

## DUALITY PROBLEM FOR THE CLASS OF LIMITED COMPLETELY CONTINUOUS OPERATORS

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(Communicated by T. S. S. R. K. Rao)

*Abstract.* We establish necessary and sufficient conditions under which the class  $L_{cc}(E, F)$  of limited completely continuous operators between two Banach lattices verifies the direct and reciprocal duality property.

### 1. Introduction

In this paper, we investigate the direct and reciprocal duality property of  $lcc$  operators. This class of operators was recently introduced and studied in [9]. We prove the direct duality property for the class  $L_{cc}(E, F)$ ,  $E$  Banach lattice with the  $DP^*$  property, and  $F$  Banach lattice if, and only if, the norm of  $E'$  is order continuous or  $F'$  has the GP-property (Corollary 3.1). The second main result is the reciprocal duality property for the class  $L_{cc}(E, F)$ , more precisely, we prove the equivalence between  $L_{cc}(E, F)$ ,  $E$  and  $F$  are Dedekind  $\sigma$ -complete Banach lattices, verifies the reciprocal duality and the norm of  $E$  or  $F$  is order continuous (Corollary 3.3).

The article is organized as follows, after the introduction section, we give in preliminaries section all common notations and definitions of Banach lattice theory. In the main results section, we study in the first subsection the direct duality property for the class of  $lcc$  operators and in the second subsection the reciprocal duality property of that class of operators.

### 2. Preliminaries

A norm bounded subset  $A$  of a Banach space  $X$  is said limited set if every weak\* null sequence  $(f_n)$  of  $X'$  converges uniformly on  $A$ , that is,  $\limsup_{n \rightarrow \infty} \sup_{x \in A} |\langle f_n, x \rangle| = 0$ . Note that every relatively compact set is limited but the converse is not true in general. Indeed, the set  $\{e_n : n \in \mathbb{N}\}$  of unit coordinate vectors is a limited set in  $\ell^\infty$  which is not relatively compact. If every limited subset of a Banach space  $X$  is relatively compact then,  $X$  has the Gelfand-Phillips property (in short GP-property). Alternatively, the Banach space  $X$  has the GP-property if, and only if, every limited and weakly null

*Mathematics subject classification* (2010): 46B42, 47B60, 47B65.

*Keywords and phrases:* Limited operator, limited sets, Dunford-Pettis\* property, Order continuous norm.

sequence  $(x_n)$  in  $X$  is norm null [7]. As example, the classical Banach spaces  $c_0$  and  $\ell^1$  have the GP-property but the Banach space  $\ell^\infty$  does not have the GP-property.

Let us recall from [9] that the operator  $T$  between two Banach spaces is called limited completely continuous (in short *lcc*), if  $T$  carries limited and weakly null sequences in  $X$  to norm null ones. Alternatively, an operator  $T$  on a Banach space  $X$  is *lcc* if, and only if, for each limited set  $A \subset X$ , the set  $T(A)$  is relatively compact [9, Theorem 2.1]. We denote this class by  $L_{cc}(X, Y)$ . This class of operators is a two-sided ideal in the space of all operators between Banach spaces. That is, if  $S$  and  $T$  are two operators between Banach spaces such that  $S$  or  $T$  is *lcc* then, the product  $S \circ T$  is *lcc*.

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space  $(E, \|\cdot\|)$  such that  $E$  is a vector lattice and its norm satisfies the following property: for each  $x, y \in E$  such that  $|x| \leq |y|$ , we have  $\|x\| \leq \|y\|$ . If  $E$  is a Banach lattice, its topological dual  $E'$ , endowed with the dual norm, is also a Banach lattice. A norm  $\|\cdot\|$  of a Banach lattice  $E$  is order continuous if for each generalized sequence  $(x_\alpha)$  such that  $x_\alpha \downarrow 0$  in  $E$ , the sequence  $(x_\alpha)$  converges to 0 for the norm  $\|\cdot\|$ , where the notation  $x_\alpha \downarrow 0$  means that the sequence  $(x_\alpha)$  is decreasing, its infimum exists and  $\inf(x_\alpha) = 0$ .

