

## PSEUDOSPECTRUM AND CONDITION SPECTRUM

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*Abstract.* For  $0 < \varepsilon < 1$ , the  $\varepsilon$ -condition spectrum of an element  $A \in \mathbb{C}^{N \times N}$ , a generalization of eigenvalues, is denoted by  $\sigma_\varepsilon(A)$ , and is defined as ([7]),

$$\sigma_\varepsilon(A) := \left\{ z \in \mathbb{C} : zI - A \text{ is not invertible or } \|zI - A\| \|(zI - A)^{-1}\| \geq \frac{1}{\varepsilon} \right\}.$$

Several results on spectrum and  $\varepsilon$ -pseudospectrum are generalized to  $\varepsilon$ -condition spectrum. The  $\varepsilon$ -condition spectrum is a useful tool in the numerical solution of operator equations. In [3], the authors have given an analogue of the Spectral Mapping Theorem for condition spectrum. This paper is a continuation of the papers [5] and [3], generalizing the Spectral Mapping Theorem for eigenvalues. In this paper we are studying size of the components of condition spectrum of a matrix. The main contribution of this paper consists of asymptotic expansions of quantities which determine the size of components of condition spectral sets. A relation connecting pseudospectrum and condition spectrum of a matrix is given as set inclusions. Using this relation a weak version of component wise condition Spectral Mapping Theorem is given. Examples are given to illustrate the theory developed.

### 1. Introduction

Let  $A \in \mathbb{C}^{N \times N}$ . The set of all eigenvalues or spectrum is denoted by  $\sigma(A)$  and is defined as

$$\sigma(A) = \{z \in \mathbb{C} : zI - A \text{ is not invertible}\}.$$

Let  $f$  be an analytic function on some open set  $\Omega$  containing  $\sigma(A)$ . By functional calculus,  $f(A)$  is defined as

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(zI - A)^{-1} dz,$$

where  $\Gamma$  is any closed contour containing  $\sigma(A)$ . The Spectral Mapping Theorem is a fundamental result in functional analysis of great importance. Given a matrix  $A$  and a function  $f$  which is analytic on an open set containing  $\sigma(A)$ , the theorem asserts that

$$f(\sigma(A)) = \sigma(f(A)).$$

There are several generalizations of the concept of the spectrum in literature such as Ransford spectrum [8], pseudospectrum [9], condition spectrum [7], etc. It is natural to ask whether there are any results similar to the Spectral Mapping Theorem for these sets.

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DEFINITION 1. Let  $A \in \mathbb{C}^{N \times N}$  and  $0 < \varepsilon < 1$ . The  $\varepsilon$ -condition spectrum of  $A$  is denoted by  $\sigma_\varepsilon(A)$  and is defined as,

$$\sigma_\varepsilon(A) = \left\{ z \in \mathbb{C} : \|zI - A\| \|(zI - A)^{-1}\| \geq \frac{1}{\varepsilon} \right\},$$

with the convention that  $\|zI - A\| \|(zI - A)^{-1}\| = \infty$ , if  $zI - A$  is not invertible.

Note that because of the above convention,  $\sigma(A) \subseteq \sigma_\varepsilon(A)$  for all  $0 < \varepsilon < 1$ . Whenever  $z \notin \sigma_\varepsilon(A)$ , we are guaranteed a stable solution to the linear system  $(A - zI)x = b$ . This fact makes the  $\varepsilon$ -condition spectrum a useful tool in the numerical solution of operator equations.

In [5], asymptotic expansions of the sizes of components of pseudospectra are given using some tools developed in [1]. In [3], the authors gives an analogue of the Spectral Mapping Theorem for condition spectrum in Banach algebras. We state the main theorem proved in [3], removing an unnecessary assumption there that  $f$  must be injective.

THEOREM 1. (Condition Spectral Mapping Theorem) *Let  $A$  be a complex Banach algebra with unit  $e$ . For  $a \in A$  that is not a scalar multiple of the unit,  $0 < \varepsilon < 1$  sufficiently small,  $\Omega$  a bounded open subset of  $\mathbb{C}$  containing  $\sigma_\varepsilon(a)$  and  $f$  an analytic function on  $\Omega$ , define*

$$\phi(\varepsilon) = \sup_{\lambda \in \sigma_\varepsilon(a)} \left\{ \frac{1}{\|f(\lambda) - f(a)\| \|[f(\lambda) - f(a)]^{-1}\|} \right\}.$$

*If  $f(a)$  is not a scalar multiple of unit, then  $\phi(\varepsilon)$  is well defined,  $0 \leq \phi(\varepsilon) \leq 1$ ,  $\lim_{\varepsilon \rightarrow 0} \phi(\varepsilon) = 0$  and for  $\varepsilon$  satisfying  $\phi(\varepsilon) < 1$ , we have*

$$f(\sigma_\varepsilon(a)) \subseteq \sigma_{\phi(\varepsilon)}(f(a)).$$

*Furthermore suppose there exists  $\varepsilon_0$  with  $0 < \varepsilon_0 < 1$  such that  $\sigma_{\varepsilon_0}(f(a)) \subseteq f(\Omega)$ . For  $0 < \varepsilon \leq \varepsilon_0$  define*

$$\psi(\varepsilon) = \sup_{\mu \in f^{-1}(\sigma_\varepsilon(f(a))) \cap \Omega} \left\{ \frac{1}{\|\mu - a\| \|(\mu - a)^{-1}\|} \right\}.$$

*Then  $\psi(\varepsilon)$  is well defined,  $0 \leq \psi(\varepsilon) \leq 1$ ,  $\lim_{\varepsilon \rightarrow 0} \psi(\varepsilon) = 0$  and for  $0 < \varepsilon \leq \varepsilon_0$  satisfying  $\psi(\varepsilon) < 1$ , we have*

$$\sigma_\varepsilon(f(a)) \subseteq f(\sigma_{\psi(\varepsilon)}(a)).$$

When  $A$  is a normal matrix, its  $\varepsilon$ -condition spectrum is union of closed disks containing eigenvalues. For a non-normal matrix, its  $\varepsilon$ -condition spectrum may be much larger than this union.

Let  $A \in \mathbb{C}^{N \times N}$  with distinct eigenvalues  $\{\lambda_j : j = 1, \dots, k\}$  each having some positive algebraic multiplicity. When  $\varepsilon$  is small,  $\sigma_\varepsilon(A)$  consist of  $k$  disjoint components

each containing an eigenvalue. In the condition Spectral Mapping Theorem, the sizes of condition spectra are characterized by one pair of functions  $\phi$  and  $\psi$ . Our first order business is to characterize each component by functions  $\phi_j$  and  $\psi_j$ , offering a sharper bound than the one in the condition Spectral Mapping Theorem. The functions  $\phi_j$  and  $\psi_j$  are continuous and monotonically non-decreasing and depends only on the eigenvalue  $\lambda_j$ .

The following is the outline of the paper. In Section 2, the general theorem (Theorem 2) in the form of two set inclusions for each component is stated and proved. In the theorem, we derive the exact expression for  $\phi_j$  and  $\psi_j$  mentioned in the above paragraph. We derive the usual Spectral Mapping Theorem from this general theorem (Remark 2). We also consider a normal matrix and analytically find the values of  $\phi_j$  and  $\psi_j$  and illustrate the theory developed (Example 1). In Section 3, we determine the size of each component of condition spectrum of  $f(A)$  for a diagonalizable matrix  $A$  (Corollary 1) and a more general case (Corollary 2). In section 4, we derive the asymptotic expansion of  $f(A)$  for a diagonalizable matrix (Theorem 5) and a more general case (Theorem 6). We give some examples to illustrate the given theory. In section 5, an improved relation connecting pseudospectrum and condition spectrum is proved (Lemma 1). A weak version of the Spectral Mapping Theorem for each components of condition spectrum is proved (Theorem 8) using the component-wise relation connecting pseudospectrum and condition spectrum of a matrix (Theorem 7). Some examples are also given to illustrate the theory.

### 2. A component-wise condition Spectral Mapping Theorem

The following is a sharper version of the condition Spectral Mapping Theorem for complex analytic functions discussed in [3]. The proof is similar as that in [3] for the original theorem and is included here for completeness. The proof is an easy consequence of the definition of the functions in the statement of the theorem.

As already mentioned in the introduction, when  $\varepsilon$  is small,  $\sigma_\varepsilon(A)$  is a disjoint union of sets each containing exactly one eigenvalue. Denote the component containing the eigenvalue  $\lambda_j$  by  $\sigma_\varepsilon(A, \lambda_j)$ . *Throughout this paper, we shall be assuming that the parameter  $\varepsilon$  is sufficiently small so that the components of condition spectral sets are pairwise disjoint.* The value of  $\varepsilon$  may need to be restricted further, this point will be elaborated upon later. In case  $f(\lambda_j) = f(\lambda_k)$  for some  $\lambda_j \neq \lambda_k$ , we identify the two components  $\sigma_\varepsilon(f(A), f(\lambda_j))$  and  $\sigma_\varepsilon(f(A), f(\lambda_k))$ , for all  $0 < \varepsilon < 1$ . Let  $I$  be the identity matrix and  $\mathcal{S} = \{\alpha I : \alpha \in \mathbb{C}\}$ .

**THEOREM 2.** *Let  $A \in \mathbb{C}^{N \times N} \setminus \mathcal{S}$ ,  $\{\lambda_1, \dots, \lambda_k\}$  be distinct eigenvalues of  $A$  and  $f$  be an analytic function defined on an open set  $\Omega$  containing  $\sigma(A)$ . For each  $j$  and  $0 < \varepsilon < 1$  sufficiently small, define*

$$\phi_j(\varepsilon) = \sup_{\xi \in \sigma_\varepsilon(A, \lambda_j)} \left\{ \frac{1}{\|f(\xi)I - f(A)\| \|(f(\xi)I - f(A))^{-1}\|} \right\}.$$

*If  $f$  is a non-constant function then  $\phi_j(\varepsilon)$  is well defined,  $0 \leq \phi_j(\varepsilon) \leq 1$ ,  $\lim_{\varepsilon \rightarrow 0} \phi_j(\varepsilon) = 0$*

and for  $\varepsilon$  satisfying  $\phi_j(\varepsilon) < 1$ , we have

$$f(\sigma_\varepsilon(A, \lambda_j)) \subseteq \sigma_{\phi_j(\varepsilon)}(f(A), f(\lambda_j)).$$

Furthermore, suppose that there exist  $\varepsilon_0$  with  $0 < \varepsilon_0 < 1$  such that  $\sigma_{\varepsilon_0}(f(A)) \subseteq f(\Omega)$ . For  $0 < \varepsilon \leq \varepsilon_0$ , define

$$\psi_j(\varepsilon) = \sup_{\mu \in f^{-1}(\sigma_\varepsilon(f(A), f(\lambda_j))) \cap \Omega} \left\{ \frac{1}{\|\mu I - A\| \|(\mu I - A)^{-1}\|} \right\}.$$

