

STRONG CONTINUITY OF THE LIDSTONE EIGENVALUES OF THE BEAM EQUATION IN POTENTIALS

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Abstract. In this paper we study the dependence of the Lidstone eigenvalues $\lambda_m(q)$, $m \in \mathbb{N}$, of the fourth-order beam equation on potentials $q \in L^p[0, 1]$, $1 \leq p \leq \infty$. The first result is that $\lambda_m(q)$ have a strongly continuous dependence on potentials, i.e., as nonlinear functionals, $\lambda_m(q)$ are continuous in $q \in L^p[0, 1]$ when the weak topology is considered. The second result is that $\lambda_m(q)$ are continuously Fréchet differentiable in potentials $q \in L^p[0, 1]$ when the L^p norm is considered. These results will be used in studying the optimal estimations for these eigenvalues in later works.

1. Introduction

Recently, we have undertaken a systematic study in papers [8, 9, 15, 18] on the dependence of solutions and eigenvalues of the second-order Sturm-Liouville operators on potentials. It has been shown that eigenvalues of Sturm-Liouville operators have the strongly continuous dependence on potentials, i.e., as nonlinear functionals of potentials, eigenvalues are continuous in potentials even when the weak topologies are considered for potentials. These strong continuity results have been applied to solve several interesting extremal problems and optimal estimations for the corresponding eigenvalues in papers [14, 19]. See also the survey article [16]. This has given another approach to solve extremal problems on eigenvalues, which is different from that in [6, 7].

In this paper, we will study the dependence of eigenvalues of the following fourth-order beam equation on potentials q

$$y^{(4)}(x) + q(x)y(x) = \lambda y(x), \quad x \in [0, 1]. \quad (1.1)$$

Here $q \in \mathcal{L}^p := L^p([0, 1], \mathbb{R})$, endowed with the L^p norm $\|\cdot\|_p$, where $1 \leq p \leq \infty$. With the Lidstone boundary condition

$$y(0) = y(1) = 0 = y''(0) = y''(1), \quad (1.2)$$

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it is well-known that problem (1.1)–(1.2) has a sequence of (real) eigenvalues

$$\lambda_1(q) < \lambda_2(q) < \dots < \lambda_m(q) < \dots$$

such that $\lim_{m \rightarrow \infty} \lambda_m(q) = +\infty$, see [5]. Notice that solutions $y(x)$ of (1.1) are in the Sobolev space $W^{4,1}([0, 1], \mathbb{R})$. For example, one has for constant potentials

$$\lambda_m(c) = (m\pi)^4 + c \quad \forall m \in \mathbb{N}, \quad c \in \mathbb{R}. \tag{1.3}$$

It is a basic result that $\lambda_m(q)$ are continuous in potentials q when the L^p norm $\|\cdot\|_p$ for $q \in \mathcal{L}^p$ is considered. For the Lebesgue spaces \mathcal{L}^p , besides the norm topologies $\|\cdot\|_p$, one has the weak topologies w_p which are defined as follows. We say that $q_n \rightarrow q$ in (\mathcal{L}^p, w_p) , if

$$\int_0^1 q_n v \, dx \rightarrow \int_0^1 q v \, dx \quad \forall v \in \mathcal{L}^{p^*},$$

where $p^* = p/(p - 1)$ is the conjugate exponent of p . A functional $f : \mathcal{L}^p \rightarrow \mathbb{R}$ is said to be strongly continuous if $f : (\mathcal{L}^p, w_p) \rightarrow \mathbb{R}$ is continuous. Evidently, strong continuity of f implies that $f : (\mathcal{L}^p, \|\cdot\|_p) \rightarrow \mathbb{R}$ is continuous.

One of the main results of this paper is the following strong continuity of $\lambda_m(q)$ in q .

THEOREM 1.1. *For each $m \in \mathbb{N}$, as a nonlinear functional, $\lambda_m(q)$ is strongly continuous in $q \in \mathcal{L}^p$, where $1 \leq p \leq \infty$.*

Another result of this paper is the following continuous Fréchet differentiability of $\lambda_m(q)$ in $q \in \mathcal{L}^p$ with the L^p norm $\|\cdot\|_p$.