Recall that a Banach space  $X$  has the Dunford-Pettis\* property (in short  $DP^*$  property) if every relatively weakly compact subset of  $X$  is limited. Also, a vector lattice  $E$  is Dedekind  $\sigma$ -complete if every majorized countable nonempty subset of  $E$  has a supremum. We will use the term operator  $T : E \rightarrow F$  between two Banach lattices to mean a bounded linear mapping. If  $T$  is an operator from a Banach lattice  $E$  into another Banach lattice  $F$  then, its dual operator  $T'$  is defined from  $F'$  into  $E'$  by  $T'(f)(x) = f(T(x))$  for each  $f \in F'$  and for each  $x \in E$ . We refer the reader to [1] for unexplained terminology from Banach lattice theory. Some useful and additional properties of limited sets and Banach spaces with the GP-property can be found in [4, 9].

Finally, we give the following definitions.

DEFINITION 2.1. Let  $E$  and  $F$  be two Banach spaces and  $\mathcal{U}(E, F)$  a class of operators from  $E$  to  $F$ ,

- we shall say that the class  $\mathcal{U}(E, F)$  satisfies the *direct duality property*, if we have  $T \in \mathcal{U}(E, F) \implies T' \in \mathcal{U}(F', E')$ .
- by duality, we say that the class  $\mathcal{U}(E, F)$  satisfies the *reciprocal duality property*, if we have  $T' \in \mathcal{U}(F', E') \implies T \in \mathcal{U}(E, F)$ .

### 3. Main results

#### 3.1. Direct duality property of *lcc* operators

Note that there exists a *lcc* operator whose adjoint is not *lcc*. Indeed, the identity operator of the Banach lattice  $\ell^1$  is *lcc* (because  $\ell^1$  has the GP-property, Theorem 2.2 [9]) but its adjoint, which is the identity operator of the Banach lattice  $\ell^\infty$ , is not *lcc* (because  $\ell^\infty$  does not have the GP-property, Theorem 2.2 [9]).

To establish our first main result we will need the following Lemma.

LEMMA 3.1. *Let  $E$  be a Banach lattice. If  $E'$  does not have the GP-property then, there exists a limited weakly null sequence  $(f_n)$  in  $E'$  satisfying  $\|f_n\| = 1$  for all  $n$ . Moreover, there exists a sequence  $(y_n)$  of  $E^+$  satisfying  $\|y_n\| \leq 1$  and  $|f_n(y_n)| \geq \frac{1}{4}$  for all  $n$ .*

*Proof.* If  $E'$  does not have the GP-property then, there exists a limited weakly null sequence  $(f_n)$  in  $E'$  satisfying  $\|f_n\| = 1$  for all  $n$ .

As  $\|f_n\| = \sup\{|f_n(y)| : y \in E, \|y\| \leq 1\}$  for each  $n$ , there exists some  $y_n \in B_E$ , where  $B_E$  denotes the closed unit ball of  $E$ , such that  $|f_n(y_n)| \geq \frac{1}{2}\|f_n\| = \frac{1}{2}$ . By observing that  $|f_n(y_n^+)| \geq \frac{1}{4}$  or  $|f_n(y_n^-)| \geq \frac{1}{4}$  and replacing  $y_n$  by  $y_n^+$  or by  $y_n^-$  we may assume that for each  $n$  there exists some  $y_n \in E^+$  with  $\|y_n\| \leq 1$  and  $|f_n(y_n)| \geq \frac{1}{4}$ .  $\square$

The following Theorem gives some necessary conditions of Banach lattices under which the adjoint of each *lcc* operator  $T : E \rightarrow F$  is also *lcc*.