Then  $\psi_j(\varepsilon)$  is well defined,  $\lim_{\varepsilon \rightarrow 0} \psi_j(\varepsilon) = 0$  and for  $\varepsilon$  satisfying  $\psi_j(\varepsilon) < 1$ , we have

$$\sigma_\varepsilon(f(A), f(\lambda_j)) \subseteq f(\sigma_{\psi_j(\varepsilon)}(A, \lambda_j)).$$

*Proof.* First we show that for each  $j$ ,  $\phi_j(\varepsilon)$  is well defined. Define  $g : \mathbb{C} \rightarrow \mathbb{R}$  by,

$$g(z) = \frac{1}{\|f(z)I - f(A)\| \|(f(z)I - f(A))^{-1}\|}.$$

Then  $g$  is continuous [3]. Next for  $0 < \varepsilon < 1$ ,  $\sigma_\varepsilon(A, \lambda_j)$  is a compact subset of  $\mathbb{C}$  and  $\phi_j(\varepsilon) = \sup\{g(z) : z \in \sigma_\varepsilon(A, \lambda_j)\}$ . Hence  $\phi_j(\varepsilon)$  is well defined, that is, finite. It is easy to observe that  $\phi_j$  is a monotonically non-decreasing function and  $\phi_j(\varepsilon)$  goes to zero as  $\varepsilon$  goes to zero. Now let  $\varepsilon$  be sufficiently small so that  $0 < \phi_j(\varepsilon) < 1$  and let  $z \in \sigma_\varepsilon(A, \lambda_j)$ . Then  $g(z) \leq \phi_j(\varepsilon)$ . Hence

$$\|f(z)I - f(A)\| \|(f(z)I - f(A))^{-1}\| = \frac{1}{g(z)} \geq \frac{1}{\phi_j(\varepsilon)}.$$

This means that  $f(z) \in \sigma_{\phi_j(\varepsilon)}(f(A), f(\lambda_j))$ . Thus

$$f(\sigma_\varepsilon(A, \lambda_j)) \subseteq \sigma_{\phi_j(\varepsilon)}(f(A), f(\lambda_j)).$$

Next assume that there exists  $\varepsilon_0$  with  $0 < \varepsilon_0 < 1$  such that  $\sigma_{\varepsilon_0}(f(A)) \subseteq f(\Omega)$ . We show that for each  $j$  and  $0 < \varepsilon \leq \varepsilon_0$ ,  $\psi_j(\varepsilon)$  is well defined. Define  $h : \mathbb{C} \rightarrow \mathbb{R}$  by,

$$h(\mu) = \frac{1}{\|\mu I - A\| \|(\mu I - A)^{-1}\|}.$$

Then  $h$  is continuous [3]. Since  $h(\mu) \leq 1$  for all  $\mu \in \mathbb{C}$ ,  $\psi_j(\varepsilon)$  is well defined and  $0 \leq \psi_j(\varepsilon) \leq 1$ . It is also observed that  $\psi_j$  is a monotonically non-decreasing function and  $\psi_j(\varepsilon)$  goes to zero as  $\varepsilon$  goes to zero. Now let  $\varepsilon$  be sufficiently small so that  $0 < \psi_j(\varepsilon) < 1$ . Let  $z \in \sigma_\varepsilon(f(A), f(\lambda_j)) \subseteq \sigma_{\varepsilon_0}(f(A), f(\lambda_j)) \subseteq f(\Omega)$ . Consider  $\mu \in \Omega$  such that  $z = f(\mu)$ . Then  $\mu \in f^{-1}(\sigma_\varepsilon(f(A), f(\lambda_j)))$ , hence  $h(\mu) \leq \psi_j(\varepsilon)$ , that is,

$$\|\mu I - A\| \|(\mu I - A)^{-1}\| = \frac{1}{h(\mu)} \geq \frac{1}{\psi_j(\varepsilon)}.$$

Thus  $\mu \in \sigma_{\psi_j(\varepsilon)}(A, \lambda_j)$ . Hence  $z = f(\mu) \in f(\sigma_{\psi_j(\varepsilon)}(A, \lambda_j))$ . This proves

$$\sigma_\varepsilon(f(A), f(\lambda_j)) \subseteq f(\sigma_{\psi_j(\varepsilon)}(A, \lambda_j)). \quad \square$$

REMARK 1. Combining the two inclusions, we get

$$f(\sigma_\varepsilon(A, \lambda_j)) \subseteq \sigma_{\phi_j(\varepsilon)}(f(A), f(\lambda_j)) \subseteq f(\sigma_{\psi_j(\phi_j(\varepsilon))}(A, \lambda_j)),$$

and

$$\sigma_\varepsilon(f(A), f(\lambda_j)) \subseteq f(\sigma_{\psi_j(\varepsilon)}(A, \lambda_j)) \subseteq \sigma_{\phi_j(\psi_j(\varepsilon))}(f(A), f(\lambda_j)).$$

NOTE 1. In [3], it is shown by an example that the assumption  $f(A)$  is not a scalar multiple of the identity cannot be dropped in Theorem 2.

REMARK 2. Since for every  $A \in \mathbb{C}^{N \times N} \setminus \mathcal{S}$ ,  $\lim_{\varepsilon \rightarrow 0} \phi_j(\varepsilon) = 0 = \lim_{\varepsilon \rightarrow 0} \psi_j(\varepsilon)$ , we have

$$\{\lambda_j\} = \bigcap_{0 < \varepsilon < 1} \sigma_\varepsilon(A, \lambda_j).$$

Thus the usual Spectral Mapping Theorem can be deduced from Theorem 2.

REMARK 3. Let  $A \in \mathbb{C}^{N \times N} \setminus \mathcal{S}$  and  $f(z) = \alpha + \beta z$  where  $\alpha, \beta$  are complex numbers with  $\beta \neq 0$ . Then

$$\begin{aligned} \phi_j(\varepsilon) &= \sup_{z \in \sigma_\varepsilon(A, \lambda_j)} \frac{1}{\|\beta z I - \beta A\| \|( \beta z I - \beta A )^{-1}\|} \\ &= \sup_{\lambda \in \sigma_\varepsilon(A, \lambda_j)} \frac{1}{\|z I - A\| \|(z I - A)^{-1}\|} \\ &= \varepsilon. \end{aligned}$$

In a similar way we have  $\psi_j(\varepsilon) = \varepsilon$ . Thus  $\sigma_\varepsilon(\alpha I + \beta A, \alpha + \beta \lambda_j) = \alpha + \beta \sigma_\varepsilon(A, \lambda_j)$ .

In the following we consider a  $2 \times 2$  normal matrix and give estimates for the functions  $\phi_j$  and  $\psi_j$ .

EXAMPLE 1. Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . The matrix is normal and everything can be worked out analytically. The eigenvalues are  $\lambda_1 = 1, \lambda_2 = -1$ . Take  $f(z) = z^2 + 2z$ . First consider  $\lambda_1 = 1$ :

$$\begin{aligned} \phi_1(\varepsilon) &= \sup_{\xi \in \sigma_\varepsilon(A, 1)} \|f(\xi)I - f(A)\|^{-1} \|(f(\xi)I - f(A))^{-1}\|^{-1} \\ &= \sup_{\xi \in \sigma_\varepsilon(A, 1)} \left\| \begin{bmatrix} \xi^2 + 2\xi - 3 & 0 \\ 0 & \xi^2 + 2\xi + 1 \end{bmatrix} \right\|^{-1} \left\| \begin{bmatrix} \xi^2 + 2\xi - 3 & 0 \\ 0 & \xi^2 + 2\xi + 1 \end{bmatrix}^{-1} \right\|^{-1} \\ &= \sup_{\xi \in D\left(\frac{1+\varepsilon^2}{1-\varepsilon^2}, \frac{2\varepsilon}{1-\varepsilon^2}\right)} \frac{|\xi^2 + 2\xi - 3|}{|\xi^2 + 2\xi + 1|} = \varepsilon(2 - \varepsilon), \end{aligned}$$

where  $D(z, r)$  is the disk of radius  $r$  and center at  $z$ . Hence  $\sigma_\varepsilon(A, 1)^2 + 2\sigma_\varepsilon(A, 1) \subseteq \sigma_{\phi_1(\varepsilon)}(A^2 + 2A, 3)$ . Also

$$\begin{aligned} \psi_1(\varepsilon) &= \sup_{\mu^2+2\mu \in \sigma_\varepsilon(A^2+2A,3)} \|\mu I - A\|^{-1} \|(\mu I - A)^{-1}\|^{-1} \\ &= \sup_{\mu \in \sigma_\varepsilon(A^2+2A,3)} \left\| \begin{bmatrix} \sqrt{1+\mu} - 2 & 0 \\ 0 & \sqrt{1+\mu} \end{bmatrix} \right\|^{-1} \left\| \begin{bmatrix} \sqrt{1+\mu} - 2 & 0 \\ 0 & \sqrt{1+\mu} \end{bmatrix}^{-1} \right\|^{-1} \\ &= \sup_{\mu \in D\left(\frac{3+\varepsilon^2}{1-\varepsilon^2}, \frac{4\varepsilon}{1-\varepsilon^2}\right)} \frac{|\sqrt{1+\mu} - 2|}{|\sqrt{1+\mu}|} = 1 - \sqrt{1-\varepsilon}. \end{aligned}$$

Hence  $\sigma_\varepsilon(A^2 + 2A, 3) \subseteq \sigma_{\psi_1(\varepsilon)}(A, 1)^2 + 2\sigma_{\psi_1(\varepsilon)}(A, 1)$ . Next, consider the eigenvalue  $\lambda_2 = -1$ .

$$\begin{aligned} \phi_2(\varepsilon) &= \sup_{\xi \in \sigma_\varepsilon(A, -1)} \|f(\xi)I - f(A)\|^{-1} \|(f(\xi)I - f(A))^{-1}\|^{-1} \\ &= \sup_{\xi \in \sigma_\varepsilon(A, -1)} \left\| \begin{bmatrix} \xi^2 + 2\xi - 3 & 0 \\ 0 & \xi^2 + 2\xi + 1 \end{bmatrix} \right\|^{-1} \left\| \begin{bmatrix} \xi^2 + 2\xi - 3 & 0 \\ 0 & \xi^2 + 2\xi + 1 \end{bmatrix}^{-1} \right\|^{-1} \\ &= \sup_{\xi \in D\left(\frac{-(1+\varepsilon^2)}{1-\varepsilon^2}, \frac{2\varepsilon}{1-\varepsilon^2}\right)} \frac{|\xi^2 + 2\xi + 1|}{|\xi^2 + 2\xi - 3|} = \frac{\varepsilon^2}{1-2\varepsilon}. \end{aligned}$$

Hence  $\sigma_\varepsilon(A, -1)^2 + 2\sigma_\varepsilon(A, -1) \subseteq \sigma_{\phi_1(\varepsilon)}(A^2 + 2A, -1)$ . Also

$$\begin{aligned} \psi_2(\varepsilon) &= \sup_{\mu^2+2\mu \in \sigma_\varepsilon(A^2+2A,-1)} \|\mu I - A\|^{-1} \|(\mu I - A)^{-1}\|^{-1} \\ &= \sup_{\mu \in \sigma_\varepsilon(A^2+2A,-1)} \left\| \begin{bmatrix} \sqrt{1+\mu} - 2 & 0 \\ 0 & \sqrt{1+\mu} \end{bmatrix} \right\|^{-1} \left\| \begin{bmatrix} \sqrt{1+\mu} - 2 & 0 \\ 0 & \sqrt{1+\mu} \end{bmatrix}^{-1} \right\|^{-1} \\ &= \sup_{\mu \in D\left(\frac{-(1+3\varepsilon^2)}{1-\varepsilon^2}, \frac{4\varepsilon}{1-\varepsilon^2}\right)} \frac{|\sqrt{1+\mu}|}{|\sqrt{1+\mu} - 2|} = \sqrt{\varepsilon}. \end{aligned}$$

### 3. The size of condition spectral component of $f(A)$

In this section we estimate the size of each component of the condition spectrum of  $A$  and  $f(A)$ , where  $f$  is analytic. The  $N \times N$  identity matrix is denoted by  $I$ . For  $m < N$ , the  $m \times m$  identity matrix is denoted by  $I_m$ . For any set  $S$ , the boundary of the set is denoted by  $\partial S$ . We start our discussion about the size of the condition spectrum components of a diagonalizable matrix near an eigenvalue.