THEOREM 1.2. *As a nonlinear functional of $q \in (\mathcal{L}^p, \|\cdot\|_p)$, eigenvalue $\lambda_m(q)$ is continuously Fréchet differentiable. Moreover, the Fréchet derivative is given by*

$$\partial_q \lambda_m(q) = E_m^2(\cdot, q), \tag{1.4}$$

where $E_m(\cdot, q)$ is an eigenfunction associated with $\lambda_m(q)$ satisfying the normalization condition:

$$\|E_m(\cdot, q)\|_2 = \left(\int_0^1 E_m^2(x, q) \, dx \right)^{\frac{1}{2}} = 1.$$

Here (1.4) is understood as the following bounded linear functional ℓ of $(\mathcal{L}^p, \|\cdot\|_p)$

$$\ell(h) = \int_0^1 E_m^2(x, q) h(x) \, dx \quad \forall h \in \mathcal{L}^p. \tag{1.5}$$

The continuity and differentiability results of this paper are the basis to study eigenvalues in a quantitative way. As did in [14, 15, 16, 19] for the second-order systems, we will undertake quantitative analysis for eigenvalues of the fourth-order beam equation in future works.

The paper is organized as follows. In Section 2, after recalling some basic facts on fundamental solutions and eigenvalues of linear systems, we will prove that the first eigenvalue $\lambda_1(q)$ of (1.1)–(1.2) is strongly continuous in $q \in \mathcal{L}^p$. In Section 3, we will use the induction principle to give the proof of Theorem 1.1. In these proofs, different from the approaches used in [8, 11, 15, 18] for the second-order equations, we will extensively exploit the minimax characterization for eigenvalues $\lambda_m(q)$. Such a technique is also used in [4, 12] to obtain some strong continuity of weighted eigenvalues and the first non-trivial Fučík curve of the Laplacian in weights. Finally, Theorem 1.2 will be proved at the end of Section 3.

We end the introduction with the following remark. We can argue in a similar way to prove that the strong continuity and Fréchet differentiability of eigenvalues in potentials obtained above also hold for other self-adjoint boundary conditions, such as

$$y(0) = y(1) = 0 = y'(0) = y'(1),$$

or

$$y(0) = y(1) = 0 = y''(0) = y''(1).$$

2. Preliminary results

Given $q \in \mathcal{L}^p$, where $1 \leq p \leq \infty$, and $\lambda \in \mathbb{R}$. We consider equation (1.1). Let $\varphi_i(x, \lambda, q)$ be the fundamental solution of Eq. (1.1) satisfying

$$(y(0), y'(0), y''(0), y'''(0))^T = e_i,$$

where $1 \leq i \leq 4$. Results in [8, 18] show that solutions of (1.1) have strongly continuous dependence on potentials q .

LEMMA 2.1. *As nonlinear operators, the following solution mappings*

$$\mathbb{R} \times (\mathcal{L}^p, w_p) \rightarrow (C^3, \|\cdot\|_{C^3}), \quad (\lambda, q) \rightarrow \varphi_i(\cdot, \lambda, q), \quad (2.1)$$

are continuous, where $1 \leq i \leq 4$. Here $C^3 = C^3([0, 1], \mathbb{R})$.

As for the first eigenvalue $\lambda_1(q)$, one has the following minimization characterization.

LEMMA 2.2. ([1]) *There holds*

$$\lambda_1(q) = \min_{\substack{u \in C_0^2 \\ u \neq 0}} \frac{\int_0^1 ((u'')^2 + qu^2) \, dx}{\int_0^1 u^2 \, dx}, \quad (2.2)$$

where

$$C_0^2 := \{u \in C^2([0, 1], \mathbb{R}) : u(0) = u(1) = u''(0) = u''(1) = 0\}.$$

The following lemma shows that $\lambda_1(q)$ can be estimated using $\|q\|_1$ from below and from above.

LEMMA 2.3. *As a functional, $\lambda_1(q)$ is bounded for q in any bounded subset of $(\mathcal{L}^1, \|\cdot\|_1)$.*

Proof. Let us take u in (2.2) as $\psi(x) := \sqrt{2}\sin(\pi x) \in C_0^2$. Then $\|\psi\|_2 = 1$ and

$$\lambda_1(q) \leq \int_0^1 ((\psi'')^2 + q\psi^2) \, dx \leq \|\psi''\|_2^2 + \|\psi\|_\infty^2 \|q\|_1 = \pi^4 + 2\|q\|_1. \tag{2.3}$$