THEOREM 3.1. *Let  $E$  and  $F$  be two Banach lattices. If the class  $L_{cc}(E, F)$  verifies the direct duality property then, the norm of  $E'$  is order continuous or  $F'$  has the GP-property.*

*Proof.* By way of contradiction, we assume that the norm of  $E'$  is not order continuous and  $F'$  does not have the GP-property. We have to construct a *lcc* operator  $T : E \rightarrow F$  such that its adjoint  $T' : F' \rightarrow E'$  is not *lcc*.

Since the norm of  $E'$  is not order continuous then, it follows from Theorem 2.4.14 and Proposition 2.3.11 of [8] that  $E$  contains a sublattice isomorphic to  $\ell^1$  and there exists a positive projection  $P : E \rightarrow \ell^1$ .

On the other hand, since  $F'$  does not have the GP-property, it follows from Lemma 3.1 that there exists a limited weakly null sequence  $(f_n)$  in  $E'$  satisfying  $\|f_n\| = 1$  for all  $n$  and there exists a sequence  $(y_n)$  of  $E^+$  satisfying  $\|y_n\| \leq 1$  and  $|f_n(y_n)| \geq \frac{1}{4}$  for all  $n$ .

Consider the operator

$$S : \ell^1 \rightarrow F, (\lambda_n)_{n=1}^\infty \rightarrow \sum_{n=1}^\infty \lambda_n y_n.$$

This operator is well defined and it is *lcc* (because  $\ell^1$  has the GP-property), but its adjoint defined by

$$S' : F' \rightarrow \ell^\infty, f \rightarrow (f(y_n))_{n=1}^\infty$$

is not *lcc*. Otherwise, since  $(f_n)$  is a limited weakly null sequence in  $F'$  then,  $S'(f_n) = (f_n(y_n))_{n=1}^\infty$  would be norm null. But this is impossible (because  $|f_n(y_n)| \geq \frac{1}{4}$  for all  $n$ ).

Now, we consider the operator product  $T = S \circ P : E \rightarrow F$ . Since  $S$  is a *lcc* operator then, the operator  $T$  is *lcc*. But the adjoint  $T' = P' \circ S'$  is not *lcc*. Otherwise, the operator  $i' \circ T' = i' \circ P' \circ S' = S'$  would be *lcc*, where  $i : \ell^1 \rightarrow E$  is the canonical injection, but this is impossible. This completes the proof.  $\square$

Recall that an operator  $T : E \rightarrow X$  from a Banach lattice  $E$  into a Banach space  $X$  is said to be  $M$ -weakly compact if for every disjoint sequence  $(x_n)$  in  $B_E$  we have  $\|T(x_n)\| \rightarrow 0$ , where  $B_E$  denotes the closed unit ball of  $E$ .

Note that each  $M$ -weakly compact operator is  $lcc$  but the converse is not true in general. Indeed, the operator  $T : \ell^1 \rightarrow \ell^\infty$  defined by

$$T((\alpha_n)) = \left( \sum_{n=1}^{\infty} \alpha_n \right)_{n=1}^{\infty} = \sum_{n=1}^{\infty} \alpha_n (1, 1, 1, \dots)$$

is  $lcc$  (because  $\ell^1$  has the GP-property) but fails to be  $M$ -weakly compact).

**PROPOSITION 3.1.** *Let  $E$  be a Banach lattice and  $X$  a Banach space. If the norm of  $E'$  is order continuous and  $E$  has the  $DP^*$  property then, each  $lcc$  operator  $T : E \rightarrow X$  is  $M$ -weakly compact.*

*Proof.* Let  $T : E \rightarrow X$  be a  $lcc$  operator and let  $(x_n)$  be a norm bounded disjoint sequence in  $E$ . Since the norm of  $E'$  is order continuous, it follows from Corollary 2.9 of Dodds and Fremlin [6] that  $x_n \rightarrow 0$  for  $\sigma(E, E')$ . On the other hand, since  $E$  has the  $DP^*$  property then the sequence  $(x_n)$  is limited. As  $T$  is  $lcc$ ,  $\|T(x_n)\| \rightarrow 0$  and hence  $T$  is  $M$ -weakly compact.  $\square$