**THEOREM 3.** *Let  $A \in \mathbb{C}^{N \times N}$  and  $\{\lambda_1, \dots, \lambda_k\}$  be the distinct eigenvalues of  $A$ . Let  $\lambda \in \sigma(A)$  be of algebraic multiplicity  $m \geq 1$ . Let  $A$  be diagonalizable and  $A = QDQ^{-1}$*

where

$$D = \begin{bmatrix} \lambda I_m & \\ & D_2 \end{bmatrix}$$

and  $\lambda \notin \sigma(D_2)$ . For  $0 < \varepsilon < 1$  and  $z \in \partial\sigma_\varepsilon(A, \lambda)$ ,

$$|z - \lambda| = \varepsilon \|P\| \left\| Q \begin{bmatrix} 0 & \\ & \lambda I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\| + O(\varepsilon^2),$$

where  $P$  is the projection onto the eigenspace  $\ker(A - \lambda I)$  along the range of  $A - \lambda I$ :

$$P = Q \begin{bmatrix} I_m & \\ & 0 \end{bmatrix} Q^{-1}.$$

*Proof.* Let  $z \in \partial\sigma_\varepsilon(A, \lambda)$ . Observe that

$$\begin{aligned} \|zI - A\| &= \|zI - \lambda I + \lambda I - A\| \\ &= \|\lambda I - A\| + O(|z - \lambda|) \\ &= \left\| Q \begin{bmatrix} 0 & \\ & \lambda I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\| + O(|z - \lambda|). \end{aligned}$$

From Theorem 3.1 of [5],

$$\begin{aligned} \|(zI - A)^{-1}\| &= \left\| Q \begin{bmatrix} (z - \lambda)^{-1} I_m & \\ & (z I_{N-m} - D_2)^{-1} \end{bmatrix} Q^{-1} \right\| \\ &= \frac{\|P\|}{|z - \lambda|} + O(1). \end{aligned}$$

Since  $z \in \partial\sigma_\varepsilon(A, \lambda)$ ,

$$\begin{aligned} \frac{1}{\varepsilon} &= \|zI - A\| \|(zI - A)^{-1}\| \\ &= \frac{\|P\| \left\| Q \begin{bmatrix} 0 & \\ & \lambda I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\|}{|z - \lambda|} + O(1). \end{aligned}$$

This implies that

$$|z - \lambda| = \varepsilon \|P\| \left\| Q \begin{bmatrix} 0 & \\ & \lambda I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\| + O(\varepsilon^2). \quad \square$$

**COROLLARY 1.** Let  $A \in \mathbb{C}^{N \times N}$  and  $\lambda_1, \dots, \lambda_k$  be the distinct eigenvalues of  $A$ . Suppose  $A$  is diagonalizable and  $A = QDQ^{-1}$  for some diagonal  $D$ . Assume  $f$  is analytic on some open set containing  $\sigma(A)$ . Let  $\lambda$  be any eigenvalue of  $A$  and  $\tilde{m}$  be the multiplicity of  $f(\lambda)$  as an eigenvalue of  $f(A)$ . Define

$$\tilde{P} = Q \begin{bmatrix} \lambda I_{\tilde{m}} & \\ & 0 \end{bmatrix} Q^{-1},$$

assuming that all eigenvalues  $\mu$  so that  $f(\mu) = f(\lambda)$  are placed in the first diagonal entries of  $D$ . Let  $0 < s < 1$ . Then for any  $\zeta \in \partial\sigma_s(f(A), f(\lambda))$ ,

$$|\zeta - f(\lambda)| = s \|\tilde{P}\| \left\| Q \begin{bmatrix} 0 \\ f(\lambda)I_{N-\tilde{m}} - f(D_2) \end{bmatrix} Q^{-1} \right\| + O(s^2).$$

*Proof.* Note that

$$f(D) = \begin{bmatrix} f(\lambda)I_{\tilde{m}} \\ f(D_2) \end{bmatrix}$$

where  $D_2$  is diagonal so that  $f(\mu)$  is distinct from  $f(\lambda)$  for any diagonal entry  $\mu$  of  $D_2$ . The result now follows from a direct application of Theorem 3.  $\square$

In Corollary 1, suppose  $\lambda$  is an eigenvalue of multiplicity  $m$ . If  $A$  has an eigenvalue  $\mu$  distinct from  $\lambda$  so that  $f(\mu) = f(\lambda)$ , then  $\tilde{m} > m$ . Otherwise  $\tilde{m} = m$ .

Next we consider a matrix  $A$  belonging to a more general class and study the size of the components of condition spectrum of  $A$  and of  $f(A)$ , where  $f$  is analytic. The index of an eigenvalue is the size of the largest Jordan block associated with the eigenvalue.

**THEOREM 4.** Let  $A \in \mathbb{C}^{N \times N}$  and  $\lambda_1, \dots, \lambda_k$  be the distinct eigenvalues of  $A$ . Let  $\lambda \in \{\lambda_1, \dots, \lambda_k\}$  of index  $m > 1$  and suppose there is exactly one Jordan block associated with  $\lambda$ . Let  $A = QJQ^{-1}$ , where

$$J = \left[ \begin{array}{ccc|c} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ \hline & & & \lambda \end{array} \right] \quad (1)$$

be a Jordan block of  $A$  with first block  $m \times m$  and  $\lambda \notin \sigma(J_2)$ . For  $0 < \varepsilon < 1$  sufficiently small and any  $z \in \partial\sigma_\varepsilon(A, \lambda)$ ,

$$|z - \lambda| = \varepsilon^{1/m} \|N_1^{m-1}\|^{1/m} \left\| N_1 + Q \begin{bmatrix} 0 \\ \lambda I_{N-m} - J_2 \end{bmatrix} Q^{-1} \right\|^{1/m} + O(\varepsilon^{2/m}),$$

where

$$N_1 = Q \left[ \begin{array}{ccc|c} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \hline & & & 0 \end{array} \right] Q^{-1}. \quad (2)$$

*Proof.* Let  $z \in \partial\sigma_\varepsilon(A, \lambda)$ . Observe that

$$\begin{aligned} \|zI - A\| &= \|zI - \lambda I + \lambda I - A\| \\ &= \|\lambda I - A\| + O(|z - \lambda|) \\ &= \left\| N_1 + Q \begin{bmatrix} 0 & \\ & \lambda I_{N-m} - J_2 \end{bmatrix} Q^{-1} \right\| + O(|z - \lambda|). \end{aligned}$$

From Theorem 3.3 of [5],

$$\|(zI - A)^{-1}\| = |z - \lambda|^{-m} \|N_1^{m-1}\| + O(|z - \lambda|^{1-m}).$$

Since  $z \in \partial\sigma_\varepsilon(A, \lambda)$ ,

$$\begin{aligned} \frac{1}{\varepsilon} &= \|zI - A\| \|(zI - A)^{-1}\| \\ &= |z - \lambda|^{-m} \|N_1^{m-1}\| \left\| N_1 + Q \begin{bmatrix} 0 & \\ & \lambda I_{N-m} - J_2 \end{bmatrix} Q^{-1} \right\| + O(|z - \lambda|^{1-m}). \end{aligned}$$

This implies that

$$|z - \lambda|^m = \varepsilon \|N_1^{m-1}\| \left\| N_1 + Q \begin{bmatrix} 0 & \\ & \lambda I_{N-m} - J_2 \end{bmatrix} Q^{-1} \right\| + O(\varepsilon |z - \lambda|),$$

and the result now follows.  $\square$

In the above theorem, we assume for ease of exposition that there is only one Jordan block of size  $m$  for the eigenvalue  $\lambda$ . The result also holds if there are  $k > 1$  such Jordan blocks. In this case first diagonal block in (2) must be replicated  $k$  times.

**COROLLARY 2.** *Assume the hypotheses of the above theorem. Let  $f$  be analytic on some open set containing  $\sigma(A)$  so that  $f'(\lambda) \neq 0$ . Suppose  $f(\lambda) \neq f(\mu)$  for every eigenvalue  $\mu$  of  $A$  distinct from  $\lambda$ . For  $0 < s < 1$  sufficiently small and  $\zeta \in \partial\sigma_s(f(A), f(\lambda))$ ,*

$$|\zeta - f(\lambda)| = |f'(\lambda)|^{1-\frac{1}{m}} \|N_1^{m-1}\|^{1/m} \left\| \begin{bmatrix} 0 & -f'(\lambda) & \dots & -\frac{f^{(m-1)}(\lambda)}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & 0 & -f'(\lambda) \\ \hline & & & 0 \end{bmatrix} \left| \begin{array}{l} \\ \\ \\ f(\lambda)I_{N-m} - f(J_2) \end{array} \right. \right\|^{1/m} s^{1/m} + O(s^{2/m}),$$

where  $N_1$  is defined in (2).

