

## CHARACTERIZATION OF HIGHER DERIVATIONS ON REFLEXIVE ALGEBRAS

RUNLING AN, CHUNHUI XUE AND XU ZHANG

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*Abstract.* Let  $\text{Alg } \mathcal{L}$  be a CSL algebra. We say that a family of linear maps  $D = \{D_n, D_n : \text{Alg } \mathcal{L} \rightarrow \text{Alg } \mathcal{L}, n \in \mathbb{N}\}$  is higher derivable at  $\Omega \in \text{Alg } \mathcal{L}$  if  $D_n(AB) = \sum_{i+j=n} D_i(A)D_j(B)$  for any  $A, B \in \text{Alg } \mathcal{L}$  with  $AB = \Omega$ . In this paper, we give a necessary and sufficient condition for a family of linear maps  $D = \{D_n, D_n : \text{Alg } \mathcal{L} \rightarrow \text{Alg } \mathcal{L}, n \in \mathbb{N}\}$  being higher derivable at  $\Omega$  which satisfies one of the following properties: (1)  $\Omega = 0$ ; (2)  $\Omega$  is a left (or right) separating point of  $\text{Alg } \mathcal{L}$ ; (3)  $\Omega = P\Omega = \Omega P$  for some nontrivial projection  $P \in \mathcal{L}$  and  $P\Omega P$  is a left (or right) separating point of  $P\text{Alg } \mathcal{L}P$ . In particular, if  $\text{Alg } \mathcal{L}$  is an irreducible CDCSL algebra or a nest algebra, then  $D = \{D_n, D_n : \text{Alg } \mathcal{L} \rightarrow \text{Alg } \mathcal{L}, n \in \mathbb{N}\}$  is higher derivable at such  $\Omega$  if and only if it is a higher derivation.

### 1. Introduction

Let  $\mathcal{A}$  be a unital algebra. Recall that a linear map  $\delta : \mathcal{A} \rightarrow \mathcal{A}$  is called a derivation if  $\delta(AB) = \delta(A)B + A\delta(B)$  for all  $A, B \in \mathcal{A}$ . It is well known that derivations are very important maps in both theory and applications, and have been studied intensively. Over the past few years many mathematicians pay their attention to the question under what conditions that a linear map becomes a derivation. One popular topic is derivable maps. We say that a linear map  $\delta : \mathcal{A} \rightarrow \mathcal{A}$  is derivable at  $\Omega \in \mathcal{A}$  if  $\delta(AB) = \delta(A)B + A\delta(B)$  for any  $A, B \in \mathcal{A}$  with  $AB = \Omega$  (see [1, 2, 3, 6, 7] and references therein). In [2], we obtained a necessary and sufficient condition for a linear map  $\delta$  from a CSL algebra into itself which is derivable at an arbitrary element  $\Omega$ , and showed that “every linear map  $\delta$  from an irreducible CDCSL algebra or a nest algebra into itself which is derivable at  $\Omega \neq 0$  is a derivation”.

With the development of derivations, higher derivations have attracted much attention of mathematicians as an active subject of research in algebras. Let  $\mathcal{A}$  be a unital algebra and  $\mathbb{N}$  be the set of non-negative integers. Recall that a family of linear maps  $D = \{D_n, D_n : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}\}$  is called a higher derivation if  $D_n(ST) = \sum_{i+j=n} D_i(S)D_j(T)$  for all  $S, T \in \mathcal{A}$ , where  $D_0 = I_{\mathcal{A}}$  is the identity map on  $\mathcal{A}$ . We

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say a family of linear maps  $D = \{D_n, D_n : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}\}$  is higher derivable at  $\Omega$  if  $D_n(ST) = \sum_{i+j=n} D_i(S)D_j(T)$  for any  $S, T \in \mathcal{A}$  with  $ST = \Omega$ . It is obvious that  $D_1$  is derivable at  $\Omega$  if  $D = \{D_n, D_n : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}\}$  is higher derivable at  $\Omega$ . Motivated by the study of derivable maps on CSL algebras and higher derivable maps on triangular algebras in [1, 2, 3, 6, 7, 8], we will study higher derivable maps on CSL algebras. Note that, in previous study, the authors focus on a family of linear maps on triangular algebras which is higher derivable (derivable) at the specially selected elements, and they deduce that such map is a higher derivation (derivation). In this paper, let  $\text{Alg } \mathcal{L}$  be a CSL algebra, we will give a necessary and sufficient condition for a family of linear maps  $D = \{D_n, D_n : \text{Alg } \mathcal{L} \rightarrow \text{Alg } \mathcal{L}, n \in \mathbb{N}\}$  being derivable at  $\Omega$  which satisfies one of the following properties: (1)  $\Omega = 0$ ; (2)  $\Omega \in \text{Alg } \mathcal{L}$  is a left (or right) separating point of  $\text{Alg } \mathcal{L}$ ; (3)  $\Omega = P\Omega = \Omega P$  for some nontrivial projection  $P \in \mathcal{L}$  and  $P\Omega P$  is a left (or right) separating point of  $P\text{Alg } \mathcal{L}P$ . These results show that a family of linear maps on CSL algebras which is higher derivable (derivable) at a general element has the structure different from a higher derivation (derivation). But if  $\text{Alg } \mathcal{L}$  is an irreducible CDCSL algebra or a nest algebra, then we can show  $D = \{D_n, D_n : \text{Alg } \mathcal{L} \rightarrow \text{Alg } \mathcal{L}, n \in \mathbb{N}\}$  is higher derivable at such  $\Omega$  if and only if it is a higher derivation.

Let us recall and fix some notation in this paper. Let  $H$  be a Hilbert space over the real or complex number field  $\mathbb{F}$  and  $\mathcal{B}(H)$  be the set of all linear bounded operators on  $H$ . Recall that a subspace lattice on  $H$  is a collection  $\mathcal{L}$  of strongly closed projections on  $H$  that is closed under the usual operation  $\wedge$  and  $\vee$ , and contains  $0$  and  $I$ .  $\mathcal{L}$  is called a nontrivial subspace lattice if  $\mathcal{L} \neq \{0, I\}$ . For any  $P \in \mathcal{L}$ , we define  $P_- = \vee\{Q \in \mathcal{L} : Q \not\leq P\}$ ,  $P_+ = \wedge\{Q \in \mathcal{L} : Q \not\leq P\}$ . A totally ordered subspace lattice is called a nest. A subspace lattice  $\mathcal{L}$  is called a commutative subspace lattice, or a CSL, if the projections in  $\mathcal{L}$  commute with each other. A subspace lattice  $\mathcal{L}$  is said to be completely distributive if  $P = \vee\{Q \in \mathcal{L} : Q_- \not\leq P\}$  for every  $P \in \mathcal{L}$  with  $P \neq 0$ , which is also equivalent to the condition  $P = \wedge\{Q_- : Q \in \mathcal{L}, Q \not\leq P\}$  for every  $P \in \mathcal{L}$  with  $P \neq I$ . Given a subspace lattice  $\mathcal{L}$  on  $H$ , the associated subspace lattice algebra  $\text{Alg } \mathcal{L}$  is the set  $\text{Alg } \mathcal{L} = \{T \in \mathcal{B}(H), TP = PTP, \forall P \in \mathcal{L}\}$ . Obviously,  $\text{Alg } \mathcal{L}$  is a unital weakly closed subalgebra of  $\mathcal{B}(H)$ . Dually, if  $\mathcal{A}$  is a subalgebra of  $\mathcal{B}(H)$ , by  $\text{Lat } \mathcal{A}$  we denote the collection of projections that are left invariant by each operator in  $\mathcal{A}$ . An algebra  $\mathcal{A}$  is reflexive if  $\mathcal{A} = \text{Alg Lat } \mathcal{A}$ . Clearly, every reflexive algebra is of the form  $\text{Alg } \mathcal{L}$  for some subspace lattice  $\mathcal{L}$  and vice versa. We call  $\text{Alg } \mathcal{L}$  a nest algebra if  $\mathcal{L}$  is a nest, a CSL algebra if  $\mathcal{L}$  is a CSL, and a CDCSL algebra if  $\mathcal{L}$  is a completely distributive CSL. It is well known that every CSL algebra is a reflexive algebra. A CSL algebra  $\text{Alg } \mathcal{L}$  is irreducible if and only if its commutant is trivial, i.e.  $(\text{Alg } \mathcal{L})' = \mathbb{C}I$ .