On the other hand, for any $u \in C_0^2$ with $u \neq 0$, one has

$$\begin{aligned} \|u\|_\infty^2 &\leq \left(\int_0^1 |u'| \, dx \right)^2 \leq \int_0^1 u' u' \, dx \\ &= uu'|_0^1 - \int_0^1 uu'' \, dx \leq \int_0^1 |uu''| \, dx \\ &\leq \|u\|_2 \|u''\|_2, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \int_0^1 qu^2 \, dx &\geq -\|q\|_1 \|u\|_\infty^2 = -\frac{\|q\|_1 \|u\|_2}{\sqrt{2}} \cdot \frac{\sqrt{2} \|u\|_\infty^2}{\|u\|_2} \\ &\geq -\frac{1}{2} \left(\frac{\|q\|_1^2 \|u\|_2^2}{2} + \frac{2\|u\|_\infty^4}{\|u\|_2^2} \right). \end{aligned} \tag{2.5}$$

Let us write (2.4) as

$$\|u''\|_2 \geq \frac{\|u\|_\infty^2}{\|u\|_2}. \tag{2.6}$$

By (2.5) and (2.6), one has

$$\int_0^1 ((u'')^2 + qu^2) \, dx = \|u''\|_2^2 + \int_0^1 qu^2 \, dx \geq -\frac{\|q\|_1^2 \|u\|_2^2}{4}.$$

Thus (2.2) yields

$$\lambda_1(q) \geq -\|q\|_1^2/4. \tag{2.7}$$

Now (2.3) and (2.7) have proved the lemma. \square

For $q \in \mathcal{L}^1$, let $E_1(x, q)$ be an eigenfunction associated with $\lambda_1(q)$ satisfying the normalization condition: $\|E_1(\cdot, q)\|_2 = 1$. For definiteness, we always take the normalized eigenfunction $E_1(x, q)$ of $\lambda_1(q)$ so that $E_1'(0, q) > 0$. Then $E_1(x, q)$ is uniquely determined.

LEMMA 2.4. *Let $q_n \rightarrow q_0$ in (\mathcal{L}^1, w_1) . Then, up to a subsequence, one has some $\tilde{E} \in C^3$ such that*

$$E_1(\cdot, q_n) \rightarrow \tilde{E} \quad \text{in } (C^3, \|\cdot\|_{C^3}). \tag{2.8}$$

Proof. Since $q_n \rightarrow q_0$ in (\mathcal{L}^1, w_1) , $\|q_n\|_1$ is bounded. By Lemma 2.3, up to a subsequence, one can assume that $\lambda_1(q_n) \rightarrow \tilde{\lambda}$ for some $\tilde{\lambda} \in \mathbb{R}$.

Due to the Lidstone boundary condition (1.2), one has $a_n, b_n \in \mathbb{R}$, $n \in \mathbb{N}$, such that

$$E_1(x, q_n) \equiv a_n \varphi_2(x, \lambda_1(q_n), q_n) + b_n \varphi_4(x, \lambda_1(q_n), q_n). \quad (2.9)$$

Step 1. We claim that $\{a_n\}$ is bounded.

If not, let us assume that $|a_n| \rightarrow +\infty$. Then $\{b_n\}$ would be unbounded. Otherwise, if $\{b_n\}$ is bounded, one has $b_n/a_n \rightarrow 0$. By Lemma 2.1,

$$z_n(x) := \varphi_2(x, \lambda_1(q_n), q_n) + \frac{b_n}{a_n} \varphi_4(x, \lambda_1(q_n), q_n) \rightarrow \varphi_2(x, \tilde{\lambda}, q_0) =: \tilde{z} \neq 0 \quad (2.10)$$

in $(C^3, \|\cdot\|_{C^3})$. Recall that

$$1 = \int_0^1 (E_1(x, q_n))^2 dx = a_n^2 \cdot \int_0^1 z_n^2(x) dx. \quad (2.11)$$

This is impossible because $a_n \rightarrow \infty$ and $z_n \rightarrow \tilde{z} \neq 0$. Thus both $\{a_n\}$ and $\{b_n\}$ are unbounded in the present case. Notice that (2.11) can be rewritten as

$$1 = \int_0^1 (E_1(x, q_n))^2 dx = b_n^2 \cdot \int_0^1 \hat{z}_n^2(x) dx, \quad (2.12)$$

where

$$\hat{z}_n(x) := \frac{a_n}{b_n} \varphi_2(x, \lambda_1(q_n), q_n) + \varphi_4(x, \lambda_1(q_n), q_n).$$

