  $B$  is called an extension of  $A$ , and we write  $A \subset B$ , if  $D(A) \subset D(B)$  and if  $A$  and  $B$  coincide on  $D(A)$ . If  $A \subset B$ , then  $B^* \subset A^*$ .

The product  $AB$  of two unbounded operators  $A$  and  $B$  is defined by

$$BA(x) = B(Ax) \text{ for } x \in D(BA)$$

where

$$D(BA) = \{x \in D(A) : Ax \in D(B)\}.$$

Recall too that the unbounded operator  $A$ , defined on a Hilbert space  $H$ , is said to be invertible if there exists an *everywhere defined* (i.e. on the whole of  $H$ ) bounded operator  $B$ , which then will be designated by  $A^{-1}$ , such that

$$BA \subset AB = I$$

where  $I$  is the usual identity operator. This is the definition adopted in the present paper. It may be found in e.g. [3] or [11].

*Mathematics subject classification* (2010): Primary 47A05; Secondary 47B25.

*Keywords and phrases:* Unbounded selfadjoint and normal operators, invertible unbounded operators, operator products, strong commutativity.

Partially supported by "Laboratoire d'Analyse Mathématique et Applications".

An unbounded operator  $A$  is said to be closed if its graph is closed; symmetric if  $A \subset A^*$ ; self-adjoint if  $A = A^*$  (hence from known facts self-adjoint operators are automatically closed); normal if it is *closed* and  $AA^* = A^*A$  (this implies that  $D(AA^*) = D(A^*A)$ ). It is also worth recalling that normal operators  $A$  do obey the domain condition  $D(A) = D(A^*)$ .

Commutativity of unbounded operators must be handled with care. First, recall the definition of two strongly commuting unbounded (self-adjoint) operators (see e.g. [23]):

DEFINITION 1. Let  $A$  and  $B$  be two unbounded self-adjoint operators. We say that  $A$  and  $B$  strongly commute if all the projections in their associated projection-valued measures commute.

This is in fact equivalent to saying that  $e^{itA}e^{isB} = e^{isB}e^{itA}$  for all  $s, t \in \mathbb{R}$ .

We shall use the same definition for unbounded normal operators.

Nelson [19] showed that there exists a pair of two essentially self-adjoint operators  $A$  and  $B$  on some common domain  $D$  such that

1.  $A : D \rightarrow D, B : D \rightarrow D,$
2.  $ABx = BAx$  for all  $x \in D,$
3. but  $e^{it\bar{A}}$  and  $e^{is\bar{B}}$  do not commute, i.e.  $A$  and  $B$  do not strongly commute.

Based on the previous example, Fuglede [7] proved a similar result. Hence an expression of the type  $AB = BA$ , although being quite strong since it implies that  $D(AB) = D(BA)$ , does not necessarily mean that  $A$  and  $B$  strongly commute.

There are results (see e.g. [7], [19]) giving conditions implying the strong commutativity of  $\bar{A}$  and  $\bar{B}$ . For instance, we have

THEOREM 1. ([21]) *Let  $A$  and  $B$  be two semi-bounded operators in a Hilbert space  $H$  and let  $D$  be a dense linear manifold contained in the domains of  $AB, BA, A^2$  and  $B^2$  such that  $ABx = BAx$  for all  $x \in D$ . If the restriction of  $(A+B)^2$  to  $D$  is essentially self-adjoint, then  $A$  and  $B$  are essentially self-adjoint and  $\bar{A}$  and  $\bar{B}$  strongly commute.*

Now, let us recall results obtained by Devinatz-Nussbaum (and von Neumann) on strong commutativity:

THEOREM 2. (Devinatz-Nussbaum-von Neumann, [4] and cf. [20]) *If there exists a self-adjoint operator  $A$  such that  $A \subseteq BC$ , where  $B$  and  $C$  are self-adjoint, then  $B$  and  $C$  strongly commute.*

COROLLARY 1. *Let  $A, B$  and  $C$  be unbounded self-adjoint operators. Then*

$$A \subseteq BC \implies A = BC.$$

The following improvement of Corollary 1 appeared in [20]

