

COMMUTATION RELATIONS FOR TRUNCATED TOEPLITZ OPERATORS

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Abstract. For truncated Toeplitz operators, which are compressions of multiplication operators to model subspaces of the Hardy space H^2 , we obtain criteria for commutation relations. The results show an analogy to the case of Toeplitz matrices, and they extend the theory of Sedlock algebras.

1. Introduction

Truncated Toeplitz operators are compressions of multiplication operators to model subspaces of the Hardy space H^2 ; they represent a far reaching generalization of classical Toeplitz matrices. Although particular cases had appeared before in the literature, the general theory has been initiated in the seminal paper [14]. Since then, truncated Toeplitz operators have constituted an active area of research. We mention only a few relevant papers: [2, 3, 4, 5, 10, 17]; see also the recent survey [9] and the references within.

In particular, in [15] Sedlock has investigated when a product of truncated Toeplitz operators is itself a truncated Toeplitz operator. It turns out that this does not happen very often. More precisely, there exists a family of classes \mathcal{B}_u^α (precise definitions in the next section), where α is in the extended complex plane, such that, whenever the product of two nonscalar truncated Toeplitz operators is itself a truncated Toeplitz operator, both operators have to belong to the same class \mathcal{B}_u^α . These classes are commutative algebras, and they are the maximal subalgebras of the subspace of truncated Toeplitz operators.

On the other hand, truncated Toeplitz operators represent a far reaching generalization of classical Toeplitz matrices. Toeplitz matrices whose product is also a Toeplitz matrix are sometimes called generalized circulants [7], and a discussion of the classes \mathcal{B}_u^α for this particular case appears in [16]. A uniform procedure for imposing conditions on products of Toeplitz matrices has been devised in [12], leading to characterizations of different classes of Toeplitz matrices: normal, unitary, commuting, etc.

The purpose of the present paper is to adapt the approach in [12] to the general case of truncated Toeplitz operators on an arbitrary model space. The algebraic relations carry through neatly if we take advantage of a certain unitary operator between

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different model spaces, called the Crofoot transform. As a consequence, we obtain complete characterizations of some classes of truncated Toeplitz operators defined by commutation relations.

The plan of the paper is the following. After a preliminary section, we introduce the Sedlock classes in Section 3 and the Crofoot transform in Section 4. Section 5 is dedicated to the key technical argument, which is analogous to the one in [12]. The main results are then proved in Section 6.

2. Preliminaries

Our notations are mostly standard: \mathbb{C} is the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc, and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle. By $\hat{\mathbb{C}}$ we will denote the extended complex plane $\mathbb{C} \cup \{\infty\}$. As is customary, we will view the Hardy space H^2 on \mathbb{D} as a subspace of $L^2(\mathbb{T})$ by identifying functions analytic in \mathbb{D} with their radial limits (almost everywhere). Similarly, the algebra H^∞ of bounded analytic functions in \mathbb{D} may be viewed as a closed subalgebra of $L^\infty(\mathbb{T})$.

An inner function $u \in H^\infty$ is characterized by $|u| = 1$ almost everywhere on \mathbb{T} . If u is an inner function and $a \in \mathbb{D}$, we define the inner function u_a by

$$u_a(z) = \frac{u(z) - a}{1 - \bar{a}u(z)}.$$

If u is an inner function, the model space K_u^2 is defined by $K_u^2 = H^2 \ominus uH^2$. We denote by $P_{K_u^2}$ the orthogonal projection (in $L^2(\mathbb{T})$) onto K_u^2 .

The conjugation of $L^2(\mathbb{T})$ defined by $\tilde{f} = u\bar{z}\bar{f}$ bijectively maps K_u^2 to itself; it is this latter restriction that will appear in the sequel. The space K_u^2 is a reproducing kernel space of analytic functions on \mathbb{D} , and the reproducing kernels for points $\lambda \in \mathbb{D}$ are

$$k_\lambda^u(z) = \frac{1 - \overline{u(\lambda)}u(z)}{1 - \bar{\lambda}z}.$$

The conjugate kernels \tilde{k}_λ^u will also appear; an easy computation yields

$$\tilde{k}_\lambda^u(z) = \frac{u(z) - u(\lambda)}{z - \lambda}.$$

As shown in [1], in special cases one may have “reproducing kernels” for points $\zeta \in \mathbb{T}$. Namely, all functions in K_u^2 have a nontangential limit $f(\zeta)$ in $\zeta \in \mathbb{T}$ precisely when u has an angular derivative in the sense of Caratheodory in ζ . In this case the function

$$k_\zeta^u(z) = \frac{1 - \overline{u(\zeta)}u(z)}{1 - \bar{\zeta}z}$$

belongs to K_u^2 , and $f(\zeta) = \langle f, k_\zeta^u \rangle$ for $f \in K_u^2$.