### 2. Maps higher derivable at 0

In this section, we give a necessary and sufficient condition for a family of linear maps  $D = \{D_n, n \in \mathbb{N}\}$  on a CSL algebra  $\text{Alg } \mathcal{L}$  which is higher derivable at zero. Let  $\mathcal{A} = \text{Alg } \mathcal{L}$  be a CSL algebra. Then for every nontrivial projection  $P$  in

$\mathcal{L}$ ,  $(I - P)\mathcal{A}P = \{0\}$ . Set  $P = P_1$ ,  $P_2 = I - P$  and  $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ ,  $i, j = 1, 2$ . Then  $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{22}$ . We shall consider  $\mathcal{A}_{ij}$  as a subset of  $\mathcal{A}$  and regard  $I_1 = P_1$  and  $I_2 = P_2$  as the unit of  $\mathcal{A}_{11}$  and  $\mathcal{A}_{22}$  respectively. Since  $D_n$  is linear, for any  $A_{ij} \in \mathcal{A}_{ij}$ , we can write  $D_n(A_{ij}) = D_n^{11}(A_{ij}) + D_n^{12}(A_{ij}) + D_n^{22}(A_{ij})$ , where  $D_n^{11} : \mathcal{A}_{ij} \rightarrow \mathcal{A}_{11}$ ,  $D_n^{12} : \mathcal{A}_{ij} \rightarrow \mathcal{A}_{12}$ ,  $D_n^{22} : \mathcal{A}_{ij} \rightarrow \mathcal{A}_{22}$  are linear maps,  $i, j \in \{1, 2\}$ . Since  $D_0 = I_{\mathcal{A}}$ , thus  $D_0^{ij}$  is the identity map on  $\mathcal{A}_{ij}$  and  $D_0^{ij}$  is zero on  $\mathcal{A}_{kl}$  when  $(k, l) \neq (i, j)$ ,  $1 \leq i, j, k, l \leq 2$ . We will use this decomposition and notation in the sequel.

The following is our main result in this section.

**THEOREM 2.1.** *Let  $\mathcal{L}$  be a nontrivial CSL and  $\mathcal{A} = \text{Alg}\mathcal{L}$  be the associated CSL algebra. Suppose  $D = \{D_n, D_n : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}\}$  is a family of linear maps with  $D_n(I) = 0$  for all  $n > 0$ , then it is higher derivable at 0 if and only if the following statements hold for any  $X \in \mathcal{A}_{11}$ ,  $Y \in \mathcal{A}_{12}$  and  $Z \in \mathcal{A}_{22}$ ,*

- (1)  $\sum_{i+j=n} D_i^{11}(X_1)D_j^{11}(X_2) = 0$  for any  $X_1, X_2 \in \mathcal{A}_{11}$  with  $X_1X_2 = 0$ .
- (2)  $\sum_{i+j=n} D_i^{22}(Z_1)D_j^{22}(Z_2) = 0$  for any  $Z_1, Z_2 \in \mathcal{A}_{22}$  with  $Z_1Z_2 = 0$ .
- (3)  $D_n^{22}(X) = 0$ ,  $D_n^{11}(Y) = D_n^{22}(Y) = 0$ ,  $D_n^{11}(Z) = 0$ .
- (4)  $D_n(X + Y + Z) = D_n^{11}(X) + D_n^{12}(X) + D_n^{12}(Y) + D_n^{12}(Z) + D_n^{22}(Z)$ .
- (5)  $D_n^{12}(X) = \sum_{i+j=n} D_i^{11}(X)D_j^{12}(I_1)$ ,  $D_n^{12}(Z) = \sum_{i+j=n} D_i^{12}(I_2)D_j^{22}(Z)$ .
- (6)  $D_n^{12}(XY) = \sum_{i+j=n} D_i^{11}(X)D_j^{12}(Y)$ ,  $D_n^{12}(YZ) = \sum_{i+j=n} D_i^{12}(Y)D_j^{22}(Z)$ .

We give some explanation about conditions (1)-(6). (1)-(2) are natural conditions ensuring that  $D = \{D_n, n \in \mathbb{N}\}$  is higher derivable at 0. (3)-(5) are the structure of  $D = \{D_n, n \in \mathbb{N}\}$  to be higher derivable at 0. (6) is the condition for  $D = \{D_n, n \in \mathbb{N}\}$  connecting the different subsets  $\mathcal{A}_{ij}$ ,  $1 \leq i, j \leq 2$ . In general, these are governing conditions for  $D = \{D_n, n \in \mathbb{N}\}$  is higher derivable at 0. But in some special case, these conditions will degenerate so that such map is a higher derivation (see Theorem 2.2).

*Proof.* First we give the proof of “only if” part. Since  $D_n$  is linear, for any  $X \in \mathcal{A}_{11}$ ,  $Y \in \mathcal{A}_{12}$  and  $Z \in \mathcal{A}_{22}$  we can write  $D_n(X + Y + Z) = D_n^{11}(X) + D_n^{11}(Y) + D_n^{11}(Z) + D_n^{12}(X) + D_n^{12}(Y) + D_n^{12}(Z) + D_n^{22}(X) + D_n^{22}(Y) + D_n^{22}(Z)$ . Let  $S = X + XY$  and  $T = -YZ + Z$ . Then  $ST = 0$  and  $0 = D_n(ST) = \sum_{i+j=n} D_i(S)D_j(T) = \sum_{i+j=n} [(D_i^{11}(X) + D_i^{11}(XY) + D_i^{12}(X) + D_i^{12}(XY)) \times (-D_j^{11}(YZ) + D_j^{11}(Z) - D_j^{12}(YZ) + D_j^{12}(Z) - D_j^{22}(YZ) + D_j^{22}(Z))]$ , and thus

$$\begin{aligned} \sum_{i+j=n} [(D_i^{11}(X) + D_i^{11}(XY))(-D_j^{11}(YZ) + D_j^{11}(Z))] &= 0, \\ \sum_{i+j=n} [(D_i^{22}(X) + D_i^{22}(XY))(-D_j^{22}(YZ) + D_j^{22}(Z))] &= 0, \\ \sum_{i+j=n} [(D_i^{11}(X) + D_i^{11}(XY))(-D_j^{12}(YZ) + D_j^{12}(Z))] & \\ + (D_i^{12}(X) + D_i^{12}(XY))(-D_j^{22}(YZ) + D_j^{22}(Z))] &= 0. \end{aligned} \tag{2.1}$$