COROLLARY 2. *Let  $A$  be an unbounded self-adjoint operator and let  $B$  and  $C$  be two closed symmetric operators such that  $AB \subset C$ . If  $B$  has a bounded inverse (hence it is self-adjoint), then  $C$  is self-adjoint. Besides  $AB = C$ .*

The research work on the normality of the product of two bounded normal operators started in 1930 by the work of Gantmaher-Krein (see [8]). Then, it followed papers by Wiegmann ([31, 32]) and Kaplansky [13]. Gheondea [10] quoted that "the normality of operators in the Pauli algebra representations became of interest in connection with some questions in polarization optics" (see [29]). Similar problems also arise in Quantum Optics (see [2]).

There have been several successful attempts by the author to generalize the previous to the case where at least one operator is unbounded. See e.g. [16] and [18]. One of the important considerations of the normality of the product of unbounded normal operators (NPUNO, in short) is strong commutativity of the latter. Indeed, the following striking result (which is not known to many) shows the great interest of investigating the question of (NPUNO):

THEOREM 3. (Devinatz-Nussbaum, [5]) *If  $A$ ,  $B$  and  $N$  are unbounded normal operators obeying  $N = AB = BA$ , then  $A$  and  $B$  strongly commute.*

In this paper we weaken the condition  $AB = BA$  to  $AB \subset BA$ , say, and still derive results on the strong commutativity of  $A$  and  $B$  (in unbounded normal and self-adjoint settings).

When proving the normality of a product, we need its closedness and its adjoint. Let us thus recall known results on those two notions:

THEOREM 4. *Let  $A$  be a densely defined unbounded operator.*

1.  $(BA)^* = A^*B^*$  if  $B$  is bounded.
2.  $A^*B^* \subset (BA)^*$  for any densely unbounded  $B$  and if  $BA$  is densely defined.
3. Both  $AA^*$  and  $A^*A$  are self-adjoint whenever  $A$  is closed.

LEMMA 1. ([30]) *If  $A$  and  $B$  are densely defined and  $A$  is invertible with inverse  $A^{-1}$  in  $B(H)$ , then  $(BA)^* = A^*B^*$ .*

LEMMA 2. *The product  $AB$  (in this order) of two densely defined closed operators  $A$  and  $B$  is closed if one of the following occurs:*

1.  $A$  is invertible,
2.  $B$  is bounded.

For related work on strong commutativity, we refer the reader to [1, 20, 21, 26, 33]. For similar papers on products, the interested reader may consult [9], [12], [14], [15], [16], [25] and [28], and further bibliography cited therein.

## 2. Main results

We start by giving a result on strong commutativity of unbounded normal operators. But we first have the following result on (NPUNO):

**THEOREM 5.** *Let  $A$  and  $B$  be two unbounded normal operators verifying  $AB \subset BA$ . If  $B$  is invertible, then  $BA$  and  $\overline{AB}$  are both normal whenever  $AB$  is densely defined.*

*Proof.* Since  $B$  is invertible,

$$AB \subset BA \implies A \subset BAB^{-1} \implies B^{-1}A \subset AB^{-1}.$$

By the Fuglede theorem ([6]), we obtain

$$B^{-1}A^* \subset A^*B^{-1}.$$

Left multiplying, then right multiplying by  $B$  yield

$$A^*B \subset BA^* \text{ so that } AB^* \subset B^*A$$

by Lemma 1. Next

$$(BA)^*BA \subset (AB)^*BA = B^*A^*BA \subset B^*BA^*A.$$

But  $BA$  is closed, hence  $(BA)^*BA$  is self-adjoint. Since  $B^*B$  and  $A^*A$  are also self-adjoint, Corollary 1 gives us

$$(BA)^*BA = B^*BA^*A.$$

Very similar arguments may be applied to prove that

$$BA(BA)^* = BB^*AA^*.$$

Thus, and since  $A$  and  $B$  are normal, we obtain

$$(BA)^*BA = BA(BA)^*,$$

establishing the normality of  $BA$ .