*Proof.* Since  $\zeta \in \partial\sigma_s(f(A), f(\lambda))$ ,

$$\frac{1}{s} = \|\zeta I - f(A)\| \|\zeta I - f(A)\|^{-1} = \|Q(\zeta I - f(J))Q^{-1}\| \|Q(\zeta I - f(J))^{-1}Q^{-1}\|.$$

Let  $J_1$  be the first diagonal block of (1). Recall that

$$f(J_1) = f(\lambda)I_m + f'(\lambda)N_1 + \dots + \frac{f^{(m-1)}(\lambda)}{(m-1)!}N_1^{m-1} = \begin{bmatrix} f(\lambda) & f'(\lambda) & \dots & \frac{f^{(m-1)}(\lambda)}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & f(\lambda) & f'(\lambda) \\ & & & f(\lambda) \end{bmatrix}.$$

Thus we have,

$$\begin{aligned} \|\zeta I - f(A)\| &= \|\zeta I - f(\lambda)I + f(\lambda)I - f(A)\| \\ &= \|f(\lambda)I - f(A)\| + O(|\zeta - f(\lambda)|) \\ &= \left\| \begin{bmatrix} 0 & -f'(\lambda) & \dots & -\frac{f^{(m-1)}(\lambda)}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & 0 & -f'(\lambda) \\ & & & 0 \end{bmatrix} \right\| + O(|\zeta - f(\lambda)|). \end{aligned}$$

From Corollary 3.4 of [5],

$$\|(\zeta I - f(A))^{-1}\| = |f'(\lambda)|^{m-1} \|N_1^{m-1}\| |\zeta - f(\lambda)|^{-m} + O(|\zeta - f(\lambda)|^{1-m}).$$

Thus

$$\frac{1}{s} = |f'(\lambda)|^{m-1} \|N_1^{m-1}\| \left\| \begin{bmatrix} 0 & -f'(\lambda) & \dots & -\frac{f^{(m-1)}(\lambda)}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & 0 & -f'(\lambda) \\ & & & 0 \end{bmatrix} \right\| |\zeta - f(\lambda)|^{-m} + O(|\zeta - f(\lambda)|^{1-m}),$$

from which the desired result follows.  $\square$

We next indicate briefly what happens in case of the hypotheses in the above fail. For instance, assume  $f(\lambda) = f(\mu)$  for some eigenvalue  $\mu$  with largest index  $\tilde{m} > m$ . Suppose  $f'(\mu) \neq 0$ . Then the dominant behavior comes from the Jordan block corresponding to  $\mu$  of dimension  $\tilde{m}$ . In this case, we obtain for  $\zeta \in \partial\sigma_s(f(A), f(\lambda))$ ,

$$|\zeta - f(\lambda)| = |f'(\lambda)|^{1-\frac{1}{\tilde{m}}} \|\tilde{N}^{\tilde{m}-1}\|^{1/\tilde{m}} \left\| \begin{bmatrix} 0 & -f'(\lambda) & \dots & -\frac{f^{(\tilde{m}-1)}(\lambda)}{(\tilde{m}-1)!} \\ & \ddots & \ddots & \vdots \\ & & 0 & -f'(\lambda) \\ & & & 0 \end{bmatrix} \right\|^{1/\tilde{m}} s^{1/\tilde{m}} + O(s^{2/\tilde{m}}),$$

where  $\tilde{N}$  is the nilpotent matrix associated with the Jordan block of  $\mu$  of size  $\tilde{m}$  and  $f(\lambda) \notin \sigma(f(J_2))$ .

Next we assume that the hypotheses of Theorem 4 holds, except that  $f'(\lambda) = 0$  and  $f''(\lambda) \neq 0$ . First assume that the index of  $\lambda$  is odd:  $2k + 1$ . The dominant term

of  $(\zeta I_m - f(J_1))^{-1}$  occurs in the top right corner and is  $2^{-k} f''(\lambda)^k \delta^{-k-1} + O(|\delta|^{-k})$  where  $\delta = \zeta - f(\lambda)$ , see [5]. This leads to, for  $\zeta \in \partial\sigma_s(f(A), f(\lambda))$ ,

$$|\zeta - f(\lambda)| = s^{1/(k+1)} \left( \frac{|f''(\lambda)|}{2} \right)^{k/(k+1)} \|N_1^{m-1}\|^{1/(k+1)} \alpha^{1/(k+1)} + O(s^{2/(k+1)}),$$

where  $\alpha := \left\| \left[ \begin{array}{ccc|c} 0 & -f'(\lambda) & \dots & -\frac{f^{(m-1)}(\lambda)}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & 0 & -f'(\lambda) \\ \hline & & & 0 \end{array} \right] \begin{array}{c} \\ \\ \\ f(\lambda)I_{N-m} - f(J_2) \end{array} \right\|$ .

If the index of  $\lambda$  is even:  $m = 2k$ , then the dominant term of  $(\zeta I - f(A))^{-1}$  is  $O(|\delta|^{-k})$  and it occurs at the  $(1, m - 1), (2, m)$  and  $(1, m)$  entries of the matrix if  $m \geq 4$ . If  $m = 2$ , then the dominant term occurs at the  $(1, 2)$  entry ([5]).

### 4. Asymptotic expansions

In this section, we give asymptotic expansions for the functions  $\phi_j$  and  $\psi_j$  in Theorem 2. We first discuss the case of a diagonalizable matrix.

**THEOREM 5.** *Let  $A \in \mathbb{C}^{N \times N} \setminus \mathcal{A}$  and  $\lambda_1, \dots, \lambda_k$  be the distinct eigenvalues of  $A$ . Suppose the algebraic multiplicity of  $\lambda_1$  is  $m \geq 1$  and  $A = QDQ^{-1}$  where  $D$  is diagonal:*

$$D = \begin{bmatrix} \lambda_1 I_m & \\ & D_2 \end{bmatrix}$$

and  $D_2$  is diagonal with  $\lambda_1 \notin \sigma(D_2)$ . Let  $D_3$  be diagonal whose diagonal entries are the diagonal entries  $\mu$  of  $D$  so that  $f(\lambda_1) \neq f(\mu)$ . Let  $f$  be an analytic function in some open set containing  $\sigma(A)$ . For small  $0 < \varepsilon < 1$ ,

$$\phi_1(\varepsilon) = \begin{cases} \varepsilon \frac{\|f'(\lambda_1)\| \|P\| \left\| Q \begin{bmatrix} 0 & \\ & \lambda_1 I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\|}{\|\tilde{P}\| \left\| Q \begin{bmatrix} 0 & \\ & f(\lambda_1) I_{N-\tilde{m}} - f(D_3) \end{bmatrix} Q^{-1} \right\|} + O(\varepsilon^2), & f'(\lambda_1) \neq 0; \\ \frac{\varepsilon^2 \frac{|f''(\lambda_1)|}{2} \|P\|^2 \left\| Q \begin{bmatrix} 0 & \\ & \lambda_1 I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\|^2}{\|\tilde{P}\| \left\| Q \begin{bmatrix} 0 & \\ & f(\lambda_1) I_{N-\tilde{m}} - f(D_3) \end{bmatrix} Q^{-1} \right\|} + O(\varepsilon^3), & f'(\lambda_1) = 0, f''(\lambda_1) \neq 0; \end{cases}$$

where  $P$  and  $\tilde{P}$  are as defined in Theorem 3 and Corollary 1, respectively, and  $\tilde{m}$  is the algebraic multiplicity of  $f(\lambda_1)$  as an eigenvalue of  $f(A)$ . For  $0 < s < 1$  sufficiently

small,

$$\psi_1(s) = \begin{cases} \frac{s \|\tilde{P}\| \left\| Q \begin{bmatrix} 0 \\ f(\lambda_1)I_{N-\tilde{m}} - f(D_3) \end{bmatrix} Q^{-1} \right\|}{\|f'(\lambda_1)\| \|P\| \left\| Q \begin{bmatrix} 0 \\ \lambda_1 I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\|} + O(s^2), & f'(\lambda_1) \neq 0; \\ \frac{s^{1/2} \|\tilde{P}\|^{1/2} \left\| Q \begin{bmatrix} 0 \\ f(\lambda_1)I_{N-\tilde{m}} - f(D_3) \end{bmatrix} Q^{-1} \right\|^{1/2}}{\sqrt{\frac{|f''(\lambda_1)|}{2}} \|P\| \left\| Q \begin{bmatrix} 0 \\ \lambda_1 I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\|} + O(s), & f'(\lambda_1) = 0, f''(\lambda_1) \neq 0. \end{cases}$$

*Proof.* By definition,

$$\begin{aligned} \phi_1(\varepsilon) &= \sup_{z \in \sigma_\varepsilon(A, \lambda_1)} \frac{1}{\|f(z)I - f(A)\| \|(f(z)I - f(A))^{-1}\|} \\ &= \sup_{z \in \sigma_\varepsilon(A, \lambda_1)} \frac{1}{\|Q(f(z)I - f(D))Q^{-1}\| \|Q(f(z)I - f(D))^{-1}Q^{-1}\|}. \end{aligned}$$

Define  $\eta = f(z) - f(\lambda_1)$  for  $z \in \partial\sigma_\varepsilon(A, \lambda_1)$ . Note that

$$f(z)I - f(D) = \begin{bmatrix} \eta I_{\tilde{m}} & \\ & f(\lambda_1)I_{N-\tilde{m}} - f(D_3) \end{bmatrix} + \begin{bmatrix} 0 & \\ & \eta I_{N-\tilde{m}} \end{bmatrix}.$$

Hence

$$\|Q(f(z)I - f(D))Q^{-1}\| \|Q(f(z)I - f(D))^{-1}Q^{-1}\| = \frac{\|\tilde{P}\| \left\| Q \begin{bmatrix} 0 \\ f(\lambda_1)I_{N-\tilde{m}} - f(D_3) \end{bmatrix} Q^{-1} \right\|}{|\eta|} + O(1).$$

If  $f'(\lambda_1) \neq 0$ , then

$$\eta = f'(\lambda_1)(z - \lambda_1) + O(|z - \lambda_1|^2) = \varepsilon f'(\lambda_1) \|P\| \left\| Q \begin{bmatrix} 0 \\ \lambda_1 I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\| + O(\varepsilon^2),$$

by Theorem 3. Hence,

$$\phi_1(\varepsilon) = \frac{\varepsilon |f'(\lambda_1)| \|P\| \left\| Q \begin{bmatrix} 0 \\ \lambda_1 I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\|}{\|\tilde{P}\| \left\| Q \begin{bmatrix} 0 \\ f(\lambda_1)I_{N-\tilde{m}} - f(D_3) \end{bmatrix} Q^{-1} \right\|} + O(\varepsilon^2).$$

Now assume that  $f'(\lambda_1) = 0$  and  $f''(\lambda_1) \neq 0$ . Then

$$\eta = \frac{f''(\lambda_1)(z - \lambda_1)^2}{2} + O(|z - \lambda_1|^3).$$

The expansion for  $\phi_1(\varepsilon)$  follows easily from Theorem 3.