The truncated Toeplitz operators (TTO) are defined as follows. Note first that, since the reproducing kernels are bounded functions, $K_u^2 \cap H^\infty$ is dense in K_u^2 . If $\phi \in$

$L^2(\mathbb{T})$, we consider the map $f \mapsto P_{K_u^2} \phi f$ defined on $K_u^2 \cap H^\infty$. If this map extends to a bounded operator on K_u^2 , we denote it A_ϕ^u and call it a truncated Toeplitz operator with symbol ϕ . The set of all TTOs on K_u^2 is a weakly closed subspace of $\mathcal{L}(K_u^2)$, that we will denote by \mathcal{T}_u .

Truncated Toeplitz operators are closer to Toeplitz matrices than to Toeplitz operators. To start with, the symbol of a TTO is not uniquely defined; it is proved in [14] that $A_\phi^u = 0$ if and only if $\phi \in uH^2 + \overline{uH^2}$. It would be tempting to speak about the uniquely defined *reduced symbol* of a TTO A_ϕ^u as the projection of ϕ onto $L^2 \ominus (uH^2 + \overline{uH^2})$. This space can also be written as $(K_u^2 + \overline{K_u^2}) \ominus \mathbb{C}(k_0^u - \overline{k_0^u})$ (see [14, 15]); in particular, any TTO has a symbol in $K_u^2 + \overline{K_u^2}$. Obviously things simplify when $k_0^u = \overline{k_0^u}$, which is equivalent to $u(0) = 0$; we will have more to say about this in Section 5.

It has been shown in [14, Theorem 4.1] that TTOs may be characterized algebraically among operators on K_u^2 ; the result is the following.

LEMMA 2.1. *The bounded operator A on K_u^2 belongs to \mathcal{T}_u if and only if there are functions $\psi, \chi \in K_u^2$ such that*

$$\Delta(A) := A - S_u A S_u^* = (\psi \otimes k_0^u) + (k_0^u \otimes \chi),$$

in which case $A = A_{\psi + \overline{\chi}}^u$.

EXAMPLES.

1. If $\phi(z) = z$, then A_ϕ^u is the *model operator* [13, 18] on the space K_u^2 ; it will be denoted by S_u .
2. In [14] are identified all rank one operators in \mathcal{T}_u : they are multiples of $k_\lambda^u \otimes \overline{k_\lambda^u}$ and of their adjoints $\overline{k_\lambda^u} \otimes k_\lambda^u$, to which are added multiples of $k_\zeta^u \otimes k_\zeta^u$ whenever u has an angular derivative in the sense of Caratheodory in $\zeta \in \mathbb{T}$.
3. For $\alpha \in \overline{\mathbb{D}}$ the *modified compressed shifts* are defined by

$$S_u^\alpha = S_u + \frac{\alpha}{1 - \alpha u(0)} k_0^u \otimes \overline{k_0^u}.$$

If $\alpha \in \mathbb{D}$, then S_u^α is unitarily equivalent to S_{u_α} , and is thus a completely non-unitary contraction (whose characteristic function, in the sense of Sz. Nagy–Foias [18], is u_α). If $\alpha \in \mathbb{T}$, then S_u^α is unitary, with singular spectral measure and multiplicity one (these are precisely the Clark unitary operators defined in [6]).

3. Sedlock classes

The Sedlock classes $\mathcal{B}_u^\alpha \subset \mathcal{T}_u$, with $\alpha \in \hat{\mathbb{C}}$, have been introduced in [15] in connection to multiplication properties of TTOs. For $\alpha \in \mathbb{C}$, \mathcal{B}_u^α is the set of operators in \mathcal{T}_u which have a symbol of the form $\phi + \alpha S_u \overline{\phi} + c$, where $\phi \in K_u^2$ and $c \in \mathbb{C}$; while, for $\alpha = \infty$, \mathcal{B}_u^∞ is the set of TTOs which have an antiholomorphic symbol. The following are the main results proved in [15].

THEOREM 3.1. (i) For any $\alpha \in \hat{\mathbb{C}}$, \mathcal{B}_u^α is a commutative weakly closed algebra.

(ii) If $\alpha \neq \alpha'$, then $\mathcal{B}_u^\alpha \cap \mathcal{B}_u^{\alpha'} = \mathbb{C}I$.

(iii) $A \in \mathcal{B}_u^\alpha$ if and only if $A^* \in \mathcal{B}_u^{1/\bar{\alpha}}$.

(iv) If $\alpha \in \mathbb{D}$, then $\mathcal{B}_u^\alpha = \{S_u^\alpha\}'$ (the commutant of S_u^α).

(v) If $A, B \in \mathcal{T}_u$, then $AB \in \mathcal{T}_u$ if and only if either one of the operators is a scalar, or both belong to the same class \mathcal{B}_u^α for some $\alpha \in \hat{\mathbb{C}}$. In the last case we also have $AB \in \mathcal{B}_u^\alpha$.

(vi) The classes \mathcal{B}_u^α are precisely the maximal subalgebras of \mathcal{T}_u .

As the algebras \mathcal{B}_u^α are the commutants of modified compressed shifts, they may be given a more concrete description. This is done in [15, Section 6], and we present below a brief summary of the results therein. There are basically two distinct types of Sedlock classes, depending on whether $|\alpha| = 1$ or not, and the case $|\alpha| > 1$ is reduced to $|\alpha| < 1$ by taking adjoints.