To prove that  $\overline{AB}$  is normal, observe first that thanks to the invertibility of  $B$ , we have

$$AB \subset BA \implies A^*B^* \subset B^*A^*.$$

Since  $B$  is invertible,  $B^*$  too is invertible. Thus by the first part of the proof,  $B^*A^*$  is normal and so is  $(AB)^* = B^*A^*$ . Hence its adjoint  $(AB)^{**} = \overline{AB}$  stays normal.  $\square$

The hypothesis  $AB \subset BA$  is fundamental as seen in the following example:

EXAMPLE 1. Let  $A$  and  $B$  be defined respectively by

$$Af(x) = f'(x) \text{ and } Bf(x) = e^{|x|}f(x)$$

on

$$D(A) = H^1(\mathbb{R}) \text{ and } D(B) = \{f \in L^2(\mathbb{R}) : e^{|x|}f \in L^2(\mathbb{R})\},$$

where  $H^1(\mathbb{R})$  is the usual Sobolev space.

The operator  $A$  is known to be normal because it is unitarily equivalent (via the  $L^2(\mathbb{R})$ -Fourier transform) to a multiplication operator by a complex-valued function.

As for  $B$ , it is clearly densely defined, self-adjoint and invertible. It is also easy to see that  $AB$  and  $BA$  do not coincide on any dense set.

Finally, let us show that  $N := BA$  (which is obviously closed) is *not normal*. We have

$$Nf(x) = e^{|x|}f'(x) \text{ defined on } D(N) = D(BA) \text{ hence}$$

$$D(N) = \{f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R}), e^{|x|}f' \in L^2(\mathbb{R})\} = \{f \in L^2(\mathbb{R}) : e^{|x|}f' \in L^2(\mathbb{R})\}.$$

To compute  $N^*$ , the adjoint of  $N$ , we should do it first for  $C_0^\infty(\mathbb{R})$  functions, then proceed as in [14]. We find that

$$N^*f(x) = e^{|x|}(\mp f(x) - f'(x))$$

on

$$D(N^*) = \{f \in L^2(\mathbb{R}) : e^{|x|}f \in L^2(\mathbb{R}), e^{|x|}f' \in L^2(\mathbb{R})\}.$$

Then we easily obtain that

$$NN^*f(x) = e^{2|x|}(-f(x) \mp 2f'(x) - f''(x))$$

and

$$N^*Nf(x) = e^{2|x|}(\mp 2f'(x) - f''(x)),$$

establishing the non-normality of  $N$ .

COROLLARY 3. Let  $A$  and  $B$  be two unbounded normal operators verifying  $AB \subset BA$ . If  $B$  is invertible, then  $\overline{AB} = BA$  whenever  $AB$  is densely defined.

*Proof.* Since  $BA$  is closed, we have

$$AB \subset BA \implies \overline{AB} \subset BA.$$

But both  $\overline{AB}$  and  $BA$  are normal, and normal operators are maximally normal (see e.g. [24]), hence

$$\overline{AB} = BA. \quad \square$$

COROLLARY 4. Let  $A$  and  $B$  be two unbounded normal operators. If  $B$  is invertible and  $BA = AB$ , then  $A$  and  $B$  strongly commute whenever  $AB$  is densely defined.

*Proof.* By Theorem 5,  $BA$  and this time  $AB$  is normal too. By Theorem 3,  $A$  and  $B$  strongly commute.  $\square$

REMARK 1. We could have stated the previous corollary as: *Let  $A$  and  $B$  be two unbounded normal operators. If  $B$  is invertible and  $AB \subset BA$  and  $AB$  is closed, then  $A$  and  $B$  strongly commute whenever  $AB$  is densely defined.*

As an application of the strong commutativity of unbounded normal operators, we have the following result (cf. [17]):

PROPOSITION 1. *Let  $A$  and  $B$  be two strongly commuting unbounded normal operators. Then  $A + B$  is essentially normal.*

REMARK 2. Recall that an unbounded closeable operator is said to be essentially normal if it has a normal closure (of course, this terminology has a different signification in Banach algebras).