We distinguish two cases. The first case is that $\{a_n/b_n\}$ is bounded. Arguing as before, it follows from Lemma 2.1 that, up to a subsequence, \hat{z}_n will tend to a non-zero function of the form $c\varphi_2(x, \tilde{\lambda}, q_0) + \varphi_4(x, \tilde{\lambda}, q_0)$, where $c \in \mathbb{R}$. By (2.12), this is impossible because $\{b_n\}$ is unbounded. The second case is that $\{a_n/b_n\}$ is unbounded. Then, up to a subsequence, one has $b_n/a_n \rightarrow 0$. Thus one still has (2.10). Therefore (2.11) is impossible because $\{a_n\}$ is unbounded. These contradictions have shown that $\{a_n\}$ is necessarily bounded.

Step 2. We claim that $\{b_n\}$ is bounded.

This can be proved as in Step 1.

Step 3. From Steps 1 and 2, let us simply assume that $a_n \rightarrow a_0$ and $b_n \rightarrow b_0$. By setting

$$\tilde{E}(x) := a_0 \varphi_2(x, \tilde{\lambda}, q_0) + b_0 \varphi_4(x, \tilde{\lambda}, q_0), \quad (2.13)$$

convergence result (2.8) follows immediately from (2.1) in Lemma 2.1. \square

Based on the minimization characterization (2.2), we will prove the following strong continuity of $\lambda_1(q)$ in $q \in \mathcal{L}^1$.

LEMMA 2.5. *As a nonlinear functional, $\lambda_1(q)$ is strongly continuous in $q \in \mathcal{L}^1$. Precisely, if $q_n \rightarrow q_0$ in (\mathcal{L}^1, w_1) , one has then $\lambda_1(q_n) \rightarrow \lambda_1(q_0)$.*

Proof. Let us take the first (normalized) eigenfunctions $y_n(x) := E_1(x, q_n)$ with potentials q_n , where $n \geq 0$. From the minimization characterization for $\lambda_1(q_0)$, one has

$$\int_0^1 (y_0'')^2 dx + \int_0^1 q_0 y_0^2 dx = \lambda_1(q_0),$$

$$\int_0^1 (y_n'')^2 dx + \int_0^1 q_0 y_n^2 dx \geq \lambda_1(q_0),$$

and, from the minimization characterization for $\lambda_1(q_n)$, one has

$$\int_0^1 (y_n'')^2 dx + \int_0^1 q_n y_n^2 dx = \lambda_1(q_n),$$

$$\int_0^1 (y_0'')^2 dx + \int_0^1 q_n y_0^2 dx \geq \lambda_1(q_n).$$

Here $n \in \mathbb{N}$ is arbitrary. From these, we obtain

$$\int_0^1 (q_n - q_0) y_n^2 dx \leq \lambda_1(q_n) - \lambda_1(q_0) \leq \int_0^1 (q_n - q_0) y_0^2 dx \quad \forall n \in \mathbb{N}. \tag{2.14}$$

By the definition for $q_n \rightarrow q_0$ in (\mathcal{L}^1, w_p) , one has

$$\lim_{n \rightarrow \infty} \int_0^1 (q_n - q_0) y_0^2 dx = \lim_{n \rightarrow \infty} \left(\int_0^1 q_n y_0^2 dx - \int_0^1 q_0 y_0^2 dx \right) = 0.$$

On the other hand, by applying Lemma 2.4 to $E_1(\cdot, q_n) = y_n$, $n \in \mathbb{N}$, one has

$$\begin{aligned} \left| \int_0^1 (q_n - q_0) y_n^2 dx \right| &= \left| \int_0^1 (q_n - q_0) \tilde{E}^2 dx + \int_0^1 (q_n - q_0) (y_n^2 - \tilde{E}^2) dx \right| \\ &\leq \left| \int_0^1 q_n \tilde{E}^2 dx - \int_0^1 q_0 \tilde{E}^2 dx \right| + \|q_n - q_0\|_1 \|y_n^2 - \tilde{E}^2\|_\infty \\ &\rightarrow 0, \end{aligned}$$

because $q_n \rightarrow q_0$ in (\mathcal{L}^1, w_1) , $y_n^2 \rightarrow \tilde{E}^2$ in $(C^3, \|\cdot\|_{C^3})$ and $\|q_n - q_0\|_1$ is bounded. Now (2.14) shows that $\lim_{n \rightarrow \infty} \lambda_1(q_n) = \lambda_1(q_0)$ (for any possible convergent subsequence). The theorem is thus proved. \square

Because of Lemma 2.5, the limiting function $\tilde{E}(x)$ in result (2.8) of Lemma 2.4 is independent of the choice of subsequences and is actually $E_1(x, q_0)$. Thus Lemma 2.4 can be improved as the following strong continuity result.

