

## COMPLETELY POSITIVE LINEAR MAPS ON MAXIMAL AND MINIMAL OPERATOR SYSTEM STRUCTURES

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(Communicated by B. Magajna)

*Abstract.* An  $s$ -entanglement breaking map between operator systems is a point-norm limit of entanglement breaking maps, which are a generalization of the corresponding notion in matrix algebras. We develop some of the key properties of this map, and obtain some conditions when the completely positive linear maps between operator systems coincide with the  $s$ -entanglement breaking maps. Especially we show that a linear map from a nuclear operator system to the maximal operator system structure of an Archimedean ordered  $*$ -vector space is completely positive if and only if it is  $s$ -entanglement breaking. We also discuss the relationships between  $s$ -entanglement breaking maps and weak  $*$ -entanglement breaking maps, and nuclear maps between operator systems.

### 1. Introduction

In [4], Kadison proved that every function system, i.e., an ordered real vector space with an Archimedean order unit, can be represented as a vector subspace of the space of continuous real-valued functions on a compact Hausdorff space via an order-preserving map which carries the order unit to the constant function 1. This result motivated Choi and Effros to obtain a noncommutative analogy: any operator system, i.e., a matrix ordered  $*$ -vector space with an Archimedean matrix order unit, is completely order isomorphic to a selfadjoint subspace of a unital  $C^*$ -algebra that contains the unit [1]. Choi and Effros's characterization is a very useful tool in a number of areas: operator spaces and completely bounded maps [7], metric aspect of noncommutative spaces [11, 12], quantum information theory [10], etc..

To create a theory for operator systems themselves, Paulsen, Todorov and Tomforde introduced the operator system version of the MIN and MAX functors from the category of normed spaces into the category of operator spaces. Some key properties were developed [9]. As an application, they investigated the relation of completely positive linear maps to the entanglement breaking maps between matrix algebras. In particular, they proved that a linear map  $\phi : M_n \rightarrow M_m$  is entanglement breaking if and only if there exist positive linear functionals  $s_l : M_n \rightarrow \mathbb{C}$  and matrices  $P_l \in M_m^+$ , where  $l = 1, \dots, q$ , such that  $\phi(X) = \sum_{l=1}^q s_l(X)P_l$ . There they raised the question:

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*Mathematics subject classification* (2010): Primary 46L07; Secondary 46B40.

*Keywords and phrases:* completely positive linear map,  $s$ -entanglement breaking map, maximal operator system structure, minimal operator system structure.

The research was supported by National Natural Science Foundation of China (Grant No. 11171109).

Given an operator system  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, e)$  and an Archimedean ordered  $*$ -vector space  $(W, W^+, f)$ , is a completely positive linear map from  $\mathcal{S}$  to  $OMAX(W)$  a ‘limit’ of sums of maps of the form  $\phi(a) = s(a)P$ , where  $s$  is a positive linear functional on  $\mathcal{S}$  and  $P \in W^+$ ?

In this paper we discuss the problem above in the point-order topology on operator systems. We begin by investigating the order topology on operator systems and introducing entanglement breaking maps and  $s$ -entanglement breaking maps between operator systems. We obtain some key properties of  $s$ -entanglement breaking maps between operator systems which are the generalizations of the matrix algebra case. In Section 4 we discuss the relation of  $s$ -entanglement breaking maps to completely positive linear maps between different operator system structures. We give a nuclear characterization of  $s$ -entanglement breaking maps between operator systems in Section 5. The next section is devoted to some notions on operator system structures.

### 2. Operator system structures

In this section we recall basic definitions. Let  $W$  be a real vector space. A cone in  $W$  is a nonempty subset  $P \subseteq W$  with the properties:

1.  $\lambda p \in P$  for any  $\lambda \geq 0$  and  $p \in P$ ;
2.  $p + q \in P$  for any  $p, q \in P$ .

An *ordered real vector space*  $(V, V^+)$  is a pair consisting of a real vector space  $V$  and a cone  $V^+ \subseteq V$  satisfying  $V^+ \cap -V^+ = \{0\}$ . In an ordered real vector space  $(V, V^+)$ , an element  $e \in V$  is called an *order unit* for  $V$  if for each  $v \in V$  there exists a real number  $r > 0$  such that  $re - v \in V^+$ .

A  *$*$ -vector space* is a complex vector space  $V$  with a conjugate linear involutive map  $*$ :  $V \mapsto V$ . In a  $*$ -vector space  $V$ , the *hermitian elements* of  $V$  are the elements in the set  $V_h = \{v \in V : v^* = v\}$ . An *ordered  $*$ -vector space*  $(V, V^+)$  is a  $*$ -vector space  $V$  together with a subset  $V^+ \subseteq V_h$  satisfying the following properties

1.  $V^+$  is a cone in  $V_h$ ;
2.  $V^+ \cap -V^+ = \{0\}$ .

When we have an ordered  $*$ -vector space  $(V, V^+)$ , a partial ordering  $\geq$  is defined on  $V_h$ :  $v \geq w$  if and only if  $v - w \in V^+$ .

A hermitian element  $e$  in an ordered  $*$ -vector space  $(V, V^+)$  is called an *order unit* for  $V$  if, for any  $v \in V_h$ , there exists a real number  $r > 0$  such that  $re \geq v$ . And an order unit  $e$  for an ordered  $*$ -vector space  $(V, V^+)$  is called *Archimedean* if whenever  $v \in V$  and  $re + v \geq 0$  for all real number  $r > 0$ , we have that  $v \in V^+$ . An *Archimedean ordered  $*$ -vector space* is a triple  $(V, V^+, e)$ , where  $(V, V^+)$  is an ordered  $*$ -vector space and  $e$  is an Archimedean order unit for  $V$ . A linear functional  $f$  on an Archimedean ordered  $*$ -vector space  $(V, V^+, e)$  is *positive* if  $f(V^+) \subseteq [0, +\infty)$  and a *state* if it is positive and  $f(e) = 1$ . We let  $S(V)$  denote the set of all states on  $V$  and call it the state space of  $V$ .

Let  $V$  be a  $*$ -vector space. For  $m, n \in \mathbb{N}$ , we let  $M_{m,n}(V)$  denote the set of all  $m \times n$  matrices with entries in  $V$ . We write  $M_n(V) := M_{n,n}(V)$ ,  $M_{m,n} := M_{m,n}(\mathbb{C})$  and  $M_n := M_{n,n}$ . With the natural addition, scalar multiplication and  $*$ -operation,  $M_n(V)$  becomes a  $*$ -vector space. The multiplication by scalar matrices on the left and right can be defined in a way similar to the multiplication of two scalar matrices. We let  $0_n$  be the  $n \times n$  zero matrix.

A *matrix ordering* on a  $*$ -vector space  $V$  is a sequence of sets  $\{C_n\}_{n=1}^\infty$  satisfying

1. each  $C_n$  is a cone in  $M_n(V)_h$ ;
2.  $C_n \cap -C_n = \{0_n\}$  for all  $n \in \mathbb{N}$ ;
3.  $X^*C_nX \subseteq C_m$  for all  $m, n \in \mathbb{N}$  and any  $X \in M_{n,m}$ .