*Proof.* Since  $A$  and  $B$  are normal, by the spectral theorem we may write

$$A = \int_{\mathbb{C}} z dE_A(z) \text{ and } B = \int_{\mathbb{C}} z' dF_B(z'),$$

where  $E_A$  and  $F_B$  designate the associated spectral measures. By the strong commutativity, we have

$$E_A(I)F_B(J) = F_B(J)E_A(I)$$

for all Borel sets  $I$  and  $J$  in  $\mathbb{C}$ . Hence

$$E_{A,B}(z, z') = E_A(z)F_B(z')$$

defines a two parameter spectral measure. Thus

$$C = \int_{\mathbb{C}} \int_{\mathbb{C}} (z + z') dE_{A,B}(z, z')$$

defines a normal operator, such that  $C = \overline{A + B}$ . Therefore,  $A + B$  is essentially normal.  $\square$

COROLLARY 5. *Let  $A$  and  $B$  be two unbounded normal operators. If  $B$  is invertible and  $BA = AB$  (where  $AB$  is densely defined), then  $A + B$  is essentially normal.*

Next, we treat the case of two unbounded self-adjoint operators. We start with the following interesting result

PROPOSITION 2. *Let  $A$  and  $B$  be two unbounded self-adjoint operators such that  $B$  is invertible. If  $AB \subset BA$ , then  $A$  and  $B$  strongly commute whenever  $AB$  is densely defined.*

*Proof.* By Theorem 5 and Corollary 3, we have  $\overline{AB} = BA$ . Hence

$$(BA)^* = [\overline{AB}]^* = (AB)^* = B^*A^* = BA.$$

That is  $BA$  is self-adjoint. Whence and since  $AB \subset BA$ , Corollary 2 yields  $AB = BA$ . Thus, and by Theorem 2,  $A$  and  $B$  strongly commute.  $\square$

We have a similar result for self-adjoint operators to that of Proposition 1

**COROLLARY 6.** [cf. [1]] *Let  $A$  and  $B$  be two unbounded self-adjoint operators such that  $B$  is invertible. If  $AB \subset BA$  and  $AB$  is densely defined, then  $A + B$  is essentially self-adjoint. Moreover, for all  $t \in \mathbb{R}$ :*

$$e^{it(A+B)} = e^{itA}e^{itB} = e^{itB}e^{itA}.$$

*Proof.* By Proposition 2,  $A$  and  $B$  strongly commute. Then, apply an akin proof to that of Proposition 1. For the last displayed equation just use the Trotter product formula which may be found in [23].  $\square$

In [5] it was noted that Theorem 2 does not have an analog in the case of unbounded normal operators. The counterexample is very simple, it suffices to take the product of two non-commuting unitary operators which is unitary anyway. Nonetheless, we have the following maximality result

**THEOREM 6.** *Let  $A$  and  $B$  be two unbounded self-adjoint operators such that  $B$  is positive and invertible. Let  $C$  be an unbounded normal operator. If  $AB \subset C$ , then  $A$  and  $B$  strongly commute,  $AB$  is self-adjoint (whenever it is densely defined) and hence  $AB = C$ .*

To prove it we need the following result:

**LEMMA 3.** *Let  $S$  and  $T$  be two self-adjoint operators defined on a Hilbert space  $H$ . Assume that  $S$  is bounded which commutes with  $T$ , i.e.  $ST \subset TS$ . Then  $f(S)T \subset Tf(S)$  for any real-valued continuous function  $f$  defined on  $\sigma(S)$ . In particular, we have  $S^{\frac{1}{2}}T \subset TS^{\frac{1}{2}}$ , if  $S$  is positive.*