Replacing  $Y$  with  $2Y$  in Eq. (2.1), we get

$$\sum_{i+j=n} D_i^{11}(X)D_j^{11}(Z) = 0, \quad \sum_{i+j=n} D_i^{22}(X)D_j^{22}(Z) = 0, \tag{2.2}$$

$$\begin{aligned} \sum_{i+j=n} [-D_i^{11}(X)D_j^{11}(YZ) + D_i^{11}(XY)D_j^{11}(Z)] &= 0, \\ \sum_{i+j=n} [-D_i^{22}(X)D_j^{22}(YZ) + D_i^{22}(XY)D_j^{22}(Z)] &= 0. \end{aligned} \tag{2.3}$$

Taking  $X = I_1, Z = I_2$  respectively in Eq. (2.2) and by induction on  $n$ , we have

$$D_n^{11}(Z) = 0, D_n^{22}(X) = 0, \forall n \in \mathbb{N}, X \in \mathcal{A}_{11}, Z \in \mathcal{A}_{22}. \tag{2.4}$$

Taking  $X = I_1, Z = I_2$  respectively in Eq. (2.3), by Eq. (2.4) and induction on  $n$ , one gets

$$D_n^{11}(Y) = 0, D_n^{22}(Y) = 0, \forall n \in \mathbb{N}, Y \in \mathcal{A}_{12}. \tag{2.5}$$

Note that  $D_n(I) = 0$ , thus Eq. (2.4) implies

$$D_n^{12}(I_1) + D_n^{12}(I_2) = 0, \forall n \in \mathbb{N}. D_n^{11}(I_1) = 0, D_n^{22}(I_2) = 0, n \geq 1. \tag{2.6}$$

By Eq. (2.1), Eqs. (2.4–2.5), we obtain

$$\sum_{i+j=n} [-D_i^{11}(X)D_j^{12}(YZ) + D_i^{11}(X)D_j^{12}(Z) + D_i^{12}(X)D_j^{22}(Z) + D_i^{12}(XY)D_j^{22}(Z)] = 0.$$

Replacing  $Y$  with  $2Y$  in above equality, we get

$$\sum_{i+j=n} [-D_i^{11}(X)D_j^{12}(YZ) + D_i^{12}(XY)D_j^{22}(Z)] = 0, \sum_{i+j=n} [D_i^{11}(X)D_j^{12}(Z) + D_i^{12}(X)D_j^{22}(Z)] = 0. \tag{2.7}$$

Taking  $Z = I_2$  and  $X = I_1$  respectively in Eq. (2.7), and by Eq. (2.6) we have,

$$\begin{aligned} D_n^{12}(XY) &= \sum_{i+j=n} D_i^{11}(X)D_j^{12}(Y), D_n^{12}(YZ) = \sum_{i+j=n} D_i^{12}(Y)D_j^{22}(Z). \\ D_n^{12}(X) &= - \sum_{i+j=n} D_i^{11}(X)D_j^{12}(I_2) = \sum_{i+j=n} D_i^{11}(X)D_j^{12}(I_1). \\ D_n^{12}(Z) &= - \sum_{i+j=n} D_i^{12}(I_1)D_j^{22}(Z) = \sum_{i+j=n} D_i^{12}(I_2)D_j^{22}(Z), \forall n \in \mathbb{N}. \end{aligned} \tag{2.8}$$

For any  $X_1, X_2 \in \mathcal{A}_{11}$  with  $X_1X_2 = 0$  and  $Z_1, Z_2 \in \mathcal{A}_{22}$  with  $Z_1Z_2 = 0$ , let  $S = X_1 + Z_1$  and  $T = X_2 + Z_2$ . Then  $ST = 0$  and  $0 = D_n(ST) = \sum_{i+j=n} D_i(S)D_j(T) = \sum_{i+j=n} [(D_i^{11}(X_1) + D_i^{12}(X_1) + D_i^{12}(Z_1) + D_i^{22}(Z_1)) (D_j^{11}(X_2) + D_j^{12}(X_2) + D_j^{12}(Z_2) + D_j^{22}(Z_2))]$ . Thus

$$\sum_{i+j=n} D_i^{11}(X_1)D_j^{11}(X_2) = 0, \sum_{i+j=n} D_i^{22}(Z_1)D_j^{22}(Z_2) = 0, \forall n \in \mathbb{N}. \tag{2.9}$$

The proof of “only if” part is complete.

Now we are in a position to check the “if” part. Let  $S, T \in \mathcal{A}$  be such that  $ST = 0, S = X_1 + Y_1 + Z_1, T = X_2 + Y_2 + Z_2, X_1, X_2 \in \mathcal{A}_{11}, Y_1, Y_2 \in \mathcal{A}_{12}$  and  $Z_1, Z_2 \in \mathcal{A}_{22}$ . Then  $X_1X_2 = 0, Z_1Z_2 = 0$  and  $X_1Y_2 + Y_1Z_2 = 0$ . Assume  $D_n$  satisfies conditions (1)–(6) in “if” part. By condition (3), we have

$$\begin{aligned} \sum_{i+j=n} D_i(S)D_j(T) &= \sum_{i+j=n} [(D_i^{11}(X_1) + D_i^{12}(X_1) + D_i^{12}(Y_1) + D_i^{12}(Z_1) + D_i^{22}(Z_1)) \\ &\quad (D_j^{11}(X_2) + D_j^{12}(X_2) + D_j^{12}(Y_2) + D_j^{12}(Z_2) + D_j^{22}(Z_2))] \\ &= \sum_{i+j=n} D_i^{11}(X_1)D_j^{11}(X_2) + G_{12} + \sum_{i+j=n} D_i^{22}(Z_1)D_j^{22}(Z_2), \text{ where} \end{aligned}$$

$$\begin{aligned}
 G_{12} &= \sum_{i+j=n} [D_i^{11}(X_1)D_j^{12}(X_2) + D_i^{11}(X_1)D_j^{12}(Y_2) + D_i^{11}(X_1)D_j^{12}(Z_2) \\
 &\quad + D_i^{12}(X_1)D_j^{22}(Z_2) + D_i^{12}(Y_1)D_j^{22}(Z_2) + D_i^{12}(Z_1)D_j^{22}(Z_2)] \\
 &= \sum_{i+j=n} [D_i^{11}(X_1)D_j^{12}(X_2) + D_i^{12}(X_1)D_j^{22}(Z_2) + D_i^{11}(X_1)D_j^{12}(Z_2) \\
 &\quad + D_i^{12}(Z_1)D_j^{22}(Z_2) + D_n^{12}(X_1Y_2 + Y_1Z_2)].
 \end{aligned}$$

By conditions (5)-(6), we have

$$\begin{aligned}
 &\sum_{i+j=n} [(D_i^{11}(X_1)D_j^{12}(X_2) + D_i^{11}(X_1)D_j^{12}(Z_2) + D_i^{12}(X_1)D_j^{22}(Z_2) + D_i^{12}(Z_1)D_j^{22}(Z_2)] \\
 &= \sum_{i+j=n} \sum_{k+l=j} D_i^{11}(X_1)D_k^{11}(X_2)D_l^{12}(I_1) - \sum_{i+j=n} \sum_{k+l=j} D_i^{11}(X_1)D_k^{12}(I_1)D_l^{22}(Z_2) \\
 &\quad + \sum_{i+j=n} \sum_{k+l=i} D_k^{11}(X_1)D_l^{12}(I_1)D_j^{22}(Z_2) - \sum_{i+j=n} \sum_{k+l=i} D_k^{12}(I_1)D_l^{22}(Z_1)D_j^{22}(Z_2) \\
 &= \sum_{k+l+j=n} D_k^{11}(X_1)D_l^{11}(X_2)D_j^{12}(I_1) - \sum_{i+k+l=n} D_i^{12}(I_1)D_k^{22}(Z_1)D_l^{22}(Z_2) = 0.
 \end{aligned}$$

Therefore  $G_{12} = 0$ . By conditions (1)-(2) we have  $\sum_{i+j=n} D_i(S)D_j(T) = 0 = D_n(ST)$  for any  $S, T \in \mathcal{A}$  with  $ST = 0$ . That is,  $D = \{D_n, D_n : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}\}$  is higher derivable at 0. The proof of this theorem is complete.  $\square$

By Theorem 2.1, we see that, in general, a family of linear maps  $D = \{D_n, n \in \mathbb{N}\}$  on CSL algebras being higher derivable at 0 is not a higher derivation. But in the case of triangular algebras, such map is a higher derivation.