1. If $|\alpha| = 1$, then S_u^α is a unitary operator of multiplicity one, with singular spectral measure μ_α . Thus $\mathcal{B}_u^\alpha = \{S_u^\alpha\}'$ is a maximal abelian subalgebra of $\mathcal{L}(K_u^2)$, and its elements may be described as functions $\Phi(S_u^\alpha)$ with $\Phi \in L^\infty(\mu_\alpha)$.
2. If $|\alpha| \neq 1$, suppose first that $|\alpha| < 1$. Then S_u^α is a completely nonunitary contraction, that has a functional calculus with functions in H^∞ [18]. Its commutant \mathcal{B}_u^α is a weakly closed nonselfadjoint algebra; its elements are the functions $\Psi(S_u^\alpha)$ with $\Psi \in H^\infty$, and we may identify their symbols as TTOs by the formula

$$\Psi(S_u^\alpha) = A_u^\Psi \frac{1}{1-\alpha\bar{z}}.$$

If $|\alpha| > 1$, then $S_u^{1/\bar{\alpha}}$ is a completely nonunitary contraction, and using Theorem 3.1 (iii) the elements of \mathcal{B}_u^α may be described as

$$\Psi(S_u^{1/\bar{\alpha}})^* = A_u^\Psi \frac{\bar{z}}{\alpha - z}$$

for $\Psi \in H^\infty$.

It is worth mentioning the following simple corollary, which determines when the product of two TTOs is zero.

COROLLARY 3.2. If A_ϕ^u, A_ψ^u are nonzero operators in \mathcal{T}_u and $A_\phi^u A_\psi^u = 0$, then there is $\alpha \in \hat{\mathbb{C}}$ such that $A_\phi^u, A_\psi^u \in \mathcal{B}_u^\alpha$. Moreover:

1. If $|\alpha| = 1$, then $A_\phi^u = \Phi(S_u^\alpha)$, $A_\psi^u = \Psi(S_u^\alpha)$, with $\Phi, \Psi \in L^\infty(\mu_\alpha)$ and $\Phi\Psi = 0$ μ_α -almost everywhere.
2. If $|\alpha| < 1$, then $A_\phi^u = \Phi(S_u^\alpha)$, $A_\psi^u = \Psi(S_u^\alpha)$, with $\Phi, \Psi \in H^\infty$, and the inner function u_α divides $\Phi\Psi$.

3. If $|\alpha| > 1$, then $A_\phi^u = \Phi(S_u^{1/\bar{\alpha}})^*$, $A_\psi^u = \Psi(S_u^{1/\bar{\alpha}})^*$, with $\Phi, \Psi \in H^\infty$, and the inner function $u_{1/\bar{\alpha}} = \frac{1-\bar{\alpha}u}{u-\alpha}$ divides $\Phi\Psi$.

Proof. Most of the statements are immediate consequences of the remarks above. For point (ii), one should note that if $h \in H^\infty$ and $h(S_u^\alpha) = 0$, then u_α divides h . This is proved directly in [15, Section 6]; alternately, it follows from the fact, noted above, that the characteristic function of S_u^α is u_α . \square

We end this section with a continuity property of Sedlock classes.

LEMMA 3.3. Suppose $\alpha_n, \alpha \in \mathbb{C}$, $\alpha_n \rightarrow \alpha$, $A_n \in \mathcal{B}_u^{\alpha_n}$, and $A_n \rightarrow A$. Then $A \in \mathcal{B}_u^\alpha$.

Proof. For $\alpha_n, \alpha \in \bar{\mathbb{D}}$ the result follows from Theorem 3.1 (iv), once we note that $\alpha_n \rightarrow \alpha$ implies $S_{u_n}^{\alpha_n} \rightarrow S_u^\alpha$. If $\alpha \notin \mathbb{D}$, we use Theorem 3.1 (iii) to reduce it to the previous case. \square

In the sequel we will usually assume that $\alpha \in \mathbb{C}$; the obvious modifications of the arguments required when $\alpha = \infty$ are left to the reader.

4. The Crofoot transform

Let u be an inner function and $a \in \mathbb{D}$. The Crofoot transform $J = J(u, a)$ is the unitary operator $J : K_u^2 \rightarrow K_{u_a}^2$ defined by

$$J(f) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}u} f.$$

It is proved in [14, Theorem 13.2] that

$$J\mathcal{T}_u J^* = \mathcal{T}_{u_a}. \tag{1}$$

The next result could be obtained by tedious calculations, but we prefer a shorter argument based on the previous section.

THEOREM 4.1. If $\alpha \in \hat{\mathbb{C}}$, then $J\mathcal{B}_u^\alpha J^* = \mathcal{B}_{u_a}^\beta$, where $\beta = \frac{\alpha-a}{1-\bar{a}\alpha}$.