Next we find the asymptotic expansion for  $\psi_1$  assuming  $f'(\lambda_1) \neq 0$ . Let  $\zeta_1 = f(\lambda_1)$  and  $\zeta = f(z)$  for  $z \in \partial\sigma_r(A, \lambda_1)$  for some small  $0 < r < 1$ . The inverse function

theorem states that the inverse of  $f$  is well defined near  $\zeta_1$ . We define  $f^{-1}(\zeta)$  as the unique element near  $\lambda_1$ . Let  $\delta = f^{-1}(\zeta) - f^{-1}(\zeta_1)$ . By definition,

$$\begin{aligned}
 \psi_1(s) &= \sup_{z \in f^{-1}(\sigma_s(f(A), \zeta_1))} \frac{1}{\|zI - A\| \|(zI - A)^{-1}\|} \\
 &= \sup_{\zeta \in \sigma_s(f(A), \zeta_1)} \frac{1}{\|Q(f^{-1}(\zeta)I - D)Q^{-1}\| \|Q(f^{-1}(\zeta)I - D)^{-1}Q^{-1}\|} \\
 &= \sup_{\zeta \in \sigma_s(f(A), \zeta_1)} \left\| Q \left( f^{-1}(\zeta)I - \begin{bmatrix} \lambda_1 I_m & \\ & D_2 \end{bmatrix} \right) Q^{-1} \right\|^{-1} \left\| Q \left( f^{-1}(\zeta)I - \begin{bmatrix} \lambda_1 I_m & \\ & D_2 \end{bmatrix} \right)^{-1} Q^{-1} \right\|^{-1} \\
 &= \sup_{\zeta \in \sigma_s(f(A), \zeta_1)} \left\| Q \left( \begin{bmatrix} \delta I_m & \\ & \lambda_1 I_{N-m} - D_2 + \delta I_{N-m} \end{bmatrix} \right) Q^{-1} \right\|^{-1} \\
 &\quad \times \left\| Q \left( \begin{bmatrix} \delta^{-1} I_m & \\ & (\lambda_1 I_{N-m} - D_2 + \delta I_{N-m})^{-1} \end{bmatrix} \right) Q^{-1} \right\|^{-1} \\
 &= \sup_{\zeta \in \sigma_s(f(A), \zeta_1)} \frac{|\delta|}{\|P\| \left\| Q \begin{bmatrix} 0 & \\ & \lambda_1 I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\|} + O(|\delta|^2) \\
 &= \sup_{\zeta \in \sigma_s(f(A), \zeta_1)} \frac{|\zeta - \zeta_1|}{|f'(\lambda_1)| \|P\| \left\| Q \begin{bmatrix} 0 & \\ & \lambda_1 I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\|} + O(|\zeta - \zeta_1|^2) \\
 &= \frac{s \|\bar{P}\| \left\| Q \begin{bmatrix} 0 & \\ & f(\lambda_1) I_{N-m} - f(D_3) \end{bmatrix} Q^{-1} \right\|}{|f'(\lambda_1)| \|P\| \left\| Q \begin{bmatrix} 0 & \\ & \lambda_1 I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\|} + O(s^2).
 \end{aligned}$$

In the above we use the fact that  $\delta = \frac{\zeta - \zeta_1}{f'(\lambda_1)} + O(|\zeta - \zeta_1|^2)$  and Corollary 1. Now assume that  $f'(\lambda_1) = 0$  and  $f''(\lambda_1) \neq 0$ . Note that

$$\zeta - \zeta_1 = f(z) - f(\lambda_1) = \frac{f''(\lambda_1)(z - \lambda_1)^2}{2} + O(|z - \lambda_1|^3).$$

Given  $\zeta$  in a small neighborhood of  $f(\lambda_1)$ , there are elements  $z_{\pm} = f^{-1}(\zeta)$  in a small neighborhood of  $\lambda_1$ . They satisfy

$$|z_{\pm} - \lambda_1| = \frac{|\zeta - \zeta_1|^{1/2}}{\sqrt{|f''(\lambda_1)|}/2} + O(|\zeta - \zeta_1|).$$

Consequently,

$$\begin{aligned}
 \psi_1(s) &= \sup_{\zeta \in \sigma_s(f(A), \zeta_1)} \max_{z_{\pm} \in f^{-1}(\zeta)} \frac{1}{\|z_{\pm}I - A\| \|(z_{\pm}I - A)^{-1}\|} \\
 &= \sup_{\zeta \in \sigma_s(f(A), \zeta_1)} \max_{z_{\pm} \in f^{-1}(\zeta)} \frac{1}{\left\| Q \begin{bmatrix} 0 & \\ & \lambda_1 I_{N-m} - D_2 \end{bmatrix} Q^{-1} + (z_{\pm} - \lambda_1)I \right\|} \\
 &\quad \times \frac{1}{\left\| Q \begin{bmatrix} (z_{\pm} - \lambda_1)I_m & \\ & \lambda_1 I_{N-m} - D_2 + (z_{\pm} - \lambda_1)I \end{bmatrix}^{-1} Q^{-1} \right\|}
 \end{aligned}$$

$$\begin{aligned}
 &= \sup_{\zeta \in \sigma_\varepsilon(f(A), \zeta_1)} \max_{z_\pm \in f^{-1}(\zeta)} \frac{1}{\left\| Q \begin{bmatrix} 0 & \\ & \lambda_1 I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\|} \cdot \frac{|z_\pm - \lambda_1|}{\|P\|} + O(|z_\pm - \lambda_1|^2) \\
 &= \sup_{\zeta \in \sigma_\varepsilon(f(A), \zeta_1)} \frac{|\zeta - \zeta_1|^{1/2}}{\sqrt{\frac{|f''(\lambda_1)|}{2}} \|P\| \left\| Q \begin{bmatrix} 0 & \\ & \lambda_1 I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\|} + O(|\zeta - \zeta_1|) \\
 &= \frac{s^{1/2} \|\bar{P}\|^{1/2} \left\| Q \begin{bmatrix} 0 & \\ & f(\lambda_1) I_{N-m} - f(D_3) \end{bmatrix} Q^{-1} \right\|^{1/2}}{\sqrt{\frac{|f''(\lambda_1)|}{2}} \|P\| \left\| Q \begin{bmatrix} 0 & \\ & \lambda_1 I_{N-m} - D_2 \end{bmatrix} Q^{-1} \right\|} + O(s),
 \end{aligned}$$

using Corollary 1.  $\square$

REMARK 4. An immediate consequence of the above theorem is that

$$\phi_1(\psi_1(s)) = \begin{cases} s + O(s^2), & f'(\lambda_1) \neq 0; \\ s + O(s^{3/2}), & f'(\lambda_1) = 0, f''(\lambda_1) \neq 0; \end{cases}$$

and

$$\psi_1(\phi_1(\varepsilon)) = \varepsilon + O(\varepsilon^2),$$

as long as  $f'(\lambda_1)$  and  $f''(\lambda_1)$  are not both zero.

EXAMPLE 2. Consider the Example 1 again, where  $\lambda_1 = 1, \lambda_2 = -1$  and  $f(z) = z^2 + 2z$ . We have  $f'(\lambda_1) \neq 0$  and

$$\phi_1(\varepsilon) = 2\varepsilon - \varepsilon^2 \quad \text{and} \quad \psi_1(\varepsilon) = 1 - \sqrt{1 - \varepsilon} = \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8} + \dots$$

Thus the results agree with Theorem 5. Note  $f'(\lambda_2) = 0, f''(\lambda_2) \neq 0$  and

$$\phi_2(\varepsilon) = \varepsilon^2 + 2\varepsilon^3 + \dots \quad \text{and} \quad \psi_2(\varepsilon) = \varepsilon^{1/2}.$$

Thus the results agree with Theorem 5.

THEOREM 6. Let  $\lambda$  be an eigenvalue of the matrix  $A \in \mathbb{C}^{N \times N}$  of index  $m \geq 2$  and  $A = QJQ^{-1}$ , where  $J$  is a Jordan form defined in (1). Let  $f$  be a function analytic in some open set containing  $\sigma(A)$  and satisfying  $f'(\lambda) \neq 0$ . Suppose  $f(\lambda) \neq f(\mu)$  for any other eigenvalue  $\mu$  distinct from  $\lambda$ . For small  $0 < \varepsilon < 1$ ,

$$\phi_1(\varepsilon) = \frac{\left\| N_1 + Q \begin{bmatrix} 0 & \\ & \lambda I_{N-m} - J_2 \end{bmatrix} Q^{-1} \right\|}{\left\| \begin{array}{c|c} 0 - f'(\lambda) \cdots - \frac{f^{(m-1)}(\lambda)}{(m-1)!} & \\ \hline \ddots & \vdots \\ 0 & -f'(\lambda) \\ & 0 \end{array} \right\|} |f'(\lambda)| \varepsilon + O(\varepsilon^{1+\frac{1}{m}}),$$

where  $N_1$  has been defined in (2). For small  $0 < s < 1$ ,

$$\psi_1(s) = \frac{\left\| \begin{bmatrix} 0 & -f'(\lambda) & \cdots & -\frac{f^{(m-1)}(\lambda)}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & 0 & -f'(\lambda) \\ & & & 0 \end{bmatrix} \right\|}{\left\| N_1 + Q \begin{bmatrix} 0 & \\ & \lambda I_{N-m} - J_2 \end{bmatrix} Q^{-1} \right\|} \frac{s}{|f'(\lambda)|} + O(s^{1+\frac{1}{m}}).$$

*Proof.* Let  $\delta = f(z) - f(\lambda) = f'(\lambda)(z - \lambda) + O(|z - \lambda|^2)$  for  $z \in \partial\sigma_\varepsilon(A, \lambda)$ . By definition

$$\begin{aligned} \phi_1(\varepsilon) &= \sup_{z \in \sigma_\varepsilon(A, \lambda)} \frac{1}{\|f(z)I - f(A)\| \| (f(z)I - f(A))^{-1} \|} \\ &= \sup_{z \in \sigma_\varepsilon(A, \lambda)} \frac{1}{\|Q(f(z)I - f(J))Q^{-1}\| \|Q(f(z)I - f(J))^{-1}Q^{-1}\|} \\ &= \sup_{z \in \sigma_\varepsilon(A, \lambda)} \frac{|\delta|^m}{|f'(\lambda)|^{m-1} \|N_1^{m-1}\| \left\| \begin{bmatrix} 0 & -f'(\lambda) & \cdots & -\frac{f^{(m-1)}(\lambda)}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & 0 & -f'(\lambda) \\ & & & 0 \end{bmatrix} \right\|} + O(|\delta|^{m+1}) \\ &= \sup_{z \in \sigma_\varepsilon(A, \lambda)} \frac{|f'(\lambda)(z - \lambda)|^m}{|f'(\lambda)|^{m-1} \|N_1^{m-1}\| \left\| \begin{bmatrix} 0 & -f'(\lambda) & \cdots & -\frac{f^{(m-1)}(\lambda)}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & 0 & -f'(\lambda) \\ & & & 0 \end{bmatrix} \right\|} + O(|z - \lambda|^{m+1}) \\ &= \frac{\left\| N_1 + Q \begin{bmatrix} 0 & \\ & \lambda I_{N-m} - J_2 \end{bmatrix} Q^{-1} \right\|}{\left\| \begin{bmatrix} 0 & -f'(\lambda) & \cdots & -\frac{f^{(m-1)}(\lambda)}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & 0 & -f'(\lambda) \\ & & & 0 \end{bmatrix} \right\|} |f'(\lambda)|\varepsilon + O(\varepsilon^{1+\frac{1}{m}}), \end{aligned}$$

by Theorem 4.