The triangular algebras were firstly introduced in [4] and then studied by many authors. Let  $\mathcal{A}$  and  $\mathcal{B}$  be two unital algebras with unit  $I_1$  and  $I_2$  respectively, and let  $\mathcal{M}$  be a faithful  $(\mathcal{A}, \mathcal{B})$ -bimodule, that is,  $\mathcal{M}$  is a  $(\mathcal{A}, \mathcal{B})$ -bimodule satisfying, for  $A \in \mathcal{A}, A\mathcal{M} = \{0\} \Rightarrow A = 0$  and for  $B \in \mathcal{B}, \mathcal{M}B = \{0\} \Rightarrow B = 0$ . Recall that the algebra  $\mathcal{T} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} : A \in \mathcal{A}, M \in \mathcal{M}, B \in \mathcal{B} \right\}$  under the usual matrix addition and formal matrix multiplication is called a triangular algebra. We shall regard  $\mathcal{A}, \mathcal{B}$  and  $\mathcal{M}$  as subsets of  $\mathcal{T}$ , i.e. we shall identify them by their copies inside  $\mathcal{T}$ . It is obvious  $\mathcal{T}$  has the unit  $I = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix}$  and contains a nontrivial idempotent  $P = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$  which we call the standard idempotent. It follows from Proposition 3 in [4] that the center of  $\mathcal{T}$  is  $\mathcal{Z}(\mathcal{T}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : AM = MB, \forall M \in \mathcal{M} \right\}$ .

**THEOREM 2.2.** *Let  $\mathcal{T}$  be a triangular algebra and  $D = \{D_n, D_n : \mathcal{T} \rightarrow \mathcal{T}, n \in \mathbb{N}\}$  be a family of linear maps being higher derivable at 0. Then  $D_n(I) \in \mathcal{Z}(\mathcal{T})$ . Moreover, if  $D_n(I) = 0$  for all  $n > 0$ , then  $D = \{D_n, D_n : \mathcal{T} \rightarrow \mathcal{T}, n \in \mathbb{N}\}$  is a higher derivation.*

*Proof.* Taking  $X = I_1, Z = I_2$  in Eq.(2.7), we get

$$\sum_{i+j=n} [-D_i^{11}(I_1)D_j^{12}(M) + D_i^{12}(M)D_j^{22}(I_2)] = 0, \quad \sum_{i+j=n} [D_i^{11}(I_1)D_j^{12}(I_2) + D_i^{12}(I_1)D_j^{22}(I_2)] = 0 \tag{2.8}$$

hold for any  $n \in \mathbb{N}$ ,  $M \in \mathcal{M}$ .

By induction on  $n$ , we get  $D_n^{11}(I_1)M = MD_n^{22}(I_2)$  for all  $M \in \mathcal{M}$ . Therefore the second equality in Eq.(2.8) can be rewritten as  $\sum_{i+j=n} D_i^{11}(I_1)[D_j^{12}(I_1) + D_j^{12}(I_2)] = 0$ . By applying induction on  $n$ , we have  $D_n^{12}(I_1) + D_n^{12}(I_2) = 0$ , and hence  $D_n(I) = D_n^{11}(I_1) + D_n^{22}(I_2) \in \mathcal{L}(\mathcal{T})$ .

If  $D_n(I) = 0$  for all  $n > 0$ . Then by similar arguments as that in the proof of Theorem 2.1 in [8] we can prove  $D^{11} = \{D_n^{11}, D_n^{11} : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}\}$  and  $D^{22} = \{D_n^{22}, D_n^{22} : \mathcal{B} \rightarrow \mathcal{B}, n \in \mathbb{N}\}$  are higher derivations, and further  $D = \{D_n, D_n : \mathcal{T} \rightarrow \mathcal{T}, n \in \mathbb{N}\}$  is a higher derivation. The proof of this theorem is complete.  $\square$

By Theorem 2.2 we can characterize higher derivations on irreducible CDCSL algebras by maps which are higher derivable at 0. To prove this result, we need the following lemma which is from Theorem 3.4 in [5].

LEMMA 2.3. *Let  $\mathcal{L}$  be a nontrivial CDCSL and  $\text{Alg}\mathcal{L}$  be the associated irreducible CDCSL algebra, then there is a nontrivial projection  $P$  in  $\mathcal{L}$  such that for  $T \in \text{Alg}\mathcal{L}$ ,  $TP\text{Alg}\mathcal{L}(I - P) = \{0\}$  implies  $TP = 0$  and  $P\text{Alg}\mathcal{L}(I - P)T = \{0\}$  implies  $(I - P)T = 0$ .*

THEOREM 2.4. *Let  $\mathcal{L}$  be a nontrivial CDCSL and  $\text{Alg}\mathcal{L}$  be the associated irreducible CDCSL algebra. If a family of linear maps  $D = \{D_n, D_n : \text{Alg}\mathcal{L} \rightarrow \text{Alg}\mathcal{L}, n \in \mathbb{N}\}$  is higher derivable at 0, then  $D_n(I) = \lambda_n I$ ,  $\lambda_n \in \mathbb{F}$ . Moreover, if  $D_n(I) = 0$  for all  $n > 0$ , then  $D = \{D_n, D_n : \text{Alg}\mathcal{L} \rightarrow \text{Alg}\mathcal{L}, n \in \mathbb{N}\}$  is a higher derivation.*

*Proof.* Taking the projection  $P$  in Lemma 2.3, we can decompose  $\text{Alg}\mathcal{L}$  into a triangular algebra. Thus by Theorem 2.2, we obtain the desired result.  $\square$

It follows from [1] that every nest algebra is a triangular algebra, thus by Theorem 2.2 we have

THEOREM 2.5. *Let  $\mathcal{N}$  be a nontrivial nest and  $\text{Alg}\mathcal{N}$  be the associated nest algebra. If a family of linear maps  $D = \{D_n, D_n : \text{Alg}\mathcal{N} \rightarrow \text{Alg}\mathcal{N}, n \in \mathbb{N}\}$  is higher derivable at 0, then  $D_n(I) = \lambda_n I$ ,  $\lambda_n \in \mathbb{F}$ . Moreover, if  $D_n(I) = 0$  for all  $n > 0$ , then  $D = \{D_n, D_n : \text{Alg}\mathcal{N} \rightarrow \text{Alg}\mathcal{N}, n \in \mathbb{N}\}$  is a higher derivation.*

### 3. Maps higher derivable at separating points of $\mathcal{A}_{11}$ or $\mathcal{A}_{22}$

In this section, we give a necessary and sufficient condition for a family of linear maps on CSL algebras which is higher derivable at a left or right separating point of  $\mathcal{A}_{11}$  or  $\mathcal{A}_{22}$ .