COROLLARY 2.6. *The following (nonlinear) eigenfunction operator is continuous*

$$(\mathcal{L}^1, w_1) \rightarrow (C^3, \|\cdot\|_{C^3}), \quad q \rightarrow E_1(\cdot, q).$$

We remark that if $q_n \rightarrow q_0$ in (\mathcal{L}^1, w_1) , it is possible to use equations (1.1) for $E_1(\cdot, q_n)$ to show that

$$E_1^{(4)}(\cdot, q_n) \rightarrow E_1^{(4)}(\cdot, q_0) \quad \text{in } (\mathcal{L}^1, w_1).$$

3. Proofs of main results

For $m \in \mathbb{N}$, we choose some normalized eigenfunction $E_m(x, q)$ associated with the m th eigenvalue $\lambda_m(q)$ of problem (1.1)–(1.2). Denote

$$V_{m-1,q} = \text{span}\{E_1(\cdot, q), \dots, E_{m-1}(\cdot, q)\} \subset \mathcal{L}^2$$

and

$$V_{m-1,q}^\perp = \left\{ u \in \mathcal{L}^2 : \langle u, v \rangle := \int_0^1 uv \, dx = 0 \quad \forall v \in V_{m-1,q} \right\}.$$

Recall that $\{E_m(\cdot, q)\}_{m \in \mathbb{N}}$ are orthogonal

$$\langle E_i(\cdot, q), E_j(\cdot, q) \rangle = 0 \quad \forall i \neq j. \tag{3.1}$$

We have the following variational characterization of eigenvalues, which is a limiting case of the minimax principle [3].

LEMMA 3.1. *For $m \in \mathbb{N}$ with $m \geq 2$, one has the following minimization (or minimax) characterization*

$$\lambda_m(q) = \min_{\substack{u \in C_0^2 \cap V_{m-1,q}^\perp \\ u \neq 0}} \frac{\int_0^1 ((u'')^2 + qu^2) \, dx}{\int_0^1 u^2 \, dx}. \tag{3.2}$$

Now we are ready to prove the theorems stated in the introduction.

Proof of Theorem 1.1. Since (\mathcal{L}^1, w_1) is the weakest topology, it suffices to show the theorem for the case $p = 1$.

Suppose that $q_n \rightarrow q_0$ in (\mathcal{L}^1, w_1) . We claim that for $m \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \lambda_m(q_n) = \lambda_m(q_0), \tag{3.3}$$

$$\lim_{n \rightarrow \infty} E_m(\cdot, q_n) = E_m(\cdot, q_0) \quad \text{in } (C^3, \|\cdot\|_{C^3}). \tag{3.4}$$

We will prove (3.3)–(3.4) by induction on $m \in \mathbb{N}$. Notice that Lemma 2.5 and Corollary 2.6 state that (3.3)–(3.4) hold for $m = 1$. Inductively, let us assume that (3.3)–(3.4) hold for all $1 \leq m \leq k - 1$. In this case, (3.4) can be rewritten as

$$\lim_{n \rightarrow \infty} V_{k-1,q_n} = V_{k-1,q_0} \quad \text{in } (C^3, \|\cdot\|_{C^3}).$$

From this, it is easy to verify that

$$f_n \in V_{k-1,q_n}^\perp \quad \text{and} \quad f_n \rightarrow f \quad \text{in } (C^3, \|\cdot\|_{C^3}) \implies f \in V_{k-1,q_0}^\perp. \tag{3.5}$$