And a  $*$ -vector space with a matrix ordering is called a *matrix ordered  $*$ -vector space*. In a matrix ordered  $*$ -vector space  $(V, \{C_n\}_{n=1}^\infty)$ , for  $v \in V$  we denote

$$v_n = \begin{pmatrix} v & & \\ & \ddots & \\ & & v \end{pmatrix} \in M_n(V).$$

An element  $e \in V_h$  is called a *matrix order unit* for  $V$  if  $e_n$  is an order unit for  $(M_n(V), C_n)$  for each  $n$ . An element  $e \in V_h$  is called an *Archimedean matrix order unit* for  $V$  if for each  $n$ ,  $e_n$  is an Archimedean order unit for  $(M_n(V), C_n)$ . An *(abstract) operator system* is a triple  $(V, \{C_n\}_{n=1}^\infty, e)$ , where  $(V, \{C_n\}_{n=1}^\infty)$  is a matrix ordered  $*$ -vector space, and  $e \in V_h$  is an Archimedean matrix order unit for  $V$ .

Given an Archimedean ordered  $*$ -vector space  $(V, V^+, e)$ . A *matrix ordering* on  $(V, V^+, e)$  is a matrix ordering  $\{C_n\}_{n=1}^\infty$  on  $V$  such that  $C_1 = V^+$ . An *operator system structure* on  $(V, V^+, e)$  is a matrix ordering  $\{C_n\}_{n=1}^\infty$  on  $(V, V^+, e)$  such that  $(V, \{C_n\}_{n=1}^\infty, e)$  is an operator system. For two matrix orderings  $\mathcal{P} = \{P_n\}_{n=1}^\infty$  and  $\mathcal{Q} = \{Q_n\}_{n=1}^\infty$  on  $(V, V^+, e)$ , we say that  $\mathcal{P}$  is *stronger* than  $\mathcal{Q}$  or  $\mathcal{Q}$  is *weaker* than  $\mathcal{P}$  if  $P_n \subseteq Q_n$  for all  $n$ . On any Archimedean ordered  $*$ -vector space  $(V, V^+, e)$  there are two operator system structures  $\mathcal{E}^{min}(V) = \{C_n^{min}(V)\}_{n=1}^\infty$ , where

$$C_n^{min}(V) = \left\{ (v_{ij}) \in M_n(V) : \sum_{i,j=1}^n \bar{\lambda}_i \lambda_j v_{ij} \in V^+ \text{ for all } \lambda_1, \dots, \lambda_n \in \mathbb{C} \right\},$$

and  $\mathcal{E}^{max}(V) = \{C_n^{max}(V)\}_{n=1}^\infty$ , where

$$C_n^{max}(V) = \{A \in M_n(V) : re_n + A \in D_n^{max}(V) \text{ for all } r > 0\}$$

and

$$D_n^{max}(V) = \left\{ \sum_{i=1}^k a_i \otimes v_i : v_i \in V^+, a_i \in M_n^+, i = 1, \dots, k, k \in \mathbb{N} \right\},$$

which represent, respectively, the weakest and strongest operator system structures on  $(V, V^+, e)$  [9]. The operator systems  $(V, \mathcal{E}^{min}(V), e)$  and  $(V, \mathcal{E}^{max}(V), e)$  will be simply denoted by  $OMIN(V)$  and  $OMAX(V)$ .

Let  $\mathcal{S}$  and  $\mathcal{T}$  be vector spaces. A linear map  $\phi : \mathcal{S} \mapsto \mathcal{T}$  induces a linear map  $\phi^{(n)} : M_n(\mathcal{S}) \mapsto M_n(\mathcal{T})$  given by

$$\phi^{(n)}((s_{i,j})_{1 \leq i,j \leq n}) = (\phi(s_{i,j}))_{1 \leq i,j \leq n}.$$

If  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, e)$  and  $(\mathcal{T}, \{D_n\}_{n=1}^\infty, f)$  are operator systems, then a linear map  $\phi : \mathcal{S} \mapsto \mathcal{T}$  is called  $n$ -positive if  $\phi^{(n)}(C_n) \subseteq D_n$  for some  $n \in \mathbb{N}$ , and *completely positive* if  $\phi^{(n)}(C_n) \subseteq D_n$  for each  $n \in \mathbb{N}$ . A 1-positive linear map will be called a *positive map*.

### 3. s-entanglement breaking maps

In this section we introduce the s-entanglement breaking maps between operator systems. Some key properties of s-entanglement breaking maps will be developed.

Let  $(V, V^+)$  be an ordered real vector space with order unit  $e$ . For  $v \in V$ , let

$$\|v\| = \inf\{r \in \mathbb{R} : re + v \geq 0 \text{ and } re - v \geq 0\}.$$

Then  $\|\cdot\|$  is a seminorm on  $V$ , the *order seminorm* on  $V$  determined by  $e$ . Moreover, when  $e$  is an Archimedean order unit,  $\|\cdot\|$  is a norm by Proposition 2.23 in [8]. In this case, it is referred to as the *order norm* and the topology induced by it is called the *order topology* on  $V$ .

Let  $(V, V^+)$  be an ordered  $*$ -vector space with order unit  $e$ , and let  $\|\cdot\|$  be the order seminorm on  $V_h$ . A seminorm  $p(\cdot)$  on  $V$  is called a  $*$ -seminorm if  $p(v^*) = p(v)$  for all  $v \in V$ . An *order seminorm* on  $V$  is a  $*$ -seminorm  $\|\cdot\|$  on  $V$  with the property that  $\|v\| = \|v\|$  for all  $v \in V_h$ . If a  $*$ -seminorm or an order seminorm on  $V$  is in fact a norm, it is called a  $*$ -norm or an *order norm* on  $V$ , respectively. In general, there are many order seminorms on an ordered  $*$ -vector space. Given an operator system  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, e)$ , we obtain an Archimedean ordered  $*$ -vector space  $(\mathcal{S}, C_1, e)$ . So we have an order norm on  $\mathcal{S}$  which is unique up to equivalence by Proposition 2.23, Proposition 4.9 and Proposition 4.11 in [8]. The topology induced by the norm will be called the *order topology* on  $\mathcal{S}$ .

**PROPOSITION 3.1.** *Let  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, e)$  be an operator system. Then each  $(M_n(\mathcal{S}), C_n, e_n)$  is an Archimedean ordered  $*$ -vector space. For any  $X_\lambda = (x_{ij}^\lambda)$  with  $\lambda \in \Lambda$  and  $X = (x_{ij})$  in  $M_n(\mathcal{S})$ , we have that  $\lim_\lambda X_\lambda = X$  in the order topology on  $M_n(\mathcal{S})$  if and only if  $\lim_\lambda x_{ij}^\lambda = x_{ij}$  in the order topology on  $\mathcal{S}$  for all  $i, j \in \{1, 2, \dots, n\}$ .*

*Proof.* Since for any  $n \in \mathbb{N}$ ,  $(M_n(\mathcal{S})_h, C_n, e_n)$  is an ordered real vector space with an Archimedean order unit  $e_n$ , an order norm  $\|\cdot\|_n$  on  $M_n(\mathcal{S})_h$  is determined by  $e_n$ . For  $Y = (y_{ij}) \in M_n(\mathcal{S})$ , we set

$$\|Y\|_n = \left\| \begin{pmatrix} 0_n & Y \\ Y^* & 0_n \end{pmatrix} \right\|_{2n}.$$

Then  $\|\cdot\|_n$  is a  $*$ -norm on  $M_n(\mathcal{S})$ . Moreover, it is an order norm on  $M_n(\mathcal{S})$ .