DEFINITION 3.0. Suppose that  $\mathcal{A}$  is a Banach algebra and  $\mathcal{M}$  is an  $\mathcal{A}$ -bimodule. Let  $W \in \mathcal{A}$ . We say that  $W$  is a left (or right) separating point of  $\mathcal{M}$ , if for every  $M \in \mathcal{M}$  the condition  $WM = 0$  (or  $MW = 0$ ) implies  $M = 0$ .

It is obvious that left (or right) invertible elements and injective operators (or operators with dense range) in  $\mathcal{A}$  are left (or right) separating points of  $\mathcal{M}$ .

**THEOREM 3.1.** *Let  $\mathcal{L}$  be a nontrivial CSL and  $\mathcal{A} = \text{Alg}\mathcal{L}$  be the associated CSL algebra. Suppose  $\Omega \in \mathcal{A}$  satisfies that  $\Omega = P\Omega = \Omega P$  for some nontrivial projection  $P \in \mathcal{L}$  and  $P\Omega P$  is a left (or right) separating point of  $P\mathcal{A}P$ . Then a family of linear maps  $D = \{D_n, D_n : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}\}$  is higher derivable at  $\Omega$  if and only if  $D_n$  satisfies conditions (2)-(6) in Theorem 2.1 and  $D_n^{11}(X_1X_2) = \sum_{i+j=n} D_i^{11}(X_1)D_j^{11}(X_2)$  for any  $X_1, X_2 \in \mathcal{A}_{11}$  with  $X_1X_2 = P\Omega P$ .*

*Proof.* First we give the proof of “only if” part. We use the same notation as that in Section 2. Since  $\Omega = P\Omega = \Omega P$  and  $P\Omega P$  is a left (or right) separating point of  $P\mathcal{A}P$ , thus we can assume  $\Omega = W \oplus 0$ ,  $W$  is a left (or right) separating point of  $P\mathcal{A}P$ . For every invertible element  $X \in \mathcal{A}_{11}$  and any  $Y \in \mathcal{A}_{12}, Z \in \mathcal{A}_{22}$ . Let  $S = X - XY$  and  $T = X^{-1}W + YZ + Z$ . Then  $ST = \Omega$  and  $D_n^{11}(W) + D_n^{12}(W) + D_n^{22}(W) = D_n(\Omega) = \sum_{i+j=n} D_i(S)D_j(T) = \sum_{i+j=n} (D_i^{11}(X) - D_i^{11}(XY) + D_i^{12}(X) - D_i^{12}(XY) + D_i^{22}(X) - D_i^{22}(XY))(D_j^{11}(X^{-1}W) + D_j^{11}(YZ) + D_j^{11}(Z) + D_j^{12}(X^{-1}W) + D_j^{12}(YZ) + D_j^{12}(Z) + D_j^{22}(X^{-1}W) + D_j^{22}(YZ) + D_j^{22}(Z))$ . Thus

$$\begin{aligned} D_n^{11}(W) &= \sum_{i+j=n} [(D_i^{11}(X) - D_i^{11}(XY))(D_j^{11}(X^{-1}W) + D_j^{11}(YZ) + D_j^{11}(Z))], \\ D_n^{22}(W) &= \sum_{i+j=n} [(D_i^{22}(X) - D_i^{22}(XY))(D_j^{22}(X^{-1}W) + D_j^{22}(YZ) + D_j^{22}(Z))], \quad (3.1) \\ D_n^{12}(W) &= \sum_{i+j=n} [(D_i^{11}(X) - D_i^{11}(XY))(D_j^{12}(X^{-1}W) + D_j^{12}(YZ) + D_j^{12}(Z)) \\ &\quad + (D_i^{12}(X) - D_i^{12}(XY))(D_j^{22}(X^{-1}W) + D_j^{22}(YZ) + D_j^{22}(Z))]. \end{aligned}$$

Replacing  $X$  and  $Y$  with  $2X$  and  $2Y$  in Eq. (3.1), we get

$$\sum_{i+j=n} D_i^{11}(X)D_j^{11}(Z) = 0, \quad \sum_{i+j=n} [-D_i^{11}(X)D_j^{11}(YZ) + D_i^{11}(XY)D_j^{11}(Z)] = 0. \quad (3.2)$$

Taking  $X = I_1$  in Eq. (3.2) and by induction on  $n$ , we have

$$D_n^{11}(Z) = 0, \quad \forall n \in \mathbb{N}, Z \in \mathcal{A}_{22}. \quad (3.3)$$

Taking  $X = I_1, Z = I_2$  in Eq. (3.2), by Eq. (3.3) and induction on  $n$ , we obtain

$$D_n^{11}(Y) = 0, \quad \forall n \in \mathbb{N}, Y \in \mathcal{A}_{12}. \quad (3.4)$$

By Eqs. (3.3–3.4), Eq. (3.1) and  $\Omega I = \Omega$ , we get  $D_n^{11}(W) = \sum_{i+j=n} D_i^{11}(W)D_j^{11}(I_1)$ .

By induction on  $n$  and by the fact that  $W$  is a left separating point (if  $W$  is a right separating point, by applying similar arguments to  $I\Omega = \Omega$ ), one gets

$$D_n^{11}(I_1) = 0, \quad n > 0. \quad (3.5)$$

Replacing  $Z$  and  $Y$  with  $2Z$  and  $2Y$  in Eq. (3.1), we get

$$\sum_{i+j=n} D_i^{22}(X)D_j^{22}(Z) = 0, \quad \sum_{i+j=n} [-D_i^{22}(X)D_j^{22}(YZ) + D_i^{22}(XY)D_j^{22}(Z)] = 0. \quad (3.6)$$

Taking  $X = I_1$  in Eq. (3.6) and by induction on  $n$  we get  $D_n^{22}(X) = 0$  for any  $n \in \mathbb{N}$  and invertible elements  $X \in \mathcal{A}_{11}$ . For any  $X \in \mathcal{A}_{11}$ , replacing  $X$  with  $mI_1 - X$ ,  $m > \|X\|$  in the above equality, we get

$$D_n^{22}(X) = 0, \quad \forall n \in \mathbb{N}, X \in \mathcal{A}_{11}. \tag{3.7}$$

Taking  $X = I_1, Z = I_2$  in Eq. (3.6), by Eq. (3.7) and induction on  $n$ , one gets

$$D_n^{22}(Y) = 0, \quad \forall n \in \mathbb{N}, Y \in \mathcal{A}_{12}. \tag{3.8}$$

By Eqs. (3.3–3.4), Eqs. (3.7–3.8) and Eq. (3.1), we obtain

$$D_n^{12}(W) = \sum_{i+j=n} [D_i^{11}(X)D_j^{12}(X^{-1}W) + D_i^{11}(X)D_j^{12}(YZ) + D_i^{11}(X)D_j^{12}(Z) + D_i^{12}(X)D_j^{22}(Z) - D_i^{12}(XY)D_j^{22}(Z)].$$

Replacing  $X$  and  $Y$  with  $2X$  and  $2Y$  in above equality, we get

$$\sum_{i+j=n} [D_i^{11}(X)D_j^{12}(Z) + D_i^{12}(X)D_j^{22}(Z)] = 0, \quad \sum_{i+j=n} [-D_i^{11}(X)D_j^{12}(YZ) + D_i^{12}(XY)D_j^{22}(Z)] = 0. \tag{3.9}$$

Taking  $X = I_1$  in Eq. (3.9) and by Eq. (3.5), we have

$$D_n^{12}(Z) = - \sum_{i+j=n} D_i^{12}(I_1)D_j^{22}(Z), \quad D_n^{12}(YZ) = \sum_{i+j=n} D_i^{12}(Y)D_j^{22}(Z), \quad \forall Z \in \mathcal{A}_{22}, Y \in \mathcal{A}_{12}. \tag{3.10}$$