Proof. Since \mathcal{B}_u^α is a maximal subalgebra of \mathcal{T}_u , it follows from (1) that $J\mathcal{B}_u^\alpha J^*$ is a maximal algebra of \mathcal{T}_{u_a} , and thus, by Theorem 3.1, it must be equal to $\mathcal{B}_{u_a}^\beta$ for some $\beta \in \hat{\mathbb{C}}$. To obtain the precise value of β , it is enough to look at the Crofoot transform of a single nonscalar operator; this we will do in the sequel. We may assume that $\dim K_u^2 > 1$, since otherwise there is nothing to prove.

Suppose first that $|\alpha| < 1$. It is shown in [15, Example 5.3] that for any $\lambda \in \mathbb{D}$ the rank one operator $\tilde{k}_\lambda^u \otimes k_\lambda^u$ belongs to $\mathcal{B}_u^{u(\lambda)}$; also, $\tilde{k}_\lambda^u \otimes k_\lambda^u$ is not scalar since $\dim K_u^2 > 1$.

If $f \in K_u^2$, then

$$f(\lambda) = \frac{1 - \overline{au}(\lambda)}{\sqrt{1 - |a|^2}} (Jf)(\lambda) = \langle Jf, \frac{1 - \overline{au}(\lambda)}{\sqrt{1 - |a|^2}} k_\lambda^{u_a} \rangle = \langle f, \frac{1 - \overline{au}(\lambda)}{\sqrt{1 - |a|^2}} J^* k_\lambda^{u_a} \rangle.$$

Therefore $J^* k_\lambda^{u_a}$ is a multiple of k_λ^u , or, equivalently, $k_\lambda^{u_a}$ is a multiple of Jk_λ^u . Since J commutes with the respective conjugations on K_u^2 and $K_{u_a}^2$, the conjugate kernel $\tilde{k}_\lambda^{u_a}$ is a multiple of $J\tilde{k}_\lambda^u$. Therefore $J(\tilde{k}_\lambda^u \otimes k_\lambda^{u_a})J^*$ is a multiple of $\tilde{k}_\lambda^{u_a} \otimes k_\lambda^{u_a}$, and thus belongs to $\mathcal{B}_{u_a}^{u_a(\lambda)}$. Since $u_a(\lambda) = \frac{u(\lambda) - a}{1 - \overline{au}(\lambda)}$, we have found, in the case $\alpha = u(\lambda)$, a nonscalar operator in the class \mathcal{B}_u^α whose Crofoot transform is in $\mathcal{B}_{u_a}^\beta$, with $\beta = \frac{\alpha - a}{1 - \overline{a\alpha}}$. By Theorem 3.1 (ii) the same must then be true for the whole class.

The result is thus proved for points in $u(\mathbb{D})$; since u is inner, this is a dense set in $\overline{\mathbb{D}}$ (see, for instance, [11, Theorem 6.6]). For $\alpha \in \overline{\mathbb{D}}$ outside this set, choose some $w \in \mathbb{D}$ such that, if $\phi = k_w^u + \alpha S_u \tilde{k}_w^u$, then A_ϕ^u is not scalar. Take a sequence $\alpha_n \rightarrow \alpha$, $\alpha_n \in u(\mathbb{D})$; then $\phi_n = k_w^u + \alpha_n S_u \tilde{k}_w^u$ tend uniformly to ϕ , and therefore $A_{\phi_n}^u \rightarrow A_\phi^u$, $JA_{\phi_n}^u J^* \rightarrow JA_\phi^u J^*$. We have $A_{\phi_n}^u \in \mathcal{B}_u^{\alpha_n}$ and $A_\phi^u \in \mathcal{B}_u^\alpha$ by the definition of the Sedlock classes. Since $\alpha_n \in u(\mathbb{D})$, $JA_{\phi_n}^u J^* \in \mathcal{B}_{u_a}^{\beta_n}$, with $\beta_n := \frac{\alpha_n - a}{1 - \overline{a\alpha_n}} \rightarrow \beta := \frac{\alpha - a}{1 - \overline{a\alpha}}$. Applying Lemma 3.3, it follows that $JA_\phi^u J^* \in \mathcal{B}_{u_a}^\beta$. So again we have found a nonscalar operator in \mathcal{B}_u^α , whose Crofoot transform is in $\mathcal{B}_{u_a}^\beta$ with $\beta = \frac{\alpha - a}{1 - \overline{a\alpha}}$, and by Theorem 3.1 (ii) the same must be true for the whole class.

Finally, if $|\alpha| > 1$, then $\alpha' = 1/\overline{\alpha} \in \mathbb{D}$, and, if $\beta' = \frac{\alpha' - a}{1 - \overline{a\alpha'}}$, then $1/\overline{\beta'} = \beta$. Therefore, using the result already proved for α' and Theorem 3.1 (iii), we obtain

$$J\mathcal{B}_u^\alpha J^* = J(\mathcal{B}_u^{\alpha'})^* J^* = (J\mathcal{B}_u^{\alpha'} J^*)^* = (\mathcal{B}_{u_a}^{\beta'})^* = \mathcal{B}_{u_a}^\beta,$$

thus ending the proof of the theorem. \square

Note that the particular case $a = \alpha$ appears in [15, Section 6]. We will only use the Crofoot transform obtained by taking $a = u(0)$; in this case $u_a(0) = 0$.