For any  $i \in \{1, 2, \dots, n\}$ , let  $E_i$  be the  $n \times 1$  matrix with  $i$ -th row is one and others are zero. Given  $Y = (y_{ij}) \in M_n(\mathcal{S})$  and any  $\varepsilon > 0$ , for any  $i, j \in \{1, 2, \dots, n\}$  we have

$$\begin{aligned} & (\|Y\|_n + \varepsilon)e_2 \pm \begin{pmatrix} 0 & y_{ij} \\ y_{ij}^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} (\|Y\|_n + \varepsilon)e & \pm E_i^* Y E_j \\ \pm E_j^* Y^* E_i & (\|Y\|_n + \varepsilon)e \end{pmatrix} \\ &= \begin{pmatrix} E_i^* (\|Y\|_n + \varepsilon)e_n E_i & \pm E_i^* Y E_j \\ \pm E_j^* Y^* E_i & E_j^* (\|Y\|_n + \varepsilon)e_n E_j \end{pmatrix} \\ &= \begin{pmatrix} E_i^* & 0 \\ 0 & E_j^* \end{pmatrix} \begin{pmatrix} (\|Y\|_n + \varepsilon)e_n & \pm Y \\ \pm Y^* & (\|Y\|_n + \varepsilon)e_n \end{pmatrix} \begin{pmatrix} E_i & 0 \\ 0 & E_j \end{pmatrix} \in C_2. \end{aligned}$$

Hence  $\|y_{ij}\|_1 \leq \|Y\|_n$ . Similarly, we can prove that  $\|E_i y_{ij} E_j^*\|_n \leq \|y_{ij}\|_1$  for any  $i, j \in \{1, 2, \dots, n\}$ . So we obtain

$$\|y_{ij}\|_1 \leq \|Y\|_n = \|\sum_{i,j=1}^n E_i y_{ij} E_j^*\|_n \leq \sum_{i,j=1}^n \|E_i y_{ij} E_j^*\|_n \leq \sum_{i,j=1}^n \|y_{ij}\|_1,$$

for any  $i, j \in \{1, 2, \dots, n\}$ . Now we have

$$\|x_{ij}^\lambda - x_{ij}\|_1 \leq \|X_\lambda - X\|_n \leq \sum_{i,j=1}^n \|x_{ij}^\lambda - x_{ij}\|_1,$$

for any  $i, j \in \{1, 2, \dots, n\}$ . This indicates that  $\lim_\lambda X_\lambda = X$  in the order topology on  $M_n(\mathcal{S})$  if and only if  $\lim_\lambda x_{ij}^\lambda = x_{ij}$  in the order topology on  $\mathcal{S}$  for all  $i, j \in \{1, 2, \dots, n\}$ .  $\square$

We recall that a positive linear functional  $f : M_n \otimes M_m \mapsto \mathbb{C}$  is called *separable* if there exist  $l \in \mathbb{N}$ , positive linear functionals  $g_i : M_n \mapsto \mathbb{C}$  and positive linear functionals  $h_i : M_m \mapsto \mathbb{C}$  for  $i = 1, 2, \dots, l$  such that  $f = \sum_{i=1}^l g_i \otimes h_i$ . A linear map  $\phi : M_k \mapsto M_m$  is called *entanglement breaking* if  $s \circ \phi^{(n)}$  is a separable state for any state  $s : M_n \otimes M_m \mapsto \mathbb{C}$  and any  $n \in \mathbb{N}$ . By Theorem 6.10 in [9] a linear map  $\phi : M_k \mapsto M_m$  is entanglement breaking if and only if there exist  $q \in \mathbb{N}$ , positive linear functionals  $f_i : M_k \mapsto \mathbb{C}$  and positive semidefinite matrices  $P_i \in M_m$  for  $i = 1, 2, \dots, q$  such that  $\phi(x) = \sum_{i=1}^q f_i(x)P_i$  for  $x \in M_k$ .

**DEFINITION 3.2.** Let  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, e)$  and  $(\mathcal{T}, \{D_n\}_{n=1}^\infty, f)$  be operator systems. A linear map  $\phi : \mathcal{S} \mapsto \mathcal{T}$  is called *entanglement breaking* if there exist  $q \in \mathbb{N}$ , positive linear functionals  $\omega_i$  on  $\mathcal{S}$  and positive elements  $p_i \in \mathcal{T}$  for  $i = 1, 2, \dots, q$  such that

$$\phi(x) = \sum_{i=1}^q \omega_i(x)p_i, \quad x \in \mathcal{S}.$$

A linear map  $\phi : \mathcal{S} \mapsto \mathcal{T}$  is called *s-entanglement breaking* if there exists a net  $\{\phi_\lambda\}_{\lambda \in \Lambda}$  of entanglement breaking maps from  $\mathcal{S}$  to  $\mathcal{T}$  such that  $\phi(x) = \lim_\lambda \phi_\lambda(x)$  for any  $x \in \mathcal{S}$  in the order topology on  $\mathcal{T}$ .

**PROPOSITION 3.3.** Let  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, e)$  and  $(\mathcal{T}, \{D_n\}_{n=1}^\infty, f)$  be operator systems, and let  $\psi$  be a linear map from  $\mathcal{S}$  to  $\mathcal{T}$ . If there exists a net of *s-entanglement breaking maps*  $\{\psi_i\}_{i \in I}$  from  $\mathcal{S}$  to  $\mathcal{T}$  such that  $\lim_i \psi_i(x) = \psi(x)$  for any  $x \in \mathcal{S}$  in the order topology on  $\mathcal{T}$ . Then  $\psi$  is *s-entanglement breaking*.

*Proof.* Since each  $\psi_i$  is s-entanglement breaking, there exist a directed set  $\Lambda_i$ , positive linear functionals  $\{\omega_{j,\lambda_i}\}_{j=1}^{n_{\lambda_i}}$  on  $\mathcal{S}$  and positive elements  $\{p_{j,\lambda_i}\}_{j=1}^{n_{\lambda_i}}$  of  $\mathcal{T}$  such that

$$\psi_{\lambda_i}(x) = \sum_{j=1}^{n_{\lambda_i}} \omega_{j,\lambda_i}(x)p_{j,\lambda_i},$$

and  $\lim_{\lambda_i} \psi_{\lambda_i}(x) = \psi_i(x)$  for all  $x \in \mathcal{S}$  in the order topology on  $\mathcal{T}$ . Now we define a new net of entanglement breaking maps  $\{\psi_\mu\}_{\mu \in \Lambda}$ , where  $\Lambda = \{(i, \lambda_i) : i \in I, \lambda_i \in \Lambda_i\}$  and  $\psi_\mu = \psi_{\lambda_i}$  for some  $\lambda_i$ , and we have

$$\lim_{\mu \in \Lambda} \psi_\mu(x) = \psi(x)$$

for any  $x \in \mathcal{S}$  in the order topology on  $\mathcal{T}$ . So  $\psi$  is s-entanglement breaking.  $\square$

**LEMMA 3.4.** *Let  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, e)$  and  $(\mathcal{T}, \{D_n\}_{n=1}^\infty, f)$  be operator systems. If  $\phi$  is a positive linear map from  $\mathcal{S}$  to  $\mathcal{T}$ , then  $\phi$  is continuous for the order topologies on  $\mathcal{S}$  and  $\mathcal{T}$ .*

*Proof.* Let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be two order norms on  $\mathcal{S}$  and  $\mathcal{T}$ , respectively. For any hermitian element  $a \in \mathcal{S}$ , we have that  $-\|a\|_1 e \leq a \leq \|a\|_1 e$ . Since  $\phi$  is positive, we obtain that  $-\|a\|_1 \phi(e) \leq \phi(a) \leq \|a\|_1 \phi(e)$ . Also we have that  $-\|\phi(e)\|_2 f \leq \phi(e) \leq \|\phi(e)\|_2 f$ . So we have

$$-\|a\|_1 \|\phi(e)\|_2 f \leq \phi(a) \leq \|a\|_1 \|\phi(e)\|_2 f.$$

Hence  $\|\phi(a)\|_2 \leq \|\phi(e)\|_2 \|a\|_1$ .