Taking  $X = I_1$  and  $Z = I_2$  in Eq. (3.9), we get

$$\sum_{i+j=n} D_i^{11}(I_1)D_j^{12}(Y) = \sum_{i+j=n} D_i^{12}(Y)D_j^{22}(I_2).$$

It follows from Eq. (3.5), Eq. (3.10) and the above equality that

$$D_n^{11}(I_1)Y = YD_n^{22}(I_2), \quad D_n^{12}(I_1) + D_n^{12}(I_2) = 0, \quad \forall Y \in \mathcal{A}_{12}. \tag{3.11}$$

Taking  $Z = I_2$  in Eq. (3.9), it follows from Eq. (3.5) and Eq. (3.11) that

$D_n^{12}(X) = \sum_{i+j=n} D_i^{11}(X)D_j^{12}(I_1)$  and  $D_n^{12}(XY) = \sum_{i+j=n} D_i^{11}(X)D_j^{12}(Y)$  hold for any invertible  $X \in \mathcal{A}_{11}$ . Therefore for any  $X \in \mathcal{A}_{11}, Y \in \mathcal{A}_{12}$ , we get

$$D_n^{12}(X) = \sum_{i+j=n} D_i^{11}(X)D_j^{12}(I_1), \quad D_n^{12}(XY) = \sum_{i+j=n} D_i^{11}(X)D_j^{12}(Y). \tag{3.12}$$

For any  $X_1, X_2 \in \mathcal{A}_{11}$  with  $X_1X_2 = W$  and  $Z_1, Z_2 \in \mathcal{A}_{22}$  with  $Z_1Z_2 = 0$ , let  $S = X_1 + Z_1$  and  $T = X_2 + Z_2$ . Then  $ST = \Omega$  and  $D_n^{11}(W) + D_n^{12}(W) = D_n(ST) = \sum_{i+j=n} D_i(S)D_j(T) = \sum_{i+j=n} (D_i^{11}(X_1) + D_i^{12}(X_1) + D_i^{12}(Z_1) + D_i^{22}(Z_1))(D_j^{11}(X_2) + D_j^{12}(X_2) + D_j^{12}(Z_2) + D_j^{22}(Z_2))$ . Thus

$$\sum_{i+j=n} D_i^{11}(X_1)D_j^{11}(X_2) = D_n^{11}(W), \quad \sum_{i+j=n} D_i^{22}(Z_1)D_j^{22}(Z_2) = 0. \tag{3.13}$$

The proof of “only if” part is complete.

For the “if” part, by similar arguments as that in the proof of Theorem 2.1, it is easy to check that  $D = \{D_n, D_n : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}\}$  is higher derivable at  $\Omega$  if it satisfies the conditions in “if” part of Theorem 3.1. The proof of this theorem is complete.  $\square$

Dually, we have

**THEOREM 3.2.** *Let  $\mathcal{L}$  be a nontrivial CSL and  $\mathcal{A} = \text{Alg}\mathcal{L}$  be the associated CSL algebra. Suppose  $\Omega \in \mathcal{A}$  satisfies  $\Omega = (I - P)\Omega = \Omega(I - P)$  for some nontrivial projection  $P \in \mathcal{L}$  and  $(I - P)\Omega(I - P)$  is a left (or right) separating point of  $(I - P)\mathcal{A}(I - P)$ . Then a family of linear maps  $D = \{D_n, D_n : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}\}$  is higher derivable at  $\Omega$  if and only if  $D_n$  satisfies conditions (1), (3)-(6) of Theorem 2.1 and  $D_n^{22}(Z_1Z_2) = \sum_{i+j=n} D_i^{22}(Z_1)D_j^{22}(Z_2)$  for any  $Z_1, Z_2 \in \mathcal{A}_{22}$  with  $Z_1Z_2 = (I - P)\Omega(I - P)$ .*

In the case of triangular algebras, we have

**THEOREM 3.3.** *Let  $\mathcal{T} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$  be a triangular algebra and  $P \in \mathcal{T}$  be the standard idempotent. Suppose  $\Omega \in \mathcal{T}$  satisfies one of the following properties: (1)  $\Omega = P\Omega = \Omega P$  and  $P\Omega P$  is a left (or right) separating point in  $\mathcal{A}$ ; (2)  $\Omega = (I - P)\Omega = \Omega(I - P)$  and  $(I - P)\Omega(I - P)$  is a left (or right) separating point in  $\mathcal{B}$ . Then a family of linear maps  $D = \{D_n, D_n : \mathcal{T} \rightarrow \mathcal{T}, n \in \mathbb{N}\}$  is higher derivable at  $\Omega$  if and only if it is a higher derivation.*

*Proof.* Only the “only if” part needs to be checked. By Eq. (3.5) and Eq. (3.11),  $0 = D_n^{11}(I_1)Y = YD_n^{22}(I_2)$ . Thus  $D_n^{22}(I_2) = D_n^{11}(I_1) = 0$ . Together with  $D_n^{12}(I_1) + D_n^{12}(I_2) = 0$  we get  $D_n(I) = 0$ . Now by similar arguments as that in the proof of Theorem 2.2, we obtain  $D = \{D_n, D_n : \mathcal{T} \rightarrow \mathcal{T}, n \in \mathbb{N}\}$  is a higher derivation.  $\square$

#### 4. Maps higher derivable at invertible elements

In this section, we give a necessary and sufficient condition for a family of linear maps on CSL algebras which is higher derivable at invertible elements.

**THEOREM 4.1.** *Let  $\mathcal{L}$  be a nontrivial CSL and  $\mathcal{A} = \text{Alg}\mathcal{L}$  be the associated CSL algebra. Assume  $\Omega = D + E + F \in \mathcal{A}$ ,  $D \in \mathcal{A}_{11}$ ,  $E \in \mathcal{A}_{12}$ ,  $F \in \mathcal{A}_{22}$  satisfies  $D$  is a left (or right) separating point of  $\mathcal{A}_{11}$  and  $F$  is a left (or right) separating point of  $\mathcal{A}_{22}$ . Then a family of linear maps  $D = \{D_n, D_n : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}\}$  is higher derivable at  $\Omega$  if and only if  $D_n$  satisfies conditions (3)-(6) of Theorem 2.1 and*

$$D_n^{11}(X_1X_2) = \sum_{i+j=n} D_i^{11}(X_1)D_j^{11}(X_2) \text{ for any } X_1, X_2 \in \mathcal{A}_{11} \text{ with } X_1X_2 = D,$$

$$D_n^{22}(Z_1Z_2) = \sum_{i+j=n} D_i^{22}(Z_1)D_j^{22}(Z_2) \text{ for any } Z_1, Z_2 \in \mathcal{A}_{22} \text{ with } Z_1Z_2 = F.$$

*Proof.* First we give the proof of “only if” part. We use the same notation as that in Section 2. Let  $X_1, X_2 \in \mathcal{A}_{11}$  with  $X_1X_2 = I_1$  and  $Z_1, Z_2 \in \mathcal{A}_{22}$  with  $Z_1Z_2 = I_2$ . For any  $Y \in \mathcal{A}_{12}$ , let  $S = X_1 + EZ_1 - X_1Y + FZ_1$  and  $T = X_2D + YZ_2 + Z_2$ . Then