5. Basic commutation formulas

In this section the inner function u is subjected to the condition $u(0) = 0$. Then $u = z u_1$, $k_0^u = \mathbf{1}$ (the constant function equal to 1), and $\tilde{k}_0^u = u_1$; also, we have the direct sum decompositions

$$K_u^2 = \mathbb{C}\mathbf{1} \oplus zK_{u_1}^2, \tag{2}$$

$$(uH^2 + \overline{uH^2})^\perp = \overline{K_u^2} + K_u^2 = \overline{zK_{u_1}^2} \oplus \mathbb{C}\mathbf{1} \oplus zK_{u_1}^2. \tag{3}$$

Any TTO has a unique symbol $\phi \in (uH^2 + \overline{uH^2})^\perp$, and according to (3) we may write

$$\phi = \phi_+ + \overline{\phi_-} + \phi_0 \tag{4}$$

with $\phi_{\pm} \in zK_{u_1}^2$ and $\phi_0 \in \mathbb{C}$. Whenever $u(0) = 0$, the operator A_{ϕ}^u will have the symbol ϕ in $\overline{K_u^2} + K_u^2$, and we will consistently use the decomposition (4). Note that $(A_{\phi}^u)^* = A_{\bar{\phi}}^u$, and $(\bar{\phi})_{\pm} = \phi_{\mp}$, $(\bar{\phi})_0 = \overline{\phi_0}$.

We define a conjugation $\check{}$ on $zK_{u_1}^2$, that we will call the *reduced conjugation*, by transporting the conjugation on $K_{u_1}^2$; that is, for $f \in zK_{u_1}^2$,

$$\check{f} = z\bar{f}u_1. \tag{5}$$

The Sedlock classes can be easily identified in terms of ϕ_{\pm} ; namely, $A_{\phi}^u \in \mathcal{B}_u^{\alpha}$ if and only if $\check{\phi}_- = \alpha\phi_+$.

Finally, let us note the formulas

$$\Delta(I) = I - S_u S_u^* = \mathbf{1} \otimes \mathbf{1}, \quad I - S_u^* S_u = u_1 \otimes u_1. \tag{6}$$

The next is the correspondent of [12, Lemma 2.3].

LEMMA 5.1. *Suppose $u(0) = 0$. If $A_{\phi}^u, A_{\psi}^u \in \mathcal{T}_u$, then*

$$\begin{aligned} \Delta(A_{\phi}^u A_{\psi}^u) &= \phi_+ \otimes \psi_- - \check{\phi}_- \otimes \check{\psi}_+ \\ &\quad + (A_{\phi}^u \psi_+ + \psi_0 \phi_+ + \phi_0 \psi_0 \mathbf{1}) \otimes \mathbf{1} + \mathbf{1} \otimes (S_u(A_{\psi}^u)^* S_u^* \phi_- + \bar{\phi}_0 \psi_-). \end{aligned}$$

Proof. Denote $\hat{\phi} = \phi - \phi(0)$, $\hat{\psi} = \psi - \psi(0)$. We have

$$\Delta(A_{\hat{\phi}}^u A_{\hat{\psi}}^u) = \Delta(A_{\phi}^u A_{\psi}^u) + \psi_0 \Delta(A_{\hat{\phi}}^u) + \phi_0 \Delta(A_{\hat{\psi}}^u) + \phi_0 \psi_0 (\mathbf{1} \otimes \mathbf{1}).$$

By Lemma 2.1, we have

$$\Delta(A_{\hat{\phi}}^u) = \phi_+ \otimes \mathbf{1} + \mathbf{1} \otimes \phi_-, \quad \Delta(A_{\hat{\psi}}^u) = \psi_+ \otimes \mathbf{1} + \mathbf{1} \otimes \psi_-, \tag{7}$$

and therefore

$$\Delta(A_{\phi}^u A_{\psi}^u) = \Delta(A_{\hat{\phi}}^u A_{\hat{\psi}}^u) + (\psi_0 \phi_+ + \phi_0 \psi_+ + \phi_0 \psi_0 \mathbf{1}) \otimes \mathbf{1} + \mathbf{1} \otimes (\bar{\psi}_0 \phi_- + \bar{\phi}_0 \psi_-). \tag{8}$$