*Proof.* We may easily show that if  $P$  is a real polynomial, then  $P(S)T \subset TP(S)$ . Now let  $f$  be a continuous function on  $\sigma(S)$ . Then (by the density of the polynomials defined on the compact  $\sigma(S)$  in the set of continuous functions with respect to the supremum norm) there exists a sequence  $(P_n)$  of polynomials converging uniformly to  $f$  so that

$$\lim_{n \rightarrow \infty} \|P_n(S) - f(S)\|_{B(H)} = 0.$$

Let  $x \in D(f(S)T) = D(T)$ . Let  $t \in D(T)$  and  $x = f(S)t$ . Setting  $x_n = P_n(S)t$ , we see that

$$Tx_n = TP_n(S)t = P_n(S)Tt \longrightarrow f(S)Tt.$$

Since  $x_n \rightarrow x$ , by the closedness of  $T$ , we get

$$x = f(S)t \in D(T) \text{ and } Tx = Tf(S)t = f(S)Tt,$$

that is, we have proved that  $f(S)T \subset Tf(S)$ .  $\square$

Now we give the proof of Theorem 6:

*Proof.* Since  $AB \subset C$  and  $B$  is invertible, we get

$$C^* \subset (AB)^* = B^*A^* = BA \text{ or } B^{-1}C^* \subset A.$$

It is also clear that  $AB \subset C$  implies that  $A \subset CB^{-1}$ . Hence, and since  $C$  is normal, the Fuglede-Putnam theorem (see [6] and [22]) allow us to write

$$B^{-1}C^* \subset CB^{-1} \implies B^{-1}C \subset C^*B^{-1}.$$

So

$$B^{-1}AB \subset BAB^{-1}.$$

Left multiplying by  $B^{-1}$ , then right multiplying by  $B^{-1}$  give us

$$(B^{-1})^2A \subset A(B^{-1})^2.$$

Since  $B^{-1}$  is bounded and positive, Lemma 3 allows us to say that  $B^{-1}$  and  $A$  commute. Hence

$$B^{-1}A \subset AB^{-1} \text{ or just } AB \subset BA.$$

By Proposition 2,  $BA$  is self-adjoint, and  $A$  strongly commutes with  $B$ . Therefore, by Corollary 2, we have  $AB = BA$ . Thus, and since self-adjoint operators are maximally normal, we deduce that  $AB = C$ .  $\square$

#### REFERENCES

- [1] A. ARAI, *Generalized weak Weyl relation and decay of quantum dynamics*, Rev. Math. Phys., **17/9** (2005), 1071–1109.
- [2] S. M. BARNETT, P. M. RADMORE, *Methods in Theoretical Quantum Optics*, Oxford 1997.
- [3] J. B. CONWAY, *A Course in Functional Analysis*, Springer, 1990 (2nd edition).
- [4] A. DEVINATZ, A. E. NUSSBAUM, J. VON NEUMANN, *On the Permutability of Self-adjoint Operators*, Ann. of Math. (2), **62** (1955), 199–203.
- [5] A. DEVINATZ, A. E. NUSSBAUM, *On the Permutability of Normal Operators*, Ann. of Math. (2), **65** (1957), 144–152.
- [6] B. FUGLEDE, *A Commutativity Theorem for Normal Operators*, Proc. Nati. Acad. Sci., **36** (1950), 35–40.
- [7] B. FUGLEDE, *Conditions for Two Selfadjoint Operators to Commute or to Satisfy the Weyl Relation*, Math. Scand., **51/1** (1982), 163–178.
- [8] F. R. GANTMAHER, M. G. KREIN, *O normalnyh operatorah v ermitovom prostranstve* [Russian], Izv. fiz-mat ob-vapri Kazanskom Univ., **1/3** (1929–1930), 71–84.
- [9] F. GESZTESY, J. A. GOLDSTEIN, H. HOLDEN, G. TESCHL, *Abstract Wave Equations and Associated Dirac-Type Operators*, Annali di Matematica. DOI 10.1007/s10231-011-0200-7.
- [10] A. GHEONDEA, *When Are the Products of Normal Operators Normal?* Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **52(100)/2** (2009), 129–150.