Now for any  $a \in \mathcal{S}$ , we have that  $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$ . So we get

$$\begin{aligned} \|\phi(a)\|_2 &\leq \left\| \phi\left(\frac{a+a^*}{2}\right) \right\|_2 + \left\| \phi\left(\frac{a-a^*}{2i}\right) \right\|_2 \\ &\leq \|\phi(e)\|_2 \left\| \frac{a+a^*}{2} \right\|_1 + \|\phi(e)\|_2 \left\| \frac{a-a^*}{2i} \right\|_1 \\ &\leq 2\|\phi(e)\|_2 \|a\|_1. \end{aligned}$$

Therefore,  $\phi$  is continuous for the order topologies on  $\mathcal{S}$  and  $\mathcal{T}$ .  $\square$

**LEMMA 3.5.** *Let  $(\mathcal{R}, \{C_n\}_{n=1}^\infty, e)$ ,  $(\mathcal{S}, \{D_n\}_{n=1}^\infty, f)$ ,  $(\mathcal{T}, \{E_n\}_{n=1}^\infty, g)$  and  $(\mathcal{U}, \{F_n\}_{n=1}^\infty, h)$  be operator systems. If  $\phi : \mathcal{S} \mapsto \mathcal{T}$  is an s-entanglement breaking map and  $\varphi : \mathcal{R} \mapsto \mathcal{S}$  and  $\psi : \mathcal{T} \mapsto \mathcal{U}$  are positive linear maps, then  $\phi \circ \varphi : \mathcal{R} \mapsto \mathcal{T}$  and  $\psi \circ \phi : \mathcal{S} \mapsto \mathcal{U}$  are also s-entanglement breaking.*

*Proof.* Since  $\phi$  is s-entanglement breaking, there exists a net of entanglement breaking maps  $\phi_\lambda(x) = \sum_{i=1}^{n_\lambda} \omega_{i,\lambda}(x)p_{i,\lambda}$  with  $\lambda \in \Lambda$ , such that  $\lim_\lambda \phi_\lambda(x) = \phi(x)$  for any  $x \in \mathcal{S}$  in the order topology on  $\mathcal{T}$ . For any  $\lambda$  and any  $r \in \mathcal{R}$ , we have

$$\phi_\lambda(\varphi(r)) = \sum_{i=1}^{n_\lambda} \omega_{i,\lambda}(\varphi(r))p_{i,\lambda}.$$

Since  $\varphi$  is positive, all  $\omega_{i,\lambda} \circ \varphi$  are positive linear functionals on  $\mathcal{R}$ . It is clear that

$$\lim_{\lambda} \phi_{\lambda}(\varphi(r)) = \phi(\varphi(r))$$

for any  $r \in \mathcal{R}$  in the order topology on  $\mathcal{T}$ . Hence  $\phi \circ \varphi$  is s-entanglement breaking. Similarly, since  $\psi$  is positive, we have that all  $\psi(p_{i,\lambda})$  are positive elements of  $\mathcal{U}$ , and

$$\psi(\phi_{\lambda}(x)) = \sum_{i=1}^{n_{\lambda}} \omega_{i,\lambda}(x) \psi(p_{i,\lambda})$$

for all  $\lambda$  and  $x \in \mathcal{S}$ . By Lemma 3.4,  $\psi$  is continuous for the order topologies on  $\mathcal{T}$  and  $\mathcal{U}$ , and so we have

$$\psi(\phi(x)) = \lim_{\lambda} \psi(\phi_{\lambda}(x)) = \lim_{\lambda} \sum_{i=1}^{n_{\lambda}} \omega_{i,\lambda}(x) \psi(p_{i,\lambda})$$

for  $x \in \mathcal{S}$ . Therefore,  $\psi \circ \phi$  is s-entanglement breaking.  $\square$

Given an operator system  $(\mathcal{S}, \{C_n\}_{n=1}^{\infty}, e)$ , a positive linear functional  $\phi : M_n(\mathcal{S}) \mapsto \mathbb{C}$  is called *weak\*-separable* if it is a weak\*-limit of sums of functionals of the form  $f \otimes g$ , where  $f$  is a positive linear functional on  $M_n$  and  $g$  is a positive linear functional on  $\mathcal{S}$ . We recall that a linear map  $\phi$  from the operator system  $(\mathcal{S}, \{C_n\}_{n=1}^{\infty}, e)$  to an operator system  $(\mathcal{T}, \{D_n\}_{n=1}^{\infty}, f)$  is called *weak\*-entanglement breaking* if for every  $n \in \mathbb{N}$  and every positive linear functional  $s : M_n(\mathcal{T}) \mapsto \mathbb{C}$ , the map  $s \circ \phi^{(n)} : M_n(\mathcal{S}) \mapsto \mathbb{C}$  is weak\*-separable [9].

**THEOREM 3.6.** *Let  $(\mathcal{S}, \{C_n\}_{n=1}^{\infty}, e)$  and  $(\mathcal{T}, \{D_n\}_{n=1}^{\infty}, f)$  be operator systems, and let  $\phi$  be a linear map from  $\mathcal{S}$  to  $\mathcal{T}$ . If  $\phi$  is s-entanglement breaking, then*

1.  $\phi : OMIN(\mathcal{S}) \mapsto OMAX(\mathcal{T})$  is completely positive,
2.  $\phi : \mathcal{S} \mapsto OMAX(\mathcal{T})$  is completely positive,
3.  $\phi : OMIN(\mathcal{S}) \mapsto \mathcal{T}$  is completely positive,
4.  $\phi$  is weak\*-entanglement breaking.