Next, we find an asymptotic expansion for  $\psi_1(s)$ . Let  $\delta = f^{-1}(\zeta) - f^{-1}(\zeta_1)$  where  $\zeta \in \partial\sigma_s(f(A), \zeta_1)$  and  $\zeta_1 = f(\lambda)$ . Again,  $f^{-1}(\zeta)$  is unique in a small neigh-

borhood of  $\lambda$ . Now

$$\begin{aligned}
 \psi_1(s) &= \sup_{z \in f^{-1}(\sigma_s(f(A), \zeta_1))} \frac{1}{\|zI - A\| \|(zI - A)^{-1}\|} \\
 &= \sup_{\zeta \in \sigma_s(f(A), \zeta_1)} \frac{1}{\|Q(f^{-1}(\zeta)I - J)Q^{-1}\| \|Q(f^{-1}(\zeta)I - J)^{-1}Q^{-1}\|} \\
 &= \sup_{\zeta \in \sigma_s(f(A), \zeta_1)} \frac{|\delta|^m}{|f'(\lambda)|^m \|N_1^{m-1}\| \left\| \begin{bmatrix} N_1 + Q \begin{bmatrix} 0 & \\ & \lambda I_{N-m} - J_2 \end{bmatrix} Q^{-1} \end{bmatrix} \right\|} + O(|\delta|^{m+1}) \\
 &= \sup_{\zeta \in \sigma_s(f(A), \zeta_1)} \frac{|\zeta - \zeta_1|^m}{|f'(\lambda)|^m \|N_1^{m-1}\| \left\| \begin{bmatrix} N_1 + Q \begin{bmatrix} 0 & \\ & \lambda I_{N-m} - J_2 \end{bmatrix} Q^{-1} \end{bmatrix} \right\|} + O(|\zeta - \zeta_1|^{m+1}) \\
 &= \frac{\left\| \begin{bmatrix} 0 & -f'(\lambda) & \cdots & -\frac{f^{(m-1)}(\lambda)}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & 0 & -f'(\lambda) \\ & & & 0 \end{bmatrix} \right\|}{\left\| \begin{bmatrix} N_1 + Q \begin{bmatrix} 0 & \\ & \lambda I_{N-m} - J_2 \end{bmatrix} Q^{-1} \end{bmatrix} \right\|} \frac{s}{|f'(\lambda)|} + O(s^{1+\frac{1}{m}}),
 \end{aligned}$$

using Corollary 2.  $\square$

An immediate corollary of the above theorem is that

$$\psi_1(\phi_1(\varepsilon)) = \varepsilon + O(\varepsilon^{1+\frac{1}{m}}) \text{ and } \phi_1(\psi_1(s)) = s + O(s^{1+\frac{1}{m}}).$$

Again for the ease of exposition, we assumed that there is only one Jordan block corresponding to  $\lambda$  of size  $m$ . The result also holds in the general case of  $k \geq 1$  such Jordan blocks. Using the fact discussed immediately following Corollary 2, a similar analysis also works for the other cases where  $f(\lambda) = f(\mu)$  for  $\lambda \neq \mu$  or when  $f'(\lambda) = 0$ .

EXAMPLE 3. Let  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ . The eigenvalue  $\lambda_1 = 0$  has index  $m = 2$ . Take  $f(z) = z^2 + z$ . Observe that  $f'(0) = 1$  and  $f(A) = A$ . It is easy to check that

$$\sigma_\varepsilon(A, 0) = \sigma_\varepsilon(f(A), f(0)) = \left\{ z \in \mathbb{C} : |z| \leq \frac{\sqrt{\varepsilon(1+2\varepsilon)}}{1-\varepsilon} \right\}.$$

Now

$$\begin{aligned}
 \phi_1(\varepsilon) &= \sup_{z \in \sigma_\varepsilon(A,0)} \|f(z)I - f(A)\|^{-1} \|(f(z)I - f(A))^{-1}\|^{-1} \\
 &= \sup_{|z| \leq \frac{\sqrt{\varepsilon(1+2\varepsilon)}}{1-\varepsilon}} \frac{2|z^2 + z|^2}{1 + 2|z^2 + z| + \sqrt{1 + 4|z^2 + z|^2}} \\
 &= \sup_{|z| \leq \frac{\sqrt{\varepsilon(1+2\varepsilon)}}{1-\varepsilon}} \frac{2|z^2 + z|^2}{1 + 2|z^2 + z| + 1 + 2|z^2 + z|^2 + \dots} \\
 &= \sup_{|z| \leq \frac{\sqrt{\varepsilon(1+2\varepsilon)}}{1-\varepsilon}} |z|^2|z + 1|^2 + \dots \\
 &= \varepsilon + O(\varepsilon^{3/2}).
 \end{aligned}$$

In the calculation of  $\psi_1$  below, let  $z = f^{-1}(\zeta) = (-1 + \sqrt{1 + 4\zeta})/2 \approx \zeta - \zeta^2$  for a small  $\zeta$ .

$$\begin{aligned}
 \psi_1(s) &= \sup_{z \in f^{-1}(\sigma_s(f(A),\zeta_1))} \|zI - A\|^{-1} \|(zI - A)^{-1}\|^{-1} \\
 &= \sup_{\zeta \in \sigma_s(A,0)} \|(\zeta - \zeta^2)I - A\|^{-1} \|((\zeta - \zeta^2)I - A)^{-1}\|^{-1} + O(|\zeta|^3) \\
 &= \sup_{|\zeta| \leq \frac{\sqrt{s(1+2s)}}{1-s}} \frac{2|\zeta - \zeta^2|^2}{1 + 2|\zeta - \zeta^2| + \sqrt{1 + 4|\zeta - \zeta^2|^2}} + O(|\zeta|^3) \\
 &= \sup_{|\zeta| \leq \frac{\sqrt{s(1+2s)}}{1-s}} \frac{2|\zeta - \zeta^2|^2}{1 + 2|\zeta - \zeta^2| + 1 + 2|\zeta - \zeta^2|^2 + \dots} + O(|\zeta|^3) \\
 &= s + O(s^{3/2}).
 \end{aligned}$$

This example illustrates the correctness of Theorem 6.

### 5. Component-wise Weak Spectral Mapping Theorem

In [2], a relation connecting pseudospectrum and condition spectrum of an element in a complex unital Banach algebra is given as set inclusions. In this section we give an improved estimate on the relation between pseudospectrum and condition spectrum in the matrix algebra (Lemma 1). For a sufficiently small  $\varepsilon$ , a relation between each component of pseudospectra and condition spectra near an eigenvalue of a matrix is also given (Theorem 7). The functions  $\phi_j$  and  $\psi_j$  defined in the Theorem 1 are continuous and monotonically non-decreasing but it appears to be difficult to find the values of these functions explicitly. In this section, we replace these functions  $\phi_j$ ,  $\psi_j$  with the functions  $\gamma_j$ ,  $\delta_j$ , respectively, that are relatively easier to estimate. Using these functions, an analogue of the Spectral Mapping Theorem for the components of condition spectrum is given (Theorem 8). We also give some examples to illustrate the

theory. For  $A \in \mathbb{C}^{N \times N}$ , the minimum singular value of  $A$  is denoted by  $s_{\min}(A)$ . For  $\varepsilon \geq 0$  the  $\varepsilon$ -pseudospectrum of  $A \in \mathbb{C}^{N \times N}$  is denoted by  $\Lambda_\varepsilon(A)$  and is defined as

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : \|(zI - A)^{-1}\| \geq \varepsilon^{-1}\} = \{z \in \mathbb{C} : s_{\min}(zI - A) \leq \varepsilon\}.$$

The component of  $\Lambda_\varepsilon(A)$  near the eigenvalue  $\lambda_j$  is denoted by  $\Lambda_\varepsilon(A, \lambda_j)$ .

LEMMA 1. Let  $A \in \mathbb{C}^{N \times N}$  and  $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$  denote the distinct eigenvalues of  $A$ . Define  $\rho := \max_j \|\lambda_j I - A\|$  and  $\alpha(A) = \sup_{z \in \sigma_\varepsilon(A)} \frac{\min_j |z - \lambda_j|}{s_{\min}(zI - A)}$ . Then for  $0 < \varepsilon < 1/\alpha(A)$ ,

$$\Lambda_{\frac{\varepsilon\rho}{2}}(A) \subseteq \sigma_\varepsilon(A) \subseteq \Lambda_{\frac{\varepsilon\rho}{1-\varepsilon\alpha(A)}}(A).$$

*Proof.* By definition, for  $0 < \varepsilon < 1$ ,  $\sigma_\varepsilon(A) := \left\{z \in \mathbb{C} : \|zI - A\| \|(zI - A)^{-1}\| \geq \frac{1}{\varepsilon}\right\}$ .

Thus,

$$\begin{aligned} \sigma_\varepsilon(A) &= \{z \in \mathbb{C} : s_{\min}(zI - A) \leq \varepsilon \|zI - A\|\} \\ &\subseteq \left\{z \in \mathbb{C} : s_{\min}(zI - A) \leq \varepsilon (\min_j |z - \lambda_j| + \|\lambda_j I - A\|)\right\} \\ &\subseteq \left\{z \in \mathbb{C} : s_{\min}(zI - A) \leq \varepsilon (\min_j |z - \lambda_j| + \max_j \|\lambda_j I - A\|)\right\}. \end{aligned}$$

Note that  $\alpha(A) \geq 1$  and

$$\begin{aligned} \sigma_\varepsilon(A) &\subseteq \{z \in \mathbb{C} : s_{\min}(zI - A) \leq \varepsilon (\rho + \alpha(A) s_{\min}(zI - A))\} \\ &= \left\{z \in \mathbb{C} : s_{\min}(zI - A) \leq \frac{\varepsilon\rho}{1 - \alpha(A)\varepsilon}\right\}. \end{aligned}$$

Thus for  $0 < \varepsilon < 1/\alpha(A)$ ,

$$\sigma_\varepsilon(A) \subseteq \Lambda_{\frac{\varepsilon\rho}{1-\varepsilon\alpha(A)}}(A).$$

Let  $z \in \Lambda_{\frac{\varepsilon\rho}{2}}(A)$ . For all  $j = 1, 2, \dots, k$ ,

$$\|\lambda_j I - A\| \leq |\lambda_j - z| + \|zI - A\|.$$

Therefore

$$\|zI - A\| \geq \max_j \|\lambda_j I - A\| - |\lambda_j - z|$$

or

$$\max_j \|\lambda_j I - A\| \leq \|zI - A\| + |z - \lambda_j| \leq 2 \|zI - A\|.$$

Thus

$$\|zI - A\| \geq \frac{\rho}{2}$$

and

$$\|(zI - A)^{-1}\| \|zI - A\| \geq \frac{2}{\varepsilon\rho} \cdot \frac{\rho}{2} = \frac{1}{\varepsilon}.$$

Therefore

$$\sigma_\varepsilon(A) \supseteq \Lambda_{\frac{\varepsilon\rho}{2}}(A). \quad \square$$

The following theorem gives a relation connecting each component of pseudospectrum and condition spectrum of a matrix near an eigenvalue. Let  $D(z, r)$  denote the closed disk in the complex plane with center  $z$  and radius  $r$ .