- [11] I. GOHBERG, S. GOLDBERG, M. A. KAASHOEK, *Basic Classes of Linear Operators*, Birkhäuser Verlag, Basel, 2003.
- [12] P. E. T. JORGENSEN, *Unbounded Operators: Perturbations and Commutativity Problems*, J. Funct. Anal. **39/3** (1980), 281–307.
- [13] I. KAPLANSKY, *Products of Normal Operators*, Duke Math. J., **20/2** (1953), 257–260.
- [14] M. H. MORTAD, *An Application of the Putnam-Fuglede Theorem to Normal Products of Selfadjoint Operators*, Proc. Amer. Math. Soc. **131** (2003), 3135–3141.
- [15] M. H. MORTAD, *On Some Product of Two Unbounded Self-adjoint Operators*, Integral Equations Operator Theory, **64** (2009), 399–408.
- [16] M. H. MORTAD, *On the Closedness, the Self-adjointness and the Normality of the Product of Two Unbounded Operators*, Demonstratio Math., **45/1** (2012), 161–167.
- [17] M. H. MORTAD, *On the Normality of the Sum of Two Normal Operators*, Complex Anal. Oper. Theory, **6/1** (2012), 105–112. DOI: 10.1007/s11785-010-0072-7.
- [18] M. H. MORTAD, *On the Normality of the Unbounded Product of Two Normal Operators*, Concrete Operators, **1** (2012), 11–18. DOI: 10.2478/conop-2012-0002.
- [19] E. NELSON, *Analytic Vectors*, Ann. of Math. (2), **70** (1959), 572–615.
- [20] A. E. NUSSBAUM, *A Commutativity Theorem for Unbounded Operators in Hilbert Space*, Trans. Amer. Math. Soc. **140** (1969), 485–491.
- [21] A. E. NUSSBAUM, *A Commutativity Theorem for Semibounded Operators in Hilbert Space*, Proc. Amer. Math. Soc. **125/12** (1997), 3541–3545.
- [22] C. R. PUTNAM, *On Normal Operators in Hilbert Space*, Amer. J. Math., **73** (1951), 357–362.
- [23] M. REED, B. SIMON, *Functional Analysis, Methods of Modern Mathematical Physics, Vol. 1*, Academic Press, 1972.
- [24] W. RUDIN, *Functional Analysis*, McGraw-Hill, 1991 (2nd edition).
- [25] K. SCHMÜDGEN, J. FRIEDRICH, *On Commuting Unbounded Selfadjoint Operators II*, Integral Equations Operator Theory, **7/6** (1984), 815–867.
- [26] K. SCHMÜDGEN, *Strongly Commuting Unbounded Selfadjoint Operators and Commutants of Unbounded Operator Algebra*, Proc. Amer. Math. Soc. **102/2** (1988), 365–372.
- [27] K. SCHMÜDGEN, *Unbounded Self-adjoint Operators on Hilbert Space*, Springer GTM **265**, (2012).
- [28] Z. SEBESTYÉN, J. STOCHÉL, *On Products of Unbounded Operators*, Acta Math. Hungar., **100/1-2** (2003), 105–129.
- [29] T. TUDOR, A. GHEONDEA, *Pauli Algebraic Forms of Normal and Non-Normal Operators*, J. Opt. Soc. Am. A, **24** (2007), 204–210.
- [30] J. WEIDMANN, *Linear operators in Hilbert spaces* (translated from the German by J. Szücs), Springer-Verlag, GTM **68** (1980).
- [31] N. A. WIEGMANN, *Normal Products of Matrices*, Duke Math J., **15**, (1948), 633–638.
- [32] N. A. WIEGMANN, *A Note on Infinite Normal Matrices*, Duke Math. J., **16** (1949), 535–538.
- [33] S. M. ZAGORODNYUK, *On commuting Symmetric Operators*, Methods of Functional Analysis and Topology, **18/2** (2012), 198–200.

(Received January 23, 2013)

Mohammed Hichem Mortad  
 Department of Mathematics, University of Oran  
 B.P. 1524, El Menouar, Oran 31000, Algeria  
 and  
 B.P. 7085 Es-Seddikia, Oran 31013, Algeria  
 e-mail: mhmortad@gmail.com; mortad@univ-oran.dz