*Proof.* First we assume that  $\phi(x) = \sum_{k=1}^m \omega_k(x) p_k$ , where each  $\omega_k$  is a positive linear functional on  $\mathcal{S}$  and each  $p_k \in \mathcal{T}^+$ . For any  $n \in \mathbb{N}$  and  $X = (x_{ij}) \in C_n^{min}(\mathcal{S})$ , since every positive linear functional on any operator system is completely positive [7, Proposition 3.8], we know that  $\omega_k^{(n)}(X) = (\omega_k(x_{ij})) \in M_n^+$  for each  $k$ . So we have

$$\phi^{(n)}(X) = (\phi(x_{ij})) = \left( \sum_{k=1}^m \omega_k(x_{ij}) p_k \right) = \sum_{k=1}^m \omega_k^{(n)}(X) \otimes p_k \in D_n^{max}(\mathcal{T}).$$

For the general case, we suppose that  $\phi(x) = \lim_{\lambda} \phi_{\lambda}(x)$  for any  $x \in \mathcal{S}$  in the order topology on  $\mathcal{T}$  and each  $\phi_{\lambda}$  has the form  $\sum_{i=1}^{n_{\lambda}} \omega_{i,\lambda}(x) p_{i,\lambda}$  as above. For any  $X = (x_{ij}) \in C_n^{min}(\mathcal{S})$ , by Proposition 3.1 we have

$$\lim_{\lambda} \phi_{\lambda}^{(n)}(X) = \lim_{\lambda} (\phi_{\lambda}(x_{ij})) = (\phi(x_{ij})) = \phi^{(n)}(X)$$

in the order topology on  $M_n(\mathcal{T})$ . Since  $\phi_\lambda^{(n)}(X) \in D_n^{\max}(\mathcal{T})$  and  $C_n^{\max}(\mathcal{T})$  is the closure of  $D_n^{\max}(\mathcal{T})$  in the order topology on  $M_n(\mathcal{T})$  by Proposition 3.20 in [9], we obtain that  $\phi^{(n)}(X) \in C_n^{\max}(\mathcal{T})$ . Hence  $\phi : OMIN(\mathcal{T}) \mapsto OMAX(\mathcal{T})$  is completely positive, and so  $\phi$  satisfies (1).

Since  $\{C_n^{\min}\}_{n=1}^\infty$  is the weakest and  $\{C_n^{\max}\}_{n=1}^\infty$  is the strongest in all of the operator system structures on an Archimedean ordered  $*$ -vector space, (2) and (3) follow from (1). By Theorem 6.15 in [9], (3) and (4) are equivalent. This proves (4).  $\square$

### 4. Completely positive linear maps

In this section we discuss the relationship between completely positive linear maps and s-entanglement breaking maps between different operator system structures.

LEMMA 4.1. *Let  $(V, V^+, e)$  be an Archimedean ordered  $*$ -vector space. If  $OMAX(V) \neq OMIN(V)$ , then there exist an operator system  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, f)$  and a completely positive linear map  $\phi$  from the operator system  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, f)$  to the operator system  $OMAX(V)$  which is not s-entanglement breaking.*

*Proof.* Choosing  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, f) = OMAX(V)$  and  $\phi = id_V$ . Then  $\phi$  is a completely positive linear map from the operator system  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, f)$  to the operator system  $OMAX(V)$ . However,  $\phi$  is not s-entanglement breaking, otherwise by (1) of Theorem 3.6 we have that  $\phi : OMIN(\mathcal{S}) \mapsto OMAX(V)$  is completely positive. And by the definition of minimal operator system we have that  $OMIN(\mathcal{S}) = OMIN(V)$ . Since  $\phi$  is the identity map, we get that  $C_n^{\min}(V) = C_n^{\max}(V)$  for all  $n$ . This contradicts the assumption.  $\square$

EXAMPLE 4.2. For  $m, n \in \mathbb{N}$ , a linear operator  $a \in B(\mathbb{C}^m \otimes \mathbb{C}^n)$  is said to be *block positive* if  $\langle a(\xi \otimes \eta), \xi \otimes \eta \rangle \geq 0$  for all  $\xi \in \mathbb{C}^m$  and  $\eta \in \mathbb{C}^n$ . A linear operator  $a \in B(\mathbb{C}^m \otimes \mathbb{C}^n)$  is called *separable positive* if it can be written in the form  $a = \sum_{i=1}^k b_i \otimes c_i$  for some positive semidefinite operators  $b_i \in B(\mathbb{C}^m)$  and  $c_i \in B(\mathbb{C}^n)$  for  $i = 1, 2, \dots, k$ . It is known that  $C_m^{\min}(M_n)$  is the set of block positive linear operators on  $\mathbb{C}^m \otimes \mathbb{C}^n$  and  $C_m^{\max}(M_n)$  is the set of separable positive linear operators on  $\mathbb{C}^m \otimes \mathbb{C}^n$  [3]. For  $n \geq 2$ , let  $\{e_{ij}^{(n)}\}_{i,j=1}^n$  be the matrix units of  $M_n$ . Take

$$a = e_{11}^{(2)} \otimes e_{11}^{(n)} + e_{21}^{(2)} \otimes e_{12}^{(n)} + e_{12}^{(2)} \otimes e_{21}^{(n)} + e_{22}^{(2)} \otimes e_{22}^{(n)}.$$

For any  $\xi = (\xi_1, \xi_2)^t \in \mathbb{C}^2$  and  $\eta = (\eta_1, \dots, \eta_n)^t \in \mathbb{C}^n$ , we have

$$\langle a(\xi \otimes \eta), \xi \otimes \eta \rangle = (\xi_1 \bar{\eta}_1 + \xi_2 \bar{\eta}_2) \overline{(\xi_1 \bar{\eta}_1 + \xi_2 \bar{\eta}_2)} \geq 0.$$

So  $a \in M_2 \otimes M_n$  is block positive. Since  $-1$  is an eigenvalue of  $a$ , it is not positive semidefinite. Hence  $a$  is not separable positive. Therefore,  $C_2^{\min}(M_n) \neq C_2^{\max}(M_n)$ , i.e.,  $\mathcal{C}^{\min}(M_n) \neq \mathcal{C}^{\max}(M_n)$ . So by Lemma 4.1, there exist an operator system  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, f)$  and a completely positive linear map  $\phi$  from the operator system  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, f)$  to the operator system  $OMAX(M_n)$  which is not s-entanglement breaking.

LEMMA 4.3. *Let  $(V, V^+, e)$  be an Archimedean ordered  $*$ -vector space, and let*

$$(M_n, \{M_m^+(M_n)\}_{m=1}^\infty, 1_n)$$

*be the operator system arising from the identification of  $M_n$  with  $B(\mathbb{C}^n)$ , where  $M_m^+(M_n)$  is the set of positive semidefinite matrices in  $M_m \otimes M_n \cong M_{mn}$ . Then a linear map  $\phi : M_n \mapsto OMAX(V)$  is completely positive if and only if  $\phi$  is s-entanglement breaking.*

*Proof.* Let  $a$  be the matrix  $(e_{ij}^{(n)})_{1 \leq i, j \leq n}$  of matrix units, where  $\{e_{ij}^{(n)}\}_{i, j=1}^n$  are the matrix units of  $M_n$ . Then  $\phi$  is completely positive if and only if  $\phi^{(n)}(a) = (\phi(e_{ij}^{(n)}))_{1 \leq i, j \leq n} \in C_n^{max}(V)$  [7]. Since  $C_n^{max}(V)$  is the closure of  $D_n^{max}(V)$  in the order topology on  $M_n(V)$ ,  $\phi^{(n)}(a) \in C_n^{max}(V)$  if and only if there exists a family of elements  $\{Y_\lambda\}_{\lambda \in \Lambda}$  in  $D_n^{max}(V)$  such that  $\lim_\lambda Y_\lambda = \phi^{(n)}(a)$  in the order topology on  $M_n(V)$ . Let  $Y_\lambda = \sum_{l=1}^{k_\lambda} A_{l,\lambda} \otimes p_{l,\lambda}$  with  $A_{l,\lambda} = (a_{ij}^{(l,\lambda)}) \in M_n^+$  and  $p_{l,\lambda} \in V^+$ . We define

$$\phi_\lambda(x) = \sum_{l=1}^{k_\lambda} Tr(A_{l,\lambda}^t x) p_{l,\lambda}, \quad x \in M_n,$$