Now, using (6) and (7),

$$\begin{aligned} \Delta(A_{\hat{\phi}}^u A_{\hat{\psi}}^u) &= A_{\hat{\phi}}^u A_{\hat{\psi}}^u - S_u A_{\hat{\phi}}^u A_{\hat{\psi}}^u S_u^* \\ &= A_{\hat{\phi}}^u A_{\hat{\psi}}^u - A_{\hat{\phi}}^u S_u A_{\hat{\psi}}^u S_u^* + A_{\hat{\phi}}^u S_u A_{\hat{\psi}}^u S_u^* - S_u A_{\hat{\phi}}^u (S_u^* S_u + u_1 \otimes u_1) A_{\hat{\psi}}^u S_u^* \\ &= A_{\hat{\phi}}^u \Delta(A_{\hat{\psi}}^u) + \Delta(A_{\hat{\phi}}^u) S_u A_{\hat{\psi}}^u S_u^* - S_u A_{\hat{\phi}}^u (u_1 \otimes u_1) A_{\hat{\psi}}^u S_u^* \\ &= A_{\hat{\phi}}^u (\psi_+ \otimes \mathbf{1} + \mathbf{1} \otimes \psi_-) + (\phi_+ \otimes \mathbf{1} + \mathbf{1} \otimes \phi_-) S_u A_{\hat{\psi}}^u S_u^* - (S_u A_{\hat{\phi}}^u u_1 \otimes S_u (A_{\hat{\psi}}^u)^* u_1) \end{aligned}$$

We have $A_{\hat{\phi}}^u \mathbf{1} = \phi_+$, $S_u^* \mathbf{1} = 0$, so the sum of the first two terms on the last line is

$$A_{\hat{\phi}}^u \psi_+ \otimes \mathbf{1} + \phi_+ \otimes \psi_- + \mathbf{1} \otimes S_u (A_{\hat{\psi}}^u)^* S_u^* \phi_-.$$

Further, $A_{\phi}^u u_1 = P_{K_u^2} \hat{\phi} u_1 = P_{K_u^2} \phi_+ u_1 + P_{K_u^2} \bar{\phi}_- u_1$. Since $\phi_+ \in zK_{u_1}^2$, $\phi_+ u_1$ has $z u_1 = u$ as a factor, and thus is orthogonal to K_u^2 . Also, $\bar{\phi}_- u_1 = \bar{z} z \bar{\phi}_- u_1 = \bar{z} \check{\phi}_-$, and $\check{\phi}_- \in zK_{u_1}^2$ implies $\bar{z} \check{\phi}_- \in K_u^2$, whence $A_{\phi}^u u_1 = \bar{z} \check{\phi}_-$. Therefore $S_u A_{\phi}^u u_1 = P_{K_u^2} \check{\phi}_- = \check{\phi}_-$.

Taking into account the relation $(A_{\psi}^u)^* = A_{\bar{\psi}}^u = A_{\psi_- + \bar{\psi}_+}^u$, a similar computation yields $S_u (A_{\psi}^u)^* u_1 = \check{\psi}_+$. Therefore

$$\Delta(A_{\phi}^u A_{\psi}^u) = A_{\check{\phi}}^u \psi_+ \otimes \mathbf{1} + \phi_+ \otimes \psi_- + \mathbf{1} \otimes S_u (A_{\psi}^u)^* S_u^* \phi_- - \check{\phi}_- \otimes \check{\psi}_+. \tag{9}$$

Gathering (8) and (9) ends the proof of the lemma. \square

From here follows the basic theorem, which corresponds to [12, Theorem 3.1].

THEOREM 5.2. *Suppose $u(0) = 0$ and $A_{\phi}^u, A_{\psi}^u, A_{\zeta}^u, A_{\eta}^u \in \mathcal{T}_u$. Then $A_{\phi}^u A_{\psi}^u - A_{\zeta}^u A_{\eta}^u \in \mathcal{T}_u$ if and only if*

$$\phi_+ \otimes \psi_- - \check{\phi}_- \otimes \check{\psi}_+ = \zeta_+ \otimes \eta_- - \check{\zeta}_- \otimes \check{\eta}_+. \tag{10}$$

Proof. By Lemma 5.1, there exist $f, g \in K_u^2$ such that

$$\Delta(A_{\phi}^u A_{\psi}^u - A_{\zeta}^u A_{\eta}^u) = \phi_+ \otimes \psi_- - \check{\phi}_- \otimes \check{\psi}_+ - \zeta_+ \otimes \eta_- + \check{\zeta}_- \otimes \check{\eta}_+ + f \otimes \mathbf{1} + \mathbf{1} \otimes g.$$

From Lemma 2.1 it follows that $A_{\phi}^u A_{\psi}^u - A_{\zeta}^u A_{\eta}^u \in \mathcal{T}_u$ if and only if there exist $f_1, g_1 \in K_u^2$ such that

$$\phi_+ \otimes \psi_- - \check{\phi}_- \otimes \check{\psi}_+ - \zeta_+ \otimes \eta_- + \check{\zeta}_- \otimes \check{\eta}_+ = f_1 \otimes \mathbf{1} + \mathbf{1} \otimes g_1. \tag{11}$$

Now, if we consider the orthogonal decomposition (2), we can write operators on K_u^2 as 2×2 block matrices. With respect to this decomposition, the left hand side of (11) has zeros on the first row and column, since $\phi_{\pm}, \psi_{\pm}, \zeta_{\pm}, \eta_{\pm} \in zK_{u_1}^2$. Meanwhile, the right hand side is the general form of an operator that has zeros in the lower right corner. It follows that both sides have to be zero, so, in particular, (10) is true. \square

6. Main results

As noticed above, the Sedlock classes have been introduced in connection with multiplication properties of TTOs, and the main result in this direction is Theorem 3.1 (v). As a consequence, a characterization of unitary TTOs is obtained in [15]. In the sequel we use Theorem 5.2 in order to improve that result (see Theorem 6.3 below), as well as to obtain complete descriptions of other classes of TTOs.