For simplicity, let us write $y_n(x) = E_k(x, q_n) \in V_{k-1,q_n}^\perp$ for $n \in \mathbb{Z}^+$. By the same arguments as in Lemma 2.4, there exists ψ such that, up to a subsequence,

$$y_n \rightarrow \psi \quad \text{in } (C^3, \|\cdot\|_{C^3}).$$

By (3.1) and (3.5), one has $\psi \in V_{k-1,q_0}^\perp$. Since $\|y_n\|_2 = 1$ for all n , one has

$$\|\psi\|_2 = 1. \tag{3.6}$$

Let us decompose

$$y_0 = u_n + v_n = u_n + \sum_{i=1}^{k-1} a_i^n E_i(\cdot, q_n),$$

where $u_n \in V_{k-1,q_n}^\perp$ and $v_n = \sum_{i=1}^{k-1} a_i^n E_i(\cdot, q_n) \in V_{k-1,q_n}$. Notice that

$$1 = \|y_0\|_2^2 = \|u_n\|_2^2 + \sum_{i=1}^{k-1} (a_i^n)^2 \|E_i(\cdot, q_n)\|_2^2 = \|u_n\|_2^2 + \sum_{i=1}^{k-1} (a_i^n)^2.$$

In particular, $|a_i^n| \leq 1$ for all $i = 1, \dots, k-1$ and $n \in \mathbb{Z}^+$. Thus, up to a subsequence,

$$v_n \rightarrow v_0 \in V_{k-1,q_0} \text{ in } (C^3, \|\cdot\|_{C^3})$$

and then

$$u_n \rightarrow u_0 \in V_{k-1,q_0}^\perp \text{ in } (C^3, \|\cdot\|_{C^3}).$$

Since $y_0 = u_0 + v_0$, we have

$$u_0 = y_0 \quad \text{and} \quad v_0 = 0.$$

Then

$$u_n \rightarrow y_0 \text{ in } (C^3, \|\cdot\|_{C^3}) \quad \text{and} \quad v_n \rightarrow 0 \text{ in } (C^3, \|\cdot\|_{C^3}). \tag{3.7}$$

Let us decompose

$$y_n = w_n + z_n, \quad \text{where } w_n \in V_{k-1,q_0}^\perp \text{ and } z_n \in V_{k-1,q_0}.$$

Similarly, one has

$$w_n \rightarrow \psi \text{ in } (C^3, \|\cdot\|_{C^3}) \quad \text{and} \quad z_n \rightarrow 0 \text{ in } (C^3, \|\cdot\|_{C^3}). \tag{3.8}$$

By Lemma 3.1, one has, for all $n \in \mathbb{N}$,

$$\int_0^1 ((u_n + v_n)'')^2 dx + \int_0^1 q_0(u_n + v_n)^2 dx = \lambda_k(q_0), \tag{3.9}$$

$$\int_0^1 (w_n'')^2 dx + \int_0^1 q_0 w_n^2 dx \geq \lambda_k(q_0) \int_0^1 w_n^2 dx, \tag{3.10}$$

and

$$\int_0^1 ((w_n + z_n)'')^2 dx + \int_0^1 q_n(w_n + z_n)^2 dx = \lambda_k(q_n), \tag{3.11}$$

$$\int_0^1 (u_n'')^2 dx + \int_0^1 q_n u_n^2 dx \geq \lambda_k(q_n) \int_0^1 u_n^2 dx. \tag{3.12}$$

From (3.10) and (3.11), we have

$$\begin{aligned} & \lambda_k(q_n) - \lambda_k(q_0) \int_0^1 w_n^2 dx \\ & \geq \int_0^1 ((w_n + z_n)'')^2 dx + \int_0^1 q_n(w_n + z_n)^2 dx - \left(\int_0^1 (w_n'')^2 dx + \int_0^1 q_0 w_n^2 dx \right) \\ & = \int_0^1 2w_n'' z_n'' dx + \int_0^1 (z_n'')^2 dx + \int_0^1 (q_n - q_0) w_n^2 dx + \int_0^1 2q_n w_n z_n dx + \int_0^1 q_n z_n^2 dx. \end{aligned}$$

By (3.6) and (3.8), one has

$$\liminf_{n \rightarrow \infty} (\lambda_k(q_n) - \lambda_k(q_0)) = \liminf_{n \rightarrow \infty} \left(\lambda_k(q_n) - \lambda_k(q_0) \int_0^1 w_n^2 dx \right) \geq 0.$$