where  $A_{l,\lambda}^t$  denotes the transpose of  $A_{l,\lambda}$ . Since  $A_{l,\lambda} \geq 0$ , it is not hard to see that each  $\psi_{l,\lambda}(x) = Tr(A_{l,\lambda}^t x)$  is a positive linear functional on  $M_n$ . So each  $\phi_\lambda$  is entanglement breaking. Moreover, we have

$$\phi_\lambda(e_{ij}^{(n)}) = \sum_{l=1}^{k_\lambda} Tr(A_{l,\lambda}^t e_{ij}^{(n)}) p_{l,\lambda} = \sum_{l=1}^{k_\lambda} a_{ij}^{(l,\lambda)} p_{l,\lambda},$$

for all  $i, j = 1, 2, \dots, n$ . So

$$(\phi_\lambda(e_{ij}^{(n)}))_{1 \leq i, j \leq n} = \sum_{l=1}^{k_\lambda} (a_{ij}^{(l,\lambda)} p_{l,\lambda})_{1 \leq i, j \leq n} = \sum_{l=1}^{k_\lambda} A_{l,\lambda} \otimes p_{l,\lambda} = Y_\lambda.$$

Hence  $\lim_\lambda (\phi_\lambda(e_{ij}^{(n)})) = \phi^{(n)}(a) = (\phi(e_{ij}^{(n)}))$  in the order topology on  $M_n(V)$ . By Proposition 3.1, we have  $\lim_\lambda \phi_\lambda(e_{ij}^{(n)}) = \phi(e_{ij}^{(n)})$  in the order topology on  $V$  for any  $i, j \in \{1, 2, \dots, n\}$ . Since each  $x \in M_n$  can be written as a linear combination of  $\{e_{ij}^{(n)}\}_{i, j=1}^n$ , we have that  $\lim_\lambda \phi_\lambda(x) = \phi(x)$  in the order topology on  $V$  for each  $x \in M_n$ . So by definition,  $\phi$  is s-entanglement breaking. Conversely, if  $\phi$  is s-entanglement breaking, we can see  $\phi$  is completely positive by Theorem 3.6.  $\square$

Let  $M$  be a von Neumann algebra, and let  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, e)$  be an operator system. A linear map  $\phi$  from  $M$  to  $\mathcal{S}$  is said to be *normal* if for every net  $\{x_i\}$  in  $M$  with  $\lim_i x_i = x \in M$  in the strong operator topology on  $M$ , we have that  $\lim_i \phi(x_i) = \phi(x)$  in the order topology on  $\mathcal{S}$ .

PROPOSITION 4.4. *Let  $B(H)$  be the set of bounded linear operators on some Hilbert space  $H$ , and let  $(V, V^+, e)$  be an Archimedean ordered  $*$ -vector space. If  $\phi : B(H) \mapsto OMAX(V)$  is a normal and completely positive linear map, then  $\phi$  is s-entanglement breaking.*

*Proof.* If  $H$  is finite dimensional, by Lemma 4.3 we have that  $\phi$  is s-entanglement breaking. When  $H$  is infinite-dimensional, there exists a net of finite-dimensional projections  $\{p_i\}_{i \in \Lambda}$  in  $B(H)$  with  $\lim_i p_i = I_H$  in the strong operator topology on  $B(H)$ , where  $I_H$  is the identity operator on  $H$ . For  $i \in \Lambda$ , set

$$\phi_i(x) = \phi(p_i x p_i), \quad x \in B(p_i H).$$

Then each  $\phi_i$  is a completely positive linear map from  $B(p_i H)$  to  $OMAX(V)$ . By Lemma 4.3 each  $\phi_i$  is s-entanglement breaking. So there exist a directed set  $\Lambda_i$ , positive linear functionals  $\{\omega_{j, \lambda_i}\}_{j=1}^{n_{\lambda_i}}$  on  $B(p_i H)$  and positive elements  $\{p_{j, \lambda_i}\}_{j=1}^{n_{\lambda_i}}$  of  $V$  such that

$$\phi_{\lambda_i}(x) = \sum_{j=1}^{n_{\lambda_i}} \omega_{j, \lambda_i}(x) p_{j, \lambda_i}, \quad \lambda_i \in \Lambda_i,$$

and  $\lim_{\lambda_i} \phi_{\lambda_i}(x) = \phi_i(x)$  for all  $x \in B(p_i H)$  in the order topology on  $OMAX(V)$ . Now for each  $\lambda_i \in \Lambda_i$ , we define

$$\psi_{\lambda_i}(x) = \sum_{j=1}^{n_{\lambda_i}} \omega_{j, \lambda_i}(p_i x p_i) p_{j, \lambda_i}, \quad x \in B(H).$$

Then we have that each  $\psi_{\lambda_i}$  is entanglement breaking, and

$$\lim_{\lambda_i} \psi_{\lambda_i}(x) = \lim_{\lambda_i} \phi_{\lambda_i}(p_i x p_i) = \phi_i(p_i x p_i)$$

for any  $i \in \Lambda$  and  $x \in B(H)$  in the order topology on  $OMAX(V)$ . We define

$$\psi_i(x) = \phi_i(p_i x p_i), \quad x \in B(H)$$

for  $i \in \Lambda$ . Then each  $\psi_i$  is an s-entanglement breaking map from  $B(H)$  to  $OMAX(V)$ , and for any  $x \in B(H)$  we have

$$\lim_i \psi_i(x) = \lim_i \phi_i(p_i x p_i) = \lim_i \phi(p_i(p_i x p_i)p_i) = \lim_i \phi(p_i x p_i) = \phi(x)$$

since  $\lim_i p_i x p_i = x$  in the strong operator topology on  $B(H)$ . Therefore,  $\phi$  is s-entanglement breaking by Proposition 3.3.  $\square$

An operator system  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, e)$  is called *nuclear* if there exist nets of unital completely positive linear maps  $\phi_\lambda : \mathcal{S} \mapsto M_{n_\lambda}$  and  $\psi_\lambda : M_{n_\lambda} \mapsto \mathcal{S}$ , such that  $\lim_\lambda \psi_\lambda(\phi_\lambda(x)) = x$  for any  $x \in \mathcal{S}$  in the order topology on  $\mathcal{S}$  [2].

**THEOREM 4.5.** *Let  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, e)$  be a nuclear operator system, and let  $(V, V^+, f)$  be an Archimedean ordered  $*$ -vector space. Then a linear map  $\phi : \mathcal{S} \mapsto OMAX(V)$  is completely positive if and only if  $\phi$  is s-entanglement breaking.*

*Proof.* Since  $\mathcal{S}$  is nuclear, there exist completely positive linear maps  $\phi_\lambda : \mathcal{S} \mapsto M_{n_\lambda}$  and  $\psi_\lambda : M_{n_\lambda} \mapsto \mathcal{S}$  such that  $\lim_\lambda \psi_\lambda(\phi_\lambda(x)) = x$  for any  $x \in \mathcal{S}$  in the order topology on  $\mathcal{S}$ . If  $\phi$  is completely positive, then  $\phi \circ \psi_\lambda : M_{n_\lambda} \mapsto OMAX(V)$  is also

completely positive. So by Lemma 4.3 we have that  $\phi \circ \psi_\lambda$  is s-entanglement breaking. Set  $\phi_\lambda = \phi \circ \psi_\lambda \circ \varphi_\lambda$ , by Lemma 3.5 we see that  $\phi_\lambda$  is s-entanglement breaking. From Lemma 3.4 we know that positive linear maps between operator systems must be continuous for the order topologies on the two operator systems, we see that  $\lim_\lambda \phi_\lambda(x) = \lim_\lambda \phi(\psi_\lambda(\varphi_\lambda(x))) = \phi(x)$  for  $x \in \mathcal{S}$  in the order topology on  $V$ . So  $\phi$  is s-entanglement breaking by Proposition 3.3.