The first result discusses commuting TTOs.

THEOREM 6.1. *Let u be an inner function. If $A_{\phi}^u, A_{\psi}^u \in \mathcal{T}_u$, then the following are equivalent:*

- (i) $A_{\phi}^u A_{\psi}^u = A_{\psi}^u A_{\phi}^u$.
- (ii) $A_{\phi}^u A_{\psi}^u - A_{\psi}^u A_{\phi}^u \in \mathcal{T}_u$.
- (iii) *One of the following is true:*

- (1) There exists $\alpha \in \hat{\mathbb{C}}$ such that A_ϕ^u and A_ψ^u both belong to \mathcal{B}_u^α .
- (2) The operators I, A_ϕ^u, A_ψ^u are not linearly independent.

Proof. It is obvious that (i) \Rightarrow (ii). For (iii) \Rightarrow (i), in case (1) commutativity follows from Sedlock’s result, while in case (2) one of the TTOs is a linear combination of the identity and the other. So we are left to prove that (ii) \Rightarrow (iii).

Both conditions (ii) and (iii) are invariant if we apply a Crofoot transform: since the transform is unitary, this is obvious for (iii)(2). For (ii) it follows from (1), while for (iii)(1) it is a consequence of Lemma 4.1. So we may assume for the rest of the proof that $u(0) = 0$, and thus apply the results from Section 5.

Assume then that $A_\phi^u A_\psi^u - A_\psi^u A_\phi^u \in \mathcal{T}_u$. Applying Theorem 5.2 with $\eta = \phi$ and $\zeta = \psi$, formula (10) becomes

$$\phi_+ \otimes \psi_- - \check{\phi}_- \otimes \check{\psi}_+ = \psi_+ \otimes \phi_- - \check{\psi}_- \otimes \check{\phi}_+. \tag{12}$$

The operators on the two sides of this equality have rank at most two. If the rank is at most one, then $\{\phi_+, \check{\phi}_-\}$ and $\{\psi_+, \check{\psi}_-\}$ are both pairs of linearly dependent functions. Suppose, for instance, that $\phi_- \neq 0$ and $\check{\phi}_- = \alpha\phi_+$. Then (12) yields

$$\phi_+ \otimes (\psi_- - \bar{\alpha}\check{\psi}_+) = (\bar{\alpha}\psi_+ - \check{\psi}_-) \otimes \check{\phi}_+.$$

The equality of the rank one operators implies the existence of $a \in \mathbb{C}$ such that

$$\psi_- - \bar{\alpha}\check{\psi}_+ = a\check{\phi}_+, \quad \bar{\alpha}\psi_+ - \check{\psi}_- = \bar{a}\phi_+.$$

Applying the reduced conjugation to the first equation and comparing the result to the second, we see that $a = 0$. Thus $\check{\psi}_- = \alpha\psi_+$, and thus A_ϕ^u and A_ψ^u both belong to \mathcal{B}_u^α ; that is, (1) is true.

Suppose now that the rank of the operators in (12) is two. The spaces spanned by $\{\phi_+, \check{\phi}_-\}$ and by $\{\psi_+, \check{\psi}_-\}$ are equal, and thus there exist $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{C}$ such that

$$\psi_+ = a_{11}\phi_+ + a_{12}\check{\phi}_-, \quad \check{\psi}_- = a_{21}\phi_+ + a_{22}\check{\phi}_-.$$

Replacing these formulas in (12) yields

$$[2a_{21}\phi_+ + (a_{22} - a_{11})\check{\phi}_-] \otimes \check{\phi}_+ + [(a_{22} - a_{11})\phi_+ - 2a_{12}\check{\phi}_-] \otimes \phi_- = 0,$$

and then the linear independence of ϕ_+ and $\check{\phi}_-$ implies that $a_{12} = a_{21} = 0$ and $a_{11} = a_{22} = a$. Thus $\phi_- = a\phi_+$, $\check{\psi}_- = a\check{\phi}_-$, $\psi_- = \bar{a}\phi_-$, and

$$A_\psi^u = \psi_0 I + A_{\psi_+ + \bar{\psi}_-}^u = \psi_0 I + aA_{\phi_+ + \bar{\phi}_-}^u = aA_\phi^u + (\psi_0 - \phi_0)I.$$

Therefore in this case (2) is satisfied. This ends the proof of the theorem. \square

One can obtain as a consequence the characterization of normal TTOs.