**THEOREM 3.2.** *Let  $E$  and  $F$  be two Banach lattices. If the norm of  $E'$  is order continuous and  $E$  has the  $DP^*$  property or  $F'$  has the GP-property then, the class  $L_{cc}(E, F)$  verifies the direct duality property.*

*Proof.* (1) Let  $T : E \rightarrow F$  be a  $lcc$  operator. Since the norm of  $E'$  is order continuous and  $E$  has the  $DP^*$  property, it follows from Proposition 3.1 that  $T$  is  $M$ -weakly compact, which imply that  $T$  is weakly compact. By the Schauder's Theorem,  $T'$  is weakly compact. Finally by Corollary 2.5 of [9], we conclude that  $T'$  is  $lcc$ .

(2) In this case, each operator  $T : E \rightarrow F$  has an adjoint which is  $lcc$ .  $\square$

As consequence of Theorem 3.1 and Theorem 3.2, we obtain the following characterization.

**COROLLARY 3.1.** *Let  $E$  and  $F$  be two Banach lattices such that  $E$  has the  $DP^*$  property. Then, the following statements are equivalent:*

1. *the class  $L_{cc}(E, F)$  of  $lcc$  operators verifies the direct duality property,*
2. *one of the following is valid:*
  - (a) *the norm of  $E'$  is order continuous,*
  - (b)  *$F'$  has the GP-property.*

### 3.2. Reciprocal duality property of *lcc* operators

In this section, we give sufficient conditions under which the reciprocal duality property for *lcc* operators is guaranteed. Note that there is an operator which is not *lcc* while its adjoint is *lcc*. Indeed, the identity operator of  $\ell^\infty$  is not *lcc* even if, the identity operator of  $(\ell^\infty)'$  is *lcc*.

By a simple proof we can quarry the following proposition:

PROPOSITION 3.2. *Let  $X$  be a Banach space and let  $A$  be a bounded set of  $X$ . Then, the following statements are equivalent:*

1.  $A$  is a limited set,
2. for each sequence  $(x_n)$  in  $A$ ,  $f_n(x_n) \rightarrow 0$  for every weak\* null sequence  $(f_n)$  of  $X'$ .

As consequences of Proposition 3.2, we obtain the following result.

COROLLARY 3.2. *Let  $X$  be a Banach space and let  $(x_n)$  be a norm bounded sequence of  $X$ . Then, the following statements are equivalent:*

1. The subset  $\{x_n, n \in \mathbb{N}\}$  is limited,
2. for each sequence  $(x_n)$  in  $A$ ,  $f_n(x_n) \rightarrow 0$  for every weak\* null sequence  $(f_n)$  of  $X'$ .

Let us recall that an operator  $T$  from a Banach lattice  $E$  into a Banach space  $X$  is said to be order weakly compact if it carries each order bounded subset of  $E$  into relatively weakly compact of  $X$ . Alternatively,  $T$  is order weakly compact if, and only if,  $\lim_{n \rightarrow \infty} \|T(x_n)\| = 0$  for every order bounded disjoint sequence  $(x_n)_n \subset E$ , [5].

To give our second main result, we will need the following Lemma.

LEMMA 3.2. *Let  $E$  be a Banach lattice. Then, for every order bounded disjoint sequence  $(x_n)$  in  $E$ , the subset  $\{x_n, n \in \mathbb{N}\}$  is limited.*

*Proof.* Let  $(x_n)$  be an order bounded disjoint sequence in  $E$ . Suppose  $(f_n)$  be a weak\* null sequence of  $E'$ , we consider the operator

$$S : X \rightarrow c_0; x \mapsto (f_n(x))_{n=1}^\infty.$$

Since  $c_0$  has an order continuous norm, it follows Theorem 2.8 [2] that the operator  $S$  is order weakly compact. Hence, by Theorem 5.26 [1], we derive that  $\|S(y_k)\|_\infty = \|(f_n(y_k))_{n=0}^\infty\|_\infty \rightarrow 0$  for every order bounded disjoint sequence  $(y_k)$  in  $E$ . Now, according to Corollary 3.2 and by the inequality  $|f_n(x_n)| \leq \|(f_k(y_n))_{k=0}^\infty\|_\infty \rightarrow 0$ , we conclude that the subset  $\{x_n, n \in \mathbb{N}\}$  is limited.  $\square$