Fix  $\lambda_j \in \sigma(A)$ . We know that there are positive  $\varepsilon_j$  and  $r_j = r_j(\varepsilon_j)$  so that for all  $0 < \varepsilon \leq \varepsilon_j$ ,  $\sigma_\varepsilon(A, \lambda_j) \subseteq D(\lambda_j, r_j)$  and  $D(\lambda_j, r_j)$  has a trivial intersection with all other components of the condition spectrum:  $D(\lambda_j, r_j) \cap \sigma_\varepsilon(A, \lambda_q) = \varnothing$ , for all  $q \neq j$ .

**THEOREM 7.** *Let  $A \in \mathbb{C}^{N \times N}$  and  $\lambda_j \in \sigma(A)$ . Define  $\alpha_j(A) = \sup_{z \in \sigma_\varepsilon(A, \lambda_j)} \frac{|z - \lambda_j|}{s_{\min}(zI - A)}$ .*

Then for  $0 < \varepsilon < 1/\min\{\alpha_j(A), \varepsilon_j\}$ ,

$$\Lambda_{\frac{\varepsilon\|\lambda_j I - A\|}{2}}(A, \lambda_j) \subseteq \sigma_\varepsilon(A, \lambda_j) \subseteq \Lambda_{\frac{\varepsilon\|\lambda_j I - A\|}{1 - \varepsilon\alpha_j(A)}}(A, \lambda_j).$$

*Proof.* Let  $r_j$  be as defined in the paragraph before the statement of this theorem. Then

$$\begin{aligned} \sigma_\varepsilon(A, \lambda_j) &= \{z \in D(\lambda_j, r_j) : s_{\min}(zI - A) \leq \varepsilon \|zI - A\|\} \\ &\subseteq \{z \in D(\lambda_j, r_j) : s_{\min}(zI - A) \leq \varepsilon (|z - \lambda_j| + \|\lambda_j I - A\|)\}. \end{aligned}$$

Note that  $\alpha_j(A) \geq 1$  and

$$\begin{aligned} \sigma_\varepsilon(A, \lambda_j) &\subseteq \{z \in D(\lambda_j, r_j) : s_{\min}(zI - A) \leq \varepsilon (\|\lambda_j I - A\| + \alpha_j(A) s_{\min}(zI - A))\} \\ &= \left\{ z \in D(\lambda_j, r_j) : s_{\min}(zI - A) \leq \frac{\varepsilon \|\lambda_j I - A\|}{1 - \varepsilon \alpha_j(A)} \right\}. \end{aligned}$$

Thus for  $0 < \varepsilon < \frac{1}{\min(\alpha_j(A), \varepsilon_j)}$ ,

$$\sigma_\varepsilon(A, \lambda_j) \subseteq \Lambda_{\frac{\varepsilon\|\lambda_j I - A\|}{1 - \varepsilon\alpha_j(A)}}(A, \lambda_j).$$

Let  $z \in \Lambda_{\frac{\varepsilon\|\lambda_j I - A\|}{2}}(A, \lambda_j)$ . Then

$$\|\lambda_j I - A\| \leq \|zI - A\| + |z - \lambda_j| \leq 2 \|zI - A\|,$$

and so

$$\|zI - A\| \geq \frac{\|\lambda_j I - A\|}{2}.$$

Consequently,

$$\|zI - A\| \|(zI - A)^{-1}\| \geq \frac{\|\lambda_j I - A\|}{2} \cdot \frac{2}{\varepsilon \|\lambda_j I - A\|} = \frac{1}{\varepsilon},$$

and it follows that

$$\sigma_\varepsilon(A, \lambda_j) \supseteq \Lambda_{\frac{\varepsilon \|\lambda_j I - A\|}{2}}(A, \lambda_j). \quad \square$$

Using the relation proved between components of pseudospectrum and condition spectrum of a matrix near an eigenvalue, we give the following theorem, which is a weak version of Theorem 1. It is weak compared to Theorem 2 because in this theorem we assume that  $f$  is injective and also analytic in a bigger neighborhood. More importantly, the functions describing the sizes of the condition spectra are larger than the corresponding ones in Theorem 2. We follow the approach in [4].

**THEOREM 8.** (Weak Condition Spectral Mapping Theorem.) *Let  $A \in \mathbb{C}^{N \times N}$  and  $\lambda_j \in \sigma(A)$ . Let  $\rho, \alpha(A), \alpha_j(A)$  be as defined in Lemma 1 and Theorem 7. Let  $0 < \varepsilon < 1$  be sufficiently small and  $\Omega$  be an open subset of  $\mathbb{C}$  containing  $\sigma_{\frac{\varepsilon \rho}{1 - \varepsilon \alpha(A)}}(A)$ . Let  $f$  be an injective analytic function defined on  $\Omega$ . Assume further  $A, f(A)$  are not scalar multiple of the identity. Define*

$$\begin{aligned} \gamma_j(\varepsilon) &:= \sup \left\{ \|f(A + P) - f(A)\| : \|P\| \leq \frac{\varepsilon \|\lambda_j I - A\|}{1 - \varepsilon \alpha_j(A)} \right\}, \\ \delta_j(\varepsilon) &:= \sup \left\{ \|Q\| : \|f(A + Q) - f(A)\| \leq \frac{\varepsilon \|f(\lambda_j)I - f(A)\|}{1 - \varepsilon \alpha_j(f(A))} \right\}, \end{aligned}$$

where  $\alpha_j(f(A)) = \sup_{z \in \sigma_\varepsilon(f(A), f(\lambda_j))} \frac{|z - f(\lambda_j)|}{s_{\min}(zI - f(A))}$ . Then  $\lim_{\varepsilon \rightarrow 0} \gamma_j(\varepsilon) = 0 = \lim_{\varepsilon \rightarrow 0} \delta_j(\varepsilon)$  and the following two assertions hold:

1. Let  $0 < \varepsilon < 1 / \min\{\alpha_j(A), \varepsilon_j\}$  be such that  $\frac{2 \gamma_j(\varepsilon)}{\|f(\lambda_j)I - f(A)\|} < 1$ . Then

$$f(\sigma_\varepsilon(A, \lambda_j)) \subseteq \sigma_{\frac{2 \gamma_j(\varepsilon)}{\|f(\lambda_j)I - f(A)\|}}(f(A), f(\lambda_j));$$

2. Let  $0 < \varepsilon < 1 / \min\{\alpha_j(A), \varepsilon_j\}$  be such that  $\frac{2 \delta_j(\varepsilon)}{\|\lambda_j I - A\|} < 1$ . Then

$$\sigma_\varepsilon(f(A), f(\lambda_j)) \subseteq f \left( \sigma_{\frac{2 \delta_j(\varepsilon)}{\|\lambda_j I - A\|}}(A, \lambda_j) \right).$$

*Proof.* Since the map  $A \mapsto f(A)$  is continuous, we obtain  $\lim_{\varepsilon \rightarrow 0} \gamma_j(\varepsilon) = 0$ . Since  $f$  is injective on  $\Omega$  the inverse of  $f$  exist on  $\Omega$ . Let  $g : f(\Omega) \rightarrow \Omega$  be the inverse of  $f$ .

Since the map  $B \mapsto g(B)$  is continuous, we obtain  $\lim_{\varepsilon \rightarrow 0} \delta_j(\varepsilon) = 0$ . Let  $\varepsilon > 0$  be such that  $\frac{2\gamma_j(\varepsilon)}{\|f(\lambda_j)I - f(A)\|} < 1$  and  $z \in \sigma_\varepsilon(A, \lambda_j)$ . Recall that for any  $r > 0$ ,

$$\Lambda_r(A) = \bigcup \{ \sigma(A + E), \|E\| \leq r \}.$$

By Theorem 7, there exist  $B \in \mathbb{C}^{N \times N}$  with  $\|B\| \leq \frac{\varepsilon \|\lambda_j I - A\|}{1 - \varepsilon \alpha_j(A)}$  such that  $z \in \sigma(A + B)$ . Then by the Spectral Mapping Theorem,  $f(z) \in \sigma(f(A + B)) = \sigma(f(A) + C)$ , where  $C = f(A + B) - f(A)$ , which satisfies  $\|C\| \leq \gamma_j(\varepsilon)$ . By Theorem 7,

$$f(z) \in \Lambda_{\gamma_j(\varepsilon)}(f(A), f(\lambda_j)) \subseteq \sigma_{\frac{2\gamma_j(\varepsilon)}{\|f(\lambda_j)I - f(A)\|}}(f(A), f(\lambda_j)).$$

This proves 1.

Let  $z \in \sigma_\varepsilon(f(A), f(\lambda_j))$ . Then by Theorem 7,  $z \in \sigma(f(A) + D)$  for some  $D \in \mathbb{C}^{N \times N}$  with  $\|D\| \leq \frac{\varepsilon \|f(\lambda_j)I - f(A)\|}{1 - \varepsilon \alpha_j(f(A))}$ . By the Inverse Mapping Theorem ([6]), there exist a unique  $P \in \mathbb{C}^{N \times N}$  and  $\varepsilon_1 > 0$  such that  $\|P\| \leq \varepsilon_1$  and  $f(A + P) = f(A) + D$ . Thus by the Spectral Mapping Theorem there exists  $\mu \in \sigma(A + P)$  such that

$$f(\mu) = z \in \sigma(f(A + P)) = \sigma(f(A) + D).$$

Claim:  $\mu \in \sigma_{\frac{2\delta_j(\varepsilon)}{\|\lambda_j I - A\|}}(A, \lambda_j)$ . Observe that

$$\|D\| = \|f(A + P) - f(A)\| \leq \frac{\varepsilon \|f(\lambda_j)I - f(A)\|}{1 - \varepsilon \alpha_j(f(A))}.$$

We can take

$$\varepsilon_1 = \delta_j(\varepsilon) = \sup \left\{ \|Q\| : \|f(A + Q) - f(A)\| \leq \frac{\varepsilon \|f(\lambda_j)I - f(A)\|}{1 - \varepsilon \alpha_j(f(A))} \right\}.$$

Now by Theorem 7,  $\mu \in \Lambda_{\delta_j(\varepsilon)}(A, \lambda_j) \subseteq \sigma_{\frac{2\delta_j(\varepsilon)}{\|\lambda_j I - A\|}}(A, \lambda_j)$ . This proves the claim. Hence

$$z = f(\mu) \in f \left( \sigma_{\frac{2\delta_j(\varepsilon)}{\|\lambda_j I - A\|}}(A, \lambda_j) \right). \text{ This proves 2. } \square$$

REMARK 5. Suppose the injective function  $f$  has a bounded Fréchet derivative in a neighborhood of  $\sigma_\varepsilon(A)$  and the inverse  $f^{-1}$  also has a bounded Fréchet derivative in a neighborhood containing  $\sigma_\varepsilon(f(A))$ . Let  $(Df)_A$  and  $(Df^{-1})_{f(A)}$  denote the Fréchet derivative of  $f$  at  $A$  and Fréchet derivative of  $f^{-1}$  at  $f(A)$ , respectively. Define

$$L_j(\varepsilon) := \sup \left\{ \|(Df)_P\| : P \in \mathbb{C}^{N \times N}, \|P - A\| \leq \frac{\varepsilon \|\lambda_j I - A\|}{1 - \varepsilon \alpha_j(A)} \right\},$$

$$L_j(\varepsilon)' := \sup \left\{ \|(Df^{-1})_P\| : P \in \mathbb{C}^{N \times N}, \|P - f(A)\| \leq \frac{\varepsilon \|f(\lambda_j)I - f(A)\|}{1 - \varepsilon \alpha_j(f(A))} \right\}.$$

Then  $\gamma_j(\varepsilon), \delta_j(\varepsilon)$  can be estimated as

$$\gamma_j(\varepsilon) \leq \frac{\varepsilon \|\lambda_j I - A\| L_j(\varepsilon)}{1 - \varepsilon \alpha_j(A)} \quad \text{and} \quad \delta_j(\varepsilon) \leq \frac{\varepsilon \|f(\lambda_j)I - f(A)\| L_j(\varepsilon)'}{1 - \varepsilon \alpha_j(f(A))}.$$

In the following, we consider a general  $3 \times 3$  diagonal matrix  $A$  with positive entries and analytically find the values  $\gamma_j(A), \delta_j(A)$  in the weak condition Spectral Mapping Theorem for the function  $f(z) = z^3$ .