THEOREM 6.2. *Let u be an inner function. If $A_\phi^u \in \mathcal{T}_u$, then the following are equivalent:*

- (i) $A_\phi^u \in \mathcal{T}_u$ is normal.
- (ii) $A_\phi^u (A_\phi^u)^* - (A_\phi^u)^* A_\phi^u \in \mathcal{T}_u$.
- (iii) One of the following is true:

- (1) There exists $\alpha \in \mathbb{T}$ such that A_ϕ^u belongs to \mathcal{B}_u^α .
- (2) A_ϕ^u is a linear combination of a selfadjoint TTO and the identity.

Proof. By applying Theorem 6.1 to the case $\psi = \bar{\phi}$, we obtain the equivalence of (i), (ii), and (iii'), where (iii') states that one of the following is true:

- (1') There exists $\alpha \in \mathbb{C}$ such that A_ϕ^u and $(A_\phi^u)^*$ both belong to \mathcal{B}_u^α .
- (2') The operators $I, A_\phi^u, (A_\phi^u)^*$ are not linearly independent.

If A_ϕ^u is a multiple of the identity, then (1), (2), (1'), (2') are all satisfied. Suppose this is not the case. If $A_\phi^u \in \mathcal{B}_u^\alpha$, then $(A_\phi^u)^* \in \mathcal{B}_u^{\bar{\alpha}^{-1}}$. If (1') is true, then we must have $\bar{\alpha}^{-1} = \alpha$, or $|\alpha| = 1$; thus (1) is equivalent to (1').

If (2) is true, then $A_\phi^u = aA + bI$, with $A = A^*$ and $a \neq 0$; then $(A_\phi^u)^* = \frac{\bar{a}}{a}A_\phi^u + \frac{a\bar{b} - \bar{a}b}{a}I$, and thus (2') is true. Conversely, suppose $(A_\phi^u)^* = cA_\phi^u + dI$. Since we have assumed that T_ϕ is not a scalar, at least one of $\Re A_\phi^u, \Im A_\phi^u$ is not a scalar. Say this is $\Re A_\phi^u$; then $\Re A_\phi^u = (c + 1)A_\phi^u + dI$, with $c \neq -1$, and thus $A_\phi^u = (c + 1)^{-1}(\Re A_\phi^u - dI)$; therefore (2) is true.

Thus (1) \Leftrightarrow (1') and (2) \Leftrightarrow (2'); this ends the proof of the theorem. \square

It is proved in [15] that if a TTO A_ϕ^u is unitary then it belongs to some class \mathcal{B}_u^α for some $\alpha \in \mathbb{T}$. In this case $A_\phi^u = \Phi(S_u^\alpha)$, where $|\Phi| = 1$ μ_α -almost everywhere. With our method we can obtain a slight improvement of this result.

THEOREM 6.3. *Let u be an inner function. If $A_\phi^u \in \mathcal{T}_u$, then the following are equivalent:*

- 1. A_ϕ^u is unitary.
- 2. A_ϕ^u is an isometry.
- 3. A_ϕ^u is a coisometry.
- 4. $(A_\phi^u)^* A_\phi^u - I \in \mathcal{T}_u$.
- 5. $A_\phi^u (A_\phi^u)^* - I \in \mathcal{T}_u$.
- 6. $A_\phi^u \in \mathcal{B}_u^\alpha$ for some $\alpha \in \mathbb{T}$, and $A_\phi^u = \Phi(S_u^\alpha)$, where $|\Phi| = 1$ μ_α -almost everywhere.

Proof. The implications (i) \Rightarrow (ii), (i) \Rightarrow (iii), (ii) \Rightarrow (iv), (iii) \Rightarrow (v), and (vi) \Rightarrow (i) are all immediate.

To prove (v) \Rightarrow (vi), we may assume, as in the proof of Theorem 6.1, that $u(0) = 0$. We may then apply Theorem 5.2 to the case $\psi = \bar{\phi}$, $\zeta = \eta = \mathbf{1}$, which implies $\zeta_{\pm} = \eta_{\pm} = 0$. We obtain then

$$\phi_+ \otimes \phi_+ = \check{\phi}_- \otimes \check{\phi}_-.$$

Therefore there exists $\alpha \in \mathbb{T}$ such that $\check{\phi}_- = \alpha\phi_+$; that is, $A_{\phi}^u \in \mathcal{B}_u^{\alpha}$. The particular form of A_{ϕ}^u is then a consequence of the description of \mathcal{B}_u^{α} in Section 3.

Finally, if (iv) is true, then (v) is true for $(A_{\phi}^u)^* = A_{\bar{\phi}}^u$. Therefore the previous paragraph yields $A_{\bar{\phi}}^u \in \mathcal{B}_u^{\alpha}$ for some $\alpha \in \mathbb{T}$, whence $A_{\phi}^u \in \mathcal{B}_u^{\alpha}$. Thus (iv) \Rightarrow (vi), which ends the proof of the theorem. \square

In particular, there do not exist nonunitary isometries or coisometries in \mathcal{T}_u . This can also be obtained as a consequence of the complex symmetry of the truncated Toeplitz operators with respect to the conjugation on K_u^2 (see [8]).

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