THEOREM 3.3. *Let  $E$  and  $F$  be two Banach lattices such that  $F$  is a Dedekind  $\sigma$ -complete. If the class  $L_{cc}(E, F)$  of *lcc* operators verifies the reciprocal duality property then, the norm of  $E$ , or  $F$ , is order continuous.*

*Proof.* Assume by way of contradiction that the norms of  $E$  and  $F$  are not order continuous. We have to construct an operator  $T : E \rightarrow F$  which is not  $lcc$  but its adjoint  $T' : F' \rightarrow E'$  is  $lcc$ .

As the norm of  $E$  is not order continuous, it follows from Theorem 10.1 of [1] and Lemma 3.4 [3] the existence of an order bounded disjoint sequence  $(x_n)$  in  $E^+$  with  $\|x_n\| = 1$  for all  $n$  and there exists a disjoint sequence of positive elements  $(g_n)$  of  $E'$  with  $\|g_n\| \leq 1$  for each  $n$ , such that  $g_n(x_n) = 1$  for all  $n$  and  $g_n(x_m) = 0$  for  $n \neq m$ . We consider the operator  $S$  defined as follows:

$$S : E \rightarrow \ell^\infty, x \rightarrow S(x) = (g_n(x))_{n=1}^\infty.$$

The operator  $S$  is well defined and it is not  $lcc$ . Indeed, since  $(x_n)$  is an order bounded disjoint sequence in  $E^+$  then, it follows from a Remark of ([1], p. 185) and Theorem 4.5 [10] that  $(x_n)$  is a limited weakly null sequence. If  $S$  were  $lcc$  then,  $\lim_{n \rightarrow \infty} \|S(x_n)\| = \lim_{n \rightarrow \infty} \|g_n(x_n)\| = 0$ , which is a contradiction with  $g_n(x_n) = 1$  for all  $n$ .

Now, as the norm of  $F$  is not order continuous and  $F$  is Dedekind  $\sigma$ -complete, it follows from Corollary 2.4.3 of [8] that  $F$  contains a positively complemented closed sublattice which is order and topologically isomorphic to  $\ell^\infty$ . Let  $P : F \rightarrow \ell^\infty$  be the positive projection and  $\iota : \ell^\infty \rightarrow F$  the canonical injection.

We consider the operator product  $T = \iota \circ S$ .  $T$  is not  $lcc$ . Otherwise,  $P \circ T = S$  would be  $lcc$ , which is a contradiction.

Now, as the subspace of  $lcc$  operators is a two-sided ideal, the adjoint

$$T' = S' \circ \iota' = S' \circ Id_{(\ell^\infty)'} \circ \iota' : F' \rightarrow (\ell^\infty)' \rightarrow (\ell^\infty)' \rightarrow E'$$

is  $lcc$  (because  $(\ell^\infty)'$  has the GP-property).  $\square$

REMARK 3.1. The condition “ $F$  is Dedekind  $\sigma$ -complete” is not accessory in the above Theorem. In fact, each operator  $T : \ell^\infty \rightarrow c$  is weakly compact and hence is  $lcc$  (Corollary 2.5 [9]) but neither  $\ell^\infty$  nor  $c$  has an order continuous norm.

As consequence of Theorem 3.2 and Theorem 4.5 [10], we have the following characterization.

COROLLARY 3.3. *Let  $E$  and  $F$  be two Dedekind  $\sigma$ -complete Banach lattices. Then, the following assertions are equivalent:*

1. *the class  $L_{cc}(E, F)$  of  $lcc$  operators verifies the reciprocal duality property,*
2. *the norm of  $E$ , or  $F$ , is order continuous.*

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(Received December 5, 2012)

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