Conversely, if  $\phi$  is s-entanglement breaking, we can see  $\phi$  is completely positive by Theorem 3.6.  $\square$

We say an operator system  $(\mathcal{T}, \{C_n\}_{n=1}^\infty, e)$  is *injective* if for every pair of operator systems  $(\mathcal{R}, \{D_n\}_{n=1}^\infty, f)$  and  $(\mathcal{S}, \{E_n\}_{n=1}^\infty, f)$  such that  $\mathcal{R} \subseteq \mathcal{S}$ , and each completely positive linear map  $\phi : \mathcal{R} \mapsto \mathcal{T}$ , there exists a completely positive extension  $\psi : \mathcal{S} \mapsto \mathcal{T}$  [1]. For example,  $B(H)$  is injective.

**THEOREM 4.6.** *If  $(\mathcal{T}, \{C_n\}_{n=1}^\infty, e)$  is an injective operator system and if  $(V, V^+, e)$  is an Archimedean ordered  $*$ -vector space, then a linear map  $\phi : OMIN(V) \mapsto \mathcal{T}$  is completely positive if and only if  $\phi$  is s-entanglement breaking.*

*Proof.* Assume that  $\phi : OMIN(V) \mapsto \mathcal{T}$  is completely positive.

Since  $(\mathcal{T}, \{C_n\}_{n=1}^\infty, e)$  is injective, the map  $\phi : OMIN(V) \mapsto OMAX(\mathcal{T})$  is completely positive by Proposition 6.11 in [9]. By Theorem 3.2 in [9],  $\mathcal{E}^{min}(V)$  is the operator system structure on  $(V, V^+, e)$  induced by the inclusion  $\tau$  of  $V$  into  $C(S(V))$ . So  $\tau$  is positive. From the injectivity of  $(\mathcal{T}, \{C_n\}_{n=1}^\infty, e)$ , we can extend  $\phi$  to a completely positive map  $\phi'$  from  $C(S(V))$  to  $OMAX(\mathcal{T})$ . Since  $C(S(V))$  is nuclear as a unital  $C^*$ -algebra, it is a nuclear operator system by Remark 5.15 in [5] and Corollary 3.2 in [2]. So by Theorem 4.5,  $\phi'$  is s-entanglement breaking. It is clear that

$$\phi = \phi' \circ \tau$$

Therefore,  $\phi$  is s-entanglement breaking by Lemma 3.5.

Conversely, if  $\phi$  is s-entanglement breaking, we can see that  $\phi$  is completely positive by Theorem 3.6.  $\square$

**PROPOSITION 4.7.** *Let  $(V, V^+, e)$  be an Archimedean ordered  $*$ -vector space. If  $OMIN(V) \neq OMAX(V)$ , then  $OMAX(V)$  is not a nuclear operator system and  $OMIN(V)$  is not an injective operator system.*

*Proof.* Let  $\phi = id_V$ . Then  $\phi : OMAX(V) \mapsto OMAX(V)$  is completely positive. If  $OMAX(V)$  is nuclear, then  $\phi$  is s-entanglement breaking by Theorem 4.5. So from (3) of Theorem 3.6 we see that  $\phi : OMIN(V) \mapsto OMAX(V)$  is completely positive. Hence we obtain that  $OMIN(V) = OMAX(V)$ , which contradicts the assumptions.

Similarly, from the fact  $\phi : OMIN(V) \mapsto OMIN(V)$  is completely positive and  $OMIN(V)$  is injective, we will get that  $\phi$  is s-entanglement breaking by Theorem 4.6. So  $\phi : OMIN(V) \mapsto OMAX(V)$  is completely positive by (2) of Theorem 3.6. We obtain that  $OMIN(V) = OMAX(V)$ , which also contradicts the assumptions.  $\square$

EXAMPLE 4.8. For  $n \geq 2$ , we have that  $OMIN(M_n) \neq OMAX(M_n)$  by Example 4.2. From Proposition 4.7 and its proof we get a non-nuclear operator system  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, e) = OMAX(M_n)$ , an Archimedean ordered  $*$ -vector space  $(M_n, M_n^+, 1_n)$  and a linear map  $\phi = id_{M_n} : \mathcal{S} \mapsto OMAX(M_n)$  which is completely positive, but it is not s-entanglement breaking.

Similarly, by Proposition 4.7 and its proof we get a non-injective operator system  $(\mathcal{T}, \{D_n\}_{n=1}^\infty, f) = OMIN(M_n)$ , an Archimedean ordered  $*$ -vector space  $(M_n, M_n^+, 1_n)$  and a linear map  $\psi = id_{M_n} : OMIN(M_n) \mapsto \mathcal{T}$  which is completely positive, but it is not s-entanglement breaking.

On the relationship between weak  $*$ -entanglement breaking maps and s-entanglement breaking maps between operator systems, we have the following corollary.

COROLLARY 4.9. *Let  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, e)$  and  $(\mathcal{T}, \{D_n\}_{n=1}^\infty, f)$  be operator systems. If  $(\mathcal{T}, \{D_n\}_{n=1}^\infty, f)$  is injective, then a linear map  $\phi : \mathcal{S} \mapsto \mathcal{T}$  is weak  $*$ -entanglement breaking if and only if  $\phi$  is s-entanglement breaking.*

*Proof.* Assume first that  $\phi$  is weak  $*$ -entanglement breaking. By Theorem 6.15 in [9], we have that the map  $\phi : OMIN(\mathcal{S}) \mapsto \mathcal{T}$  is completely positive. So  $\phi$  is s-entanglement breaking by Theorem 4.6. Conversely, if  $\phi$  is s-entanglement breaking, we can see that  $\phi$  is weak  $*$ -entanglement breaking by Theorem 3.6.  $\square$

Generally, a weak  $*$ -entanglement breaking map need not to be s-entanglement breaking. In fact, we have

PROPOSITION 4.10. *Let  $(V, V^+, e)$  be an Archimedean ordered  $*$ -vector space. If  $OMAX(V) \neq OMIN(V)$ , then there exist an operator system  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, f)$  and a linear map  $\phi : \mathcal{S} \mapsto OMIN(V)$  such that  $\phi$  is weak  $*$ -entanglement breaking, but it is not s-entanglement breaking.*

*Proof.* Let  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, f) = OMIN(V)$  and  $\phi = id_V$ . Then  $\phi$  is a completely positive linear map from  $\mathcal{S}$  to  $OMIN(V)$ . So  $\phi$  is weak  $*$ -entanglement breaking by Theorem 6.15 in [9]. However,  $\phi$  is not s-entanglement breaking. In fact, if  $\phi$  is s-entanglement breaking, then by (2) of Theorem 3.6,  $\phi : OMIN(V) \mapsto OMAX(V)$  is completely positive. So we have that  $OMIN(V) = OMAX(V)$ . This contradicts the assumption.  $\square$

EXAMPLE 4.11. For  $n \geq 2$ , denote

$$(\mathcal{S}, \{C_n\}_{n=1}^\infty, e) = OMIN(M_n), \quad (\mathcal{T}, \{D_n\}_{n=1}^\infty, f) = OMIN(M_n).$$

Then from Example 4.2 and Proposition 4.7 we have that  $(\mathcal{T}, \{D_n\}_{n=1}^\infty, f)$  is not injective. It is clear that the linear map  $\phi = id_{M_n} : \mathcal{S} \mapsto \mathcal{T}$  is completely positive. By Theorem 6.15 in [9],  $\phi$  is weak  $*$ -entanglement breaking. However,  $\phi$  is not s-entanglement breaking by Example 4.8.