$ST = \Omega$  and  $D_n^{11}(D) + D_n^{11}(E) + D_n^{11}(F) + D_n^{12}(D) + D_n^{12}(E) + D_n^{12}(F) + D_n^{22}(D) + D_n^{22}(E) + D_n^{22}(F) = D_n(\Omega) = \sum_{i+j=n} D_i(S)D_j(T) = \sum_{i+j=n} [(D_i^{11}(X_1) + D_i^{11}(EZ_1 - X_1Y) + D_i^{11}(FZ_1) + D_i^{12}(X_1) + D_i^{12}(EZ_1 - X_1Y) + D_i^{12}(FZ_1))(D_j^{11}(X_2D) + D_j^{11}(YZ_2) + D_j^{11}(Z_2)) + D_j^{12}(X_2D) + D_j^{12}(YZ_2) + D_j^{12}(Z_2)]$ . Thus

$$\begin{aligned} & D_n^{11}(D) + D_n^{11}(E) + D_n^{11}(F) \\ &= \sum_{i+j=n} [(D_i^{11}(X_1) + D_i^{11}(EZ_1 - X_1Y) + D_i^{11}(FZ_1))(D_j^{11}(X_2D) + D_j^{11}(YZ_2) + D_j^{11}(Z_2))], \\ & D_n^{22}(D) + D_n^{22}(E) + D_n^{22}(F) \\ &= \sum_{i+j=n} [(D_i^{22}(X_1) + D_i^{22}(EZ_1 - X_1Y) + D_i^{22}(FZ_1))(D_j^{22}(X_2D) + D_j^{22}(YZ_2) + D_j^{22}(Z_2))], \\ & D_n^{12}(D) + D_n^{12}(E) + D_n^{12}(F) \\ &= \sum_{i+j=n} [(D_i^{11}(X_1) + D_i^{11}(EZ_1 - X_1Y) + D_i^{11}(FZ_1))(D_j^{12}(X_2D) + D_j^{12}(YZ_2) + D_j^{12}(Z_2)) \\ & \quad + (D_i^{12}(X_1) + D_i^{12}(EZ_1 - X_1Y) + D_i^{12}(FZ_1))(D_j^{22}(X_2D) + D_j^{22}(YZ_2) + D_j^{22}(Z_2))]. \end{aligned} \quad (4.1)$$

Taking  $Z_1 = Z_2 = I_2$ ,  $X_1 = kI_1$  and  $X_2 = \frac{1}{k}I_1$ ,  $k = 1, 2$  respectively in Eq. (4.1), we get

$$\sum_{i+j=n} [D_i^{11}(I_1)D_j^{11}(Y) + D_i^{11}(I_1)D_j^{11}(I_2) - D_i^{11}(Y)D_j^{11}(Y) - D_i^{11}(Y)D_j^{11}(I_2)] = 0.$$

Replacing  $Y$  with  $2Y$  in this equality, we obtain

$$\sum_{i+j=n} [D_i^{11}(I_1)D_j^{11}(Y) - D_i^{11}(Y)D_j^{11}(I_2)] = 0, \quad \sum_{i+j=n} D_i^{11}(I_1)D_j^{11}(I_2) = 0.$$

By induction on  $n$ , we get  $D_n^{11}(I_2) = 0$ , and hence  $\sum_{i+j=n} D_i^{11}(I_1)D_j^{11}(Y) = 0$ . By induction on  $n$  again, we have

$$D_n^{11}(Y) = 0, \quad \forall n \in \mathbb{N}, Y \in \mathcal{A}_{12}. \quad (4.2)$$

Therefore the first equality in Eq. (4.1) becomes into

$$D_n^{11}(D) + D_n^{11}(F) = \sum_{i+j=n} (D_i^{11}(X_1) + D_i^{11}(FZ_1))(D_j^{11}(X_2D) + D_j^{11}(Z_2)). \quad (4.3)$$

Letting  $X_1 = I_1, X_2 = \frac{1}{2}I_1$  respectively in Eq. (4.3), we get  $\sum_{i+j=n} D_i^{11}(I_1)D_j^{11}(Z_2) = 0$ , and hence  $D_n^{11}(Z_2) = 0$ . For any  $Z \in \mathcal{A}_{22}$ , replacing  $Z$  with  $mI_2 - Z$ ,  $m > \|Z\|$  in above equality, we have

$$D_n^{11}(Z) = 0, \quad \forall n \in \mathbb{N}, Z \in \mathcal{A}_{22}. \quad (4.4)$$

Thus by Eq. (4.2), Eq. (4.4) and  $\Omega I = \Omega$ , we get  $D_n^{11}(D) = \sum_{i+j=n} D_i^{11}(D)D_j^{11}(I_1)$ . By induction on  $n$  and by the fact  $D$  is a left separating point of  $\mathcal{A}_{11}$  (if  $D$  is a right separating point of  $\mathcal{A}_{11}$ , applying same arguments to  $I\Omega = \Omega$ ), we have  $D_n^{11}(I_1) = 0$ ,  $n > 0$ .

By letting  $X_1 = X_2 = I_1$ ,  $Z_1 = kI_2$  and  $Z_2 = \frac{1}{k}I_2$ ,  $k = 1, 2$  respectively in Eq. (4.1), we obtain  $\sum_{i+j=n} [D_i^{22}(I_1)D_j^{22}(Y) + D_i^{22}(I_1)D_j^{22}(I_2)] = \sum_{i+j=n} [D_i^{22}(Y)D_j^{22}(Y) + D_i^{22}(Y)D_j^{22}(I_2)]$  for any  $Y \in \mathcal{A}_{12}$ . Replacing  $Y$  with  $2Y$  in this equality, we get  $\sum_{i+j=n} D_i^{22}(I_1)D_j^{22}(I_2) = 0$  and  $\sum_{i+j=n} D_i^{22}(I_1)D_j^{22}(Y) = \sum_{i+j=n} D_i^{22}(Y)D_j^{22}(I_2)$ . Thus by induction on  $n$ , we have

$$D_n^{22}(I_1) = 0, D_n^{22}(Y) = 0, \forall n \in \mathbb{N}, Y \in \mathcal{A}_{12}. \tag{4.5}$$

Therefore the second equality in Eq.(4.1) can be rewritten as

$$D_n^{22}(D) + D_n^{22}(F) = \sum_{i+j=n} (D_i^{22}(X_1) + D_i^{22}(FZ_1))(D_j^{22}(X_2D) + D_j^{22}(Z_2)). \tag{4.6}$$

By letting  $Z_1 = kI_2, Z_2 = \frac{1}{k}I_2$ ,  $k = 1, 2$  in Eq.(4.6) and by similar arguments as above, we get  $\sum_{i+j=n} D_i^{22}(X_1)D_j^{22}(I_2) = 0$ . Thus by induction on  $n$ , we get  $D_n^{22}(X_1) = 0$  for invertible  $X_1 \in \mathcal{A}_{11}$ . For any  $X \in \mathcal{A}_{11}$ , replacing  $X$  with  $mI_1 - X$ ,  $m > \|X\|$  in above equality, we have

$$D_n^{22}(X) = 0, \forall n \in \mathbb{N}, X \in \mathcal{A}_{11}. \tag{4.7}$$

Therefore Eq.(4.6) becomes into  $D_n^{22}(F) = \sum_{i+j=n} D_i^{22}(F)D_j^{22}(I_2)$ . By induction on  $n$  and by the fact  $F$  is a left separating point of  $\mathcal{A}_{22}$  (if  $F$  is a right separating point of  $\mathcal{A}_{22}$ , applying similar arguments to  $I\Omega = \Omega$ ), we have  $D_n^{22}(I_2) = 0, n > 0$ .