EXAMPLE 4. Consider a  $3 \times 3$  diagonal matrix with distinct eigenvalues  $\lambda_1 > \lambda_2 > \lambda_3 > 0$  and  $f(z) = z^2$ . We have

$$A = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \quad \text{and} \quad f(A) = A^2 = \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{bmatrix}.$$

We have  $\|A\| = \lambda_1$  and  $\|f(A)\| = \|A^2\| = \lambda_1^2$ . For  $z \in \mathbb{C}$ ,

$$\begin{aligned} \|zI - A\| &= \max\{|z - \lambda_1|, |z - \lambda_2|, |z - \lambda_3|\}, \\ \|zI - f(A)\| &= \|zI - A^2\| = \max\{|z - \lambda_1^2|, |z - \lambda_2^2|, |z - \lambda_3^2|\}. \end{aligned}$$

For  $z \in \sigma_\varepsilon(A, \lambda_1)$  with  $\varepsilon$  small, we have  $s_{\min}(zI - A) = |z - \lambda_1|$  and  $s_{\min}(zI - f(A)) = s_{\min}(zI - A^2) = |z - \lambda_1^2|$ . This implies,

$$\begin{aligned} \alpha_1(A) &= \sup_{z \in \sigma_\varepsilon(A, \lambda_1)} \frac{|z - \lambda_1|}{s_{\min}(zI - A)} = 1, \\ \alpha_1(A^2) &= \sup_{z \in \sigma_\varepsilon(A^2, \lambda_1^2)} \frac{|z - \lambda_1^2|}{s_{\min}(zI - A^2)} = 1. \end{aligned}$$

Thus

$$\begin{aligned} \gamma_1(\varepsilon) &= \sup \left\{ \|(A + P)^2 - A^2\| : \|P\| \leq \frac{\varepsilon \|\lambda_1 I - A\|}{1 - \varepsilon \alpha_1(A)} \right\} \\ &= \sup \left\{ \|AP + PA + P^2\| : \|P\| \leq \frac{\varepsilon \|\lambda_1 I - A\|}{1 - \varepsilon} \right\} \\ &\leq \left\{ 2\|A\| \|P\| + \|P^2\| : \|P\| \leq \frac{\varepsilon \|\lambda_1 I - A\|}{1 - \varepsilon} \right\} \\ &= \frac{2\varepsilon \lambda_1 \|\lambda_1 I - A\|}{1 - \varepsilon} + \frac{(\varepsilon \|\lambda_1 I - A\|)^2}{(1 - \varepsilon)^2}, \end{aligned}$$

where  $\|\lambda_1 I - A\| = \max\{|\lambda_1 - \lambda_2|, |\lambda_1 - \lambda_3|\}$ . Hence by the Weak Condition Spectral Mapping Theorem, for sufficiently small  $\varepsilon$ ,

$$(\sigma_\varepsilon(A, \lambda_1))^2 \subseteq \sigma_{\frac{2\gamma_1(\varepsilon)}{\|f(\lambda_1)I - f(A)\|}}(A^2, \lambda_1^2),$$

where  $\|f(\lambda_1)I - f(A)\| = \|\lambda_1^2 - A^2\| = \max\{|\lambda_1^2 - \lambda_2^2|, |\lambda_1^2 - \lambda_3^2|\}$ . Also,

$$\begin{aligned} \delta_1(\varepsilon) &= \sup \left\{ \|Q\| : \|(A + Q)^2 - A^2\| \leq \frac{\varepsilon \|\lambda_1^2 I - A^2\|}{1 - \varepsilon \alpha_1(A^2)} \right\} \\ &= \sup \left\{ \|Q\| : \|AQ + QA + Q^2\| \leq \frac{\varepsilon \|\lambda_1^2 I - A^2\|}{1 - \varepsilon} \right\} \\ &\approx \sup \left\{ \|Q\| : 2\lambda_1 \|Q\| - \|Q\|^2 \leq \frac{\varepsilon \|\lambda_1^2 I - A^2\|}{1 - \varepsilon} \right\} \\ &\approx \lambda_1 - \sqrt{\lambda_1^2 - \frac{\varepsilon \|\lambda_1^2 I - A^2\|}{1 - \varepsilon}}. \end{aligned}$$

Hence by the Weak Condition Spectral Mapping Theorem we have,

$$\sigma_\varepsilon(A^2, \lambda_1^2) \subseteq \left( \sigma_{\frac{2\delta_1(\varepsilon)}{\|\lambda_1^2 I - A^2\|}}(A, \lambda_1) \right)^2,$$

where  $\|\lambda_1 I - A\|, \|\lambda_1^2 I - A^2\|$  are defined above. In a similar way we can estimate the values of  $\gamma_j(\varepsilon), \delta_j(\varepsilon)$  for  $j = 2, 3$  and a weak version of component-wise Spectral Mapping Theorem near the eigenvalues  $\lambda_2, \lambda_3$  can be derived.

Next we consider a general  $2 \times 2$  upper triangular matrix  $A$  and estimate the values  $\gamma_j(A), \delta_j(A)$  in the Weak Condition Spectral Mapping Theorem for the function  $f(z) = z^2$ .

EXAMPLE 5. Let  $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$  such that  $a, b > 0$ . The eigenvalues are  $\lambda_1 = a, \lambda_2 = b$ . Take  $f(z) = z^2$ . First consider  $\lambda_1 = a$ . We have  $f(A) = A^2 = \begin{bmatrix} a^2 & (a+b)c \\ 0 & b^2 \end{bmatrix}$ . We have

$$\begin{aligned} \|A\| &= \frac{|a|^2 + |b|^2 + |c|^2 + \sqrt{(|a|^2 + |b|^2 + |c|^2)^2 - 4|a|^2|b|^2}}{2} \\ \|aI - A\| &= \sqrt{|a - b|^2 + |c|^2} \\ \|a^2I - A^2\| &= \sqrt{|a^2 - (a+b)c|^2 + |a^2 - b^2|^2} \end{aligned}$$

For  $z \in \mathbb{C}$  close to  $a$ ,

$$s_{\min}(zI - A) = \frac{|z - a|^2 + |z - b|^2 + |c|^2 - \sqrt{(|z - a|^2 + |z - b|^2 + |c|^2)^2 - 4|z - a|^2|z - b|^2}}{2},$$

$$s_{\min}(zI - A^2) = \frac{|z - a^2|^2 + |z - b^2|^2 + |(a+b)c|^2 - \sqrt{(|z - a^2|^2 + |z - b^2|^2 + |(a+b)c|^2)^2 - 4|z - a^2|^2|z - b^2|^2}}{2}.$$

By definition,

$$\alpha_1(A) = \sup_{z \in \sigma_\varepsilon(A, a)} \frac{|z - a|}{s_{\min}(zI - A)},$$

$$\alpha_1(A^2) = \sup_{z \in \sigma_\varepsilon(A^2, a^2)} \frac{|z - a^2|}{s_{\min}(zI - A^2)}.$$

Thus

$$\begin{aligned} \gamma_1(\varepsilon) &= \sup \left\{ \|(A + P)^2 - A^2\| : \|P\| \leq \frac{\varepsilon \|aI - A\|}{1 - \varepsilon \alpha_1(A)} \right\} \\ &= \sup \left\{ \|AP + PA + P^2\| : \|P\| \leq \frac{\varepsilon \|aI - A\|}{1 - \varepsilon \alpha_1(A)} \right\} \\ &\leq \sup \left\{ 2\|A\| \|P\| + \|P\|^2 : \|P\| \leq \frac{\varepsilon \|aI - A\|}{1 - \varepsilon \alpha_1(A)} \right\}. \end{aligned}$$

Thus by the Weak Condition Spectral Mapping Theorem, for a sufficiently small  $\varepsilon$ ,

$$(\sigma_\varepsilon(A, \lambda_1))^2 \subseteq \sigma_{\frac{2\gamma_1(\varepsilon)}{\|a^2I - A^2\|}}(A^2, \lambda_1^2).$$

Also,

$$\begin{aligned} \delta_1(\varepsilon) &= \sup \left\{ \|Q\| : \|(A + Q)^2 - A^2\| \leq \frac{\varepsilon \|a^2I - A^2\|}{1 - \varepsilon \alpha_1(A^2)} \right\} \\ &= \sup \left\{ \|Q\| : \|AQ + QA + Q^2\| \leq \frac{\varepsilon \|a^2I - A^2\|}{1 - \varepsilon \alpha_1(A^2)} \right\} \\ &\approx \sup \left\{ \|Q\| : 2\|A\| \|Q\| - \|Q\|^2 \leq \frac{\varepsilon \|a^2I - A^2\|}{1 - \varepsilon \alpha_1(A^2)} \right\} \\ &\approx \|A\| - \sqrt{\|A\|^2 - \frac{\varepsilon \|a^2I - A^2\|}{1 - \varepsilon \alpha_1(A^2)}} \end{aligned}$$

Thus by the Weak Condition Spectral Mapping Theorem, for a sufficiently small  $\varepsilon$ ,

$$\sigma_\varepsilon(A^2, a^2) \subseteq \left( \sigma_{\frac{2\delta_1(\varepsilon)}{\|aI - A\|}}(A, a) \right)^2.$$

In a similar way we can analytically find the values of  $\gamma_2(\varepsilon), \delta_2(\varepsilon)$ . Thus a weak version of the component-wise Spectral Mapping Theorem near the eigenvalue  $\lambda_2 = b$  can be derived.

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