### 5. Nuclearity and s-entanglement breaking maps

In this section we give a nuclear characterization of s-entanglement breaking maps.

**PROPOSITION 5.1.** *Let  $(\mathcal{S}, \{C_n\}_{n=1}^\infty, e)$  and  $(\mathcal{T}, \{D_n\}_{n=1}^\infty, f)$  be operator systems. Then a linear map  $\phi : \mathcal{S} \mapsto \mathcal{T}$  is s-entanglement breaking if and only if we can find a net  $X_\lambda$  of compact spaces and completely positive linear maps  $\varphi_\lambda : \mathcal{S} \mapsto C(X_\lambda)$  and  $\psi_\lambda : C(X_\lambda) \mapsto \mathcal{T}$  such that  $\phi(x) = \lim_\lambda (\psi_\lambda \circ \varphi_\lambda)(x)$  for any  $x \in \mathcal{S}$  in the order topology on  $\mathcal{T}$ .*

*Proof.* Assume that  $\phi$  is s-entanglement breaking. Then there exist a directed set  $\Lambda$ , positive linear functionals  $\{\omega_{j,\lambda}\}_{j=1}^{n_\lambda}$  on  $\mathcal{S}$  and positive elements  $\{p_{j,\lambda}\}_{j=1}^{n_\lambda}$  of  $\mathcal{T}$  such that

$$\phi_\lambda(x) = \sum_{j=1}^{n_\lambda} \omega_{j,\lambda}(x) p_{j,\lambda},$$

and  $\lim_\lambda \phi_\lambda(x) = \phi(x)$  for all  $x \in \mathcal{S}$  in the order topology on  $\mathcal{T}$ . By changing coefficients of  $\omega_{j,\lambda}$  we may assume that they are states. Set  $X_\lambda = \{\omega_{1,\lambda}, \dots, \omega_{n_\lambda,\lambda}\}$ . Then  $X_\lambda \subseteq S(\mathcal{S})$ , where  $S(\mathcal{S})$  is the state space of  $\mathcal{S}$ . So each  $X_\lambda$  is compact in the weak\*-topology. Define the maps  $\varphi_\lambda : \mathcal{S} \mapsto C(X_\lambda)$  and  $\psi_\lambda : C(X_\lambda) \mapsto \mathcal{T}$  by

$$\varphi_\lambda(x)(\omega_{i,\lambda}) = \omega_{i,\lambda}(x), \quad x \in \mathcal{S}$$

and

$$\psi_\lambda(f) = \sum_{i=1}^{n_\lambda} f(\omega_{i,\lambda}) p_{i,\lambda}, \quad f \in C(X_\lambda).$$

Then  $\varphi_\lambda$  and  $\psi_\lambda$  are completely positive linear maps and  $\phi_\lambda = \psi_\lambda \circ \varphi_\lambda$ . So we get the desired factorization.

Conversely, suppose that there exist a net of compact spaces  $\{X_\lambda\}_{\lambda \in \Lambda}$  and completely positive linear maps  $\varphi_\lambda : \mathcal{S} \mapsto C(X_\lambda)$  and  $\psi_\lambda : C(X_\lambda) \mapsto \mathcal{T}$  such that  $\lim_\lambda \psi_\lambda(\varphi_\lambda(x)) = \phi(x)$  for any  $x \in \mathcal{S}$  in the order topology on  $\mathcal{T}$ . Since  $C(X_\lambda) = \text{OMIN}(C(X_\lambda))$  by Proposition 5.2 in [9], we see that  $\psi_\lambda : C(X_\lambda) \mapsto \text{OMAX}(\mathcal{T})$  is completely positive by Proposition 6.11 in [9]. Moreover  $C(X_\lambda)$  is nuclear, and so  $\psi_\lambda$  is s-entanglement breaking by Theorem 4.5. Hence  $\psi_\lambda \circ \varphi_\lambda$  is s-entanglement breaking by Lemma 3.5. Now that  $\phi$  is s-entanglement breaking follows from Proposition 3.3.  $\square$

Considered the Problem 6.17 in [9] and compared with the factorization theorem in [6], we conjecture that completely positive maps from  $\text{OMIN}(\mathcal{S})$  to  $\text{OMAX}(\mathcal{T})$  are s-entanglement breaking maps, so we have the following question.

**QUESTION 5.2.** Let  $(V, V^+, e)$  and  $(W, W^+, f)$  be Archimedean ordered \*-vector spaces. For any completely positive linear map  $\phi : \text{OMIN}(V) \mapsto \text{OMAX}(W)$ , is  $\phi$  s-entanglement breaking?

*Acknowledgement.* The authors would like to thank Bojan Magajna and the referee(s) for helpful suggestions and careful reading of the paper.

## REFERENCES

- [1] M.-D. CHOI, E. G. EFFROS, *Injectivity and operator spaces*, J. Funct. Anal., **24** (1977), no. 2, 156–209.
- [2] K. H. HAN, V. I. PAULSEN, *An approximation theorem for nuclear operator systems*, J. Funct. Anal., **261** (2011), no. 4, 999–1009.
- [3] N. JOHNSTON, E. STØRMER, *Mapping cones are operator systems*, Bull. London Math. Soc., **44** (2012), no. 4, 738–748.
- [4] R. V. KADISON, *A representation theory for commutative topological algebra*, Mem. Amer. Math. Soc., **7** (1951).
- [5] A. KAVRUK, V. I. PAULSEN, I. G. TODOROV, M. TOMFORDE, *Tensor products of operator systems*, J. Funct. Anal., **261** (2011), no. 2, 267–299.
- [6] V. I. PAULSEN, *The maximal operator space of a normed space*, Proc. Edinburgh. Math. Soc. (2), **39** (1996), no. 2, 309–323.
- [7] V. I. PAULSEN, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, **78**, Cambridge University Press, Cambridge, 2002.
- [8] V. I. PAULSEN, M. TOMFORDE, *Vector spaces with an order unit*, Indiana Univ. Math. J., **58** (2009), no. 3, 1319–1359.
- [9] V. I. PAULSEN, I. G. TODOROV, M. TOMFORDE, *Operator system structures on ordered spaces*, Proc. London Math. Soc. (3), **102** (2011), no. 1, 25–49.
- [10] E. STØRMER, *Separable states and positive maps*, J. Funct. Anal., **254** (2008), no. 8, 2303–2312.
- [11] W. WU, *Quantized Gromov-Hausdorff distance*, J. Funct. Anal., **238** (2006), no. 1, 58–98.
- [12] W. WU, *An operator Arzelà-Ascoli theorem*, Acta Math. Sin. (Engl. Ser.), **24** (2008), no. 7, 1139–1154.

(Received February 22, 2013)

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