Similarly, from (3.9) and (3.12), we have

$$\begin{aligned} & \lambda_k(q_n) \int_0^1 u_n^2 dx - \lambda_k(q_0) \\ & \leq \int_0^1 (u_n'')^2 dx + \int_0^1 q_n u_n^2 dx - \left(\int_0^1 ((u_n + v_n)'')^2 dx + \int_0^1 q_0 (u_n + v_n)^2 dx \right) \\ & = - \int_0^1 2u_n'' v_n'' dx - \int_0^1 (v_n'')^2 dx + \int_0^1 (q_n - q_0) u_n^2 dx - \int_0^1 2q_0 u_n v_n dx - \int_0^1 q_0 v_n^2 dx. \end{aligned}$$

Because of (3.7), one has $\|u_n\|_2 \rightarrow \|y_0\|_2 = 1$. Thus

$$\limsup_{n \rightarrow \infty} (\lambda_k(q_n) - \lambda_k(q_0)) = \limsup_{n \rightarrow \infty} \left(\lambda_k(q_n) \int_0^1 u_n^2 dx - \lambda_k(q_0) \right) \leq 0.$$

These have proved (3.3) for $m = k$.

Furthermore, result (3.4) can be proved by the same arguments as in Corollary 2.6. \square

Proof of Theorem 1.2. The continuous Fréchet differentiability $\lambda_m(q)$ in $q \in (\mathcal{L}^p, \|\cdot\|_p)$ is a conventional result [10, 17]. In the following we will compute the Fréchet derivatives. Notice that for any $q, h \in \mathcal{L}^p$ and $\tau \in \mathbb{R}$, $E_m(x, q + \tau h)$ satisfies

$$E_m^{(4)}(x, q + \tau h) + (q(x) + \tau h(x))E_m(x, q + \tau h) = \lambda_m(q + \tau h)E_m(x, q + \tau h) \quad (3.13)$$

for $x \in [0, 1]$ and boundary condition (1.2). To find the Fréchet derivative $\ell := \partial_q \lambda_m(q) \cdot h \in \mathbb{R}$, let us expand $E_m(x, q + \tau h)$ and $\lambda_m(q + \tau h)$ as

$$E_m(x, q + \tau h) = E_m(x, q) + \tau z(x) + o(\tau) \quad \text{and} \quad \lambda_m(q + \tau h) = \lambda_m(q) + \tau \ell + o(\tau)$$

when $\tau \rightarrow 0$. Expanding (3.13), we know that $z(x)$ satisfies the inhomogeneous beam equation

$$z^{(4)}(x) + (q(x) - \lambda_m(q))z(x) = (\ell - h(x))E_m(x, q), \quad x \in [0, 1], \quad (3.14)$$

and the boundary condition

$$z(0) = z(1) = z''(0) = z''(1) = 0. \tag{3.15}$$

In order that Eq. (3.14) has solutions $z(x)$ satisfying (3.15), by the Fredholm principle, it is necessary that the inhomogeneous term of (3.14) is orthogonal to the eigenfunction $E_m(\cdot, q)$, i.e.

$$\int_0^1 (\ell - h(x))E_m^2(x, q) \, dx = 0.$$

As $E_m(\cdot, q)$ is normalized, we know that

$$\ell = \int_0^1 E_m^2(x, q)h(x) \, dx.$$

This gives (1.4) and (1.5). \square

In conclusion, we have established for the fourth-order beam equation (1.1) the continuity of eigenvalues in weak topologies of potentials and the continuous differentiability of eigenvalues in the norms of potentials.

Like the corresponding results for eigenvalues of Sturm-Liouville operators, these results can lead to many interesting extremal problems. For example, let $1 < p < \infty$ and $r > 0$. Theorem 1.1 shows that the following extremal problems

$$\begin{aligned} \mathbf{L}_{m,p}(r) &:= \min \{ \lambda_m(q) : q \in \mathcal{L}^p, \|q\|_p \leq r \}, \\ \mathbf{M}_{m,p}(r) &:= \max \{ \lambda_m(q) : q \in \mathcal{L}^p, \|q\|_p \leq r \}, \end{aligned}$$

can be attained by some potentials, because balls in spaces \mathcal{L}^p are compact in weak topologies w_p . Moreover, the continuous differentiability of eigenvalues in Theorem 1.2 shows that these problems can be determined using the Lagrangian multiplier method. Since Eq. (1.1) is a linear Hamiltonian systems of two-degree-freedom [13], the corresponding critical equation is some nonlinear Hamiltonian system of two-degree-freedom. A complete analysis for these extremal problems is much complicated and will be given in future works.

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