By letting  $Z_1 = kI_2$  and  $Z_2 = \frac{1}{k}I_2$ ,  $k = 1, 2$  in Eq.(4.1), we get

$$D_n^{12}(X_1Y) - D_n^{12}(X_1) = \sum_{i+j=n} [D_i^{11}(X_1)D_j^{12}(Y) + D_i^{11}(X_1)D_j^{12}(I_2)]. \tag{4.8}$$

Letting  $Y = 0$  in Eq.(4.8), we have  $\sum_{i+j=n} D_i^{11}(X_1)D_j^{12}(I_2) = -D_n^{12}(X_1)$  and  $D_n^{12}(X_1Y) = \sum_{i+j=n} D_i^{11}(X_1)D_j^{12}(Y)$  for any invertible element  $X_1 \in \mathcal{A}_{11}$  and  $Y \in \mathcal{A}_{12}$ . Letting  $X_1 = I_1$  in this equality, we obtain  $D_n^{12}(I_2) = -D_n^{12}(I_1)$  for any  $n \in \mathbb{N}$ . Thus for any  $X \in \mathcal{A}_{11}, Y \in \mathcal{A}_{12}$ ,

$$D_n^{12}(XY) = \sum_{i+j=n} D_i^{11}(X)D_j^{12}(Y),$$

$$D_n^{12}(X) = - \sum_{i+j=n} D_i^{11}(X)D_j^{12}(I_2) = \sum_{i+j=n} D_i^{11}(X)D_j^{12}(I_1). \tag{4.9}$$

Letting  $X_1 = kI_1$  and  $X_2 = \frac{1}{k}I_2$ ,  $k = 1, 2$  in Eq. (4.1), we get

$$D_n^{12}(YZ_2) + D_n^{12}(Z_2) = - \sum_{i+j=n} [D_i^{12}(I_1)D_j^{22}(Z_2) - D_i^{12}(Y)D_j^{22}(Z_2)].$$

Taking  $Y = 0$  in above equality and by similar arguments as that in the proof of Eq. (4.9), we see

$$D_n^{12}(Z) = - \sum_{i+j=n} D_i^{12}(I_1)D_j^{22}(Z) = \sum_{i+j=n} D_i^{12}(I_2)D_j^{22}(Z), D_n^{12}(YZ) = \sum_{i+j=n} D_i^{12}(Y)D_j^{22}(Z). \tag{4.10}$$

hold for all  $Y \in \mathcal{A}_{12}, Z \in \mathcal{A}_{22}$ . For any  $X_1, X_2 \in \mathcal{A}_{11}$  with  $X_1X_2 = D$ , let  $S = X_1 + E + F$  and  $T = X_2 + I_2$ . Then  $ST = \Omega$  and  $D_n^{11}(D) + D_n^{12}(D) + D_n^{12}(E) + D_n^{12}(F) + D_n^{22}(F) = D_n(ST) = \sum_{i+j=n} D_i(S)D_j(T) = \sum_{i+j=n} [(D_i^{11}(X_1) + D_i^{12}(X_1) + D_i^{12}(E) + D_i^{12}(F) + D_i^{22}(F))(D_j^{11}(X_2) + D_j^{12}(X_2) + D_j^{12}(I_2) + D_j^{22}(I_2))]$ . Thus

$$D_n^{11}(D) = \sum_{i+j=n} D_i^{11}(X_1)D_j^{11}(X_2), \quad \forall X_1, X_2 \in \mathcal{A}_{11} \text{ with } X_1X_2 = D. \tag{4.11}$$

For any  $Z_1, Z_2 \in \mathcal{A}_{22}$  with  $Z_1Z_2 = F$ , let  $S = I_1 + Z_1$  and  $T = D + E + Z_2$ . Then  $ST = \Omega$  and

$$\begin{aligned} D_n^{11}(D) + D_n^{12}(D) + D_n^{12}(E) + D_n^{12}(F) + D_n^{22}(F) &= D_n(ST) \\ &= \sum_{i+j=n} D_i(S)D_j(T) = \sum_{i+j=n} [(D_i^{11}(I_1) + D_i^{12}(I_1) + D_i^{12}(Z_1) + D_i^{22}(Z_1)) \\ &\quad (D_j^{11}(D) + D_j^{12}(D) + D_j^{12}(E) + D_j^{12}(Z_2) + D_j^{22}(Z_2))]. \end{aligned}$$

Thus

$$D_n^{22}(F) = \sum_{i+j=n} D_i^{22}(Z_1)D_j^{22}(Z_2), \quad \forall Z_1, Z_2 \in \mathcal{A}_{22} \text{ with } Z_1Z_2 = F. \tag{4.12}$$

The proof of ‘‘only if’’ part is complete.

For the ‘‘if’’ part, by using similar arguments as that in the proof of Theorem 2.1, it is easy to check that  $D = \{D_n, D_n : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}\}$  is higher derivable at  $\Omega$  if  $D_n$  satisfies the conditions in ‘‘if’’ part of Theorem 4.1. The proof of this theorem is complete.  $\square$

In the case of triangular algebras, by similar arguments as that in the proof of Theorem 3.2, we have

**THEOREM 4.2.** *Let  $\mathcal{T} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$  be a triangular algebra and  $P \in \mathcal{T}$  be the standard idempotent. Suppose  $\Omega \in \mathcal{T}$  such that  $P\Omega P$  is a left (or right) separating point in  $\mathcal{A}$  and  $(I - P)\Omega(I - P)$  is a left (or right) separating point in  $\mathcal{B}$ . Then a family of linear maps  $D = \{D_n, D_n : \mathcal{T} \rightarrow \mathcal{T}, n \in \mathbb{N}\}$  is higher derivable at  $\Omega$  if and only if it is a higher derivation.*

Note that every irreducible CDCSL algebra and nest algebra is a triangular algebra. Thus by Theorem 3.3 and Theorem 4.2, we get

**THEOREM 4.3.** *Suppose  $\mathcal{A}$  is an irreducible CDCSL algebra and  $P \in \mathcal{L}$  is the projection in Lemma 2.3, or  $\mathcal{A}$  is a nest algebra and  $P$  is a nontrivial projection in the nest. Let  $D = \{D_n, D_n : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}\}$  be a family of linear maps. Then the following statements are equivalent:*

- (1)  $D = \{D_n, n \in \mathbb{N}\}$  is higher derivable at  $\Omega$  which satisfies  $\Omega = P\Omega = \Omega P$  and  $P\Omega P$  is a left (or right) separating point of  $P\mathcal{A}P$ .
- (2)  $D = \{D_n, n \in \mathbb{N}\}$  is higher derivable at  $\Omega$  which satisfies  $\Omega = (I - P)\Omega = \Omega(I - P)$  and  $(I - P)\Omega(I - P)$  is a left (or right) separating point of  $(I - P)\mathcal{A}(I - P)$ .

(3)  $D = \{D_n, n \in \mathbb{N}\}$  is higher derivable at  $\Omega$  which satisfies  $P\Omega P$  is a left (or right) separating point of  $P\mathcal{A}P$  and  $(I - P)\Omega(I - P)$  is a left (or right) separating point of  $(I - P)\mathcal{A}(I - P)$ .

(4)  $D = \{D_n, D_n : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}\}$  is a higher derivation.

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Runling An  
Department of Mathematics  
Taiyuan University of Technology  
Taiyuan, 030024, P.R. China  
e-mail: runlingan@aliyun.com

Chunhui Xue, Xu Zhang  
Department of Mathematics  
Taiyuan University of Technology  
Taiyuan, 030024, P.R. China