

## ON SPECTRA OF COMPOSITION OPERATORS

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*Abstract.* In this paper we consider composition operators  $C_\varphi$  on the Hilbert Hardy space over the unit disc, induced by analytic selfmaps  $\varphi$ . We use the fact that the operator  $C_\varphi^* C_\varphi$  is asymptotically Toeplitz to obtain information on the essential spectrum and spectrum of  $C_\varphi$ , which we are able to describe in select cases (including the case of some hypercyclic composition operators or that of composition operators with the property that the asymptotic symbol of  $C_\varphi^* C_\varphi$  is constant a.e.). One of our tools is the Nikodym derivative of the pull-back measure induced by  $\varphi$ . An alternative formula for the essential norm of a composition operator (valid in select cases), in terms of the aforementioned Nikodym derivative, is established. Estimates of the spectra of adjoints of composition operators are obtained. Based on them, we describe the spectrum of composition operators induced by maps fixing a point, whose iterates exhibit a strong form of attractiveness to that point.

### 1. Introduction

Let  $H^2$  denote the Hilbert Hardy space over  $\mathbb{U}$ , the open unit disc centered at the origin, that is,  $H^2$  is the space of analytic functions on  $\mathbb{U}$  with square summable Maclaurin coefficients.

The norm of any  $f \in H^2$  is computable with the formulas

$$\|f\| := \sup_{0 < r < 1} \left( \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) \right)^{1/2} < +\infty, \quad (1)$$

where  $m$  is the normalized arc-length measure on  $\mathbb{T}$ , the boundary of  $\mathbb{U}$ , respectively

$$\|f\| = \sqrt{\sum_{n=0}^{+\infty} |c_n|^2}, \quad (2)$$

where  $\{c_n\}$  is the sequence of Maclaurin coefficients of  $f$ .

It is well known that  $H^2$ -functions have radial limits a.e. on  $\mathbb{T}$ , the radial limit-functions being in the Lebesgue space  $L^2_{\mathbb{T}}(dm)$  and having  $\|\cdot\|_2$ -norm identical to the  $H^2$ -norm of the function itself. For that reason,  $H^2$ -functions and their radial limit functions are customarily denoted by the same symbol. This identification will be used

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throughout this paper. It embeds isometrically  $H^2$  into  $L^2_{\mathbb{T}}(dm)$ , so one can write  $H^2 \subseteq L^2_{\mathbb{T}}(dm)$ . Recall that the Poisson kernel is the function

$$P(z, u) = \Re \frac{u+z}{u-z} = \frac{1-|z|^2}{|u-z|^2} \quad z \in \mathbb{U}, \quad u \in \mathbb{T}$$

and the Poisson integral  $P_{\mu}$  of any complex Borel measure  $\mu$  is the function

$$P_{\mu}(z) = \int_{\mathbb{T}} P(z, u) d\mu(u) \quad z \in \mathbb{U}.$$

Bounded analytic functions visibly belong to  $H^2$  since it is very easy to see that any analytic function  $f$  on  $\mathbb{U}$  satisfies condition  $\|f\|_{\infty} \leq \|f\|$ , where  $\|\cdot\|_{\infty}$  denotes the supremum norm. If the radial limit-function of a bounded analytic function is unimodular a.e., that analytic function is called an *inner function*. Clearly, inner functions are analytic selfmaps of  $\mathbb{U}$ , that is analytic functions mapping  $\mathbb{U}$  into itself.

Given any analytic selfmap  $\varphi$  of  $\mathbb{U}$ , the operator

$$C_{\varphi}f = f \circ \varphi \quad f \in H^2$$

is necessarily linear. We call  $C_{\varphi}$  the *composition operator* induced by  $\varphi$  and refer to  $\varphi$  as the *symbol* of  $C_{\varphi}$ . If  $\varphi$  is a conformal automorphism, we say that  $C_{\varphi}$  is an *automorphic* composition operator or a composition operator with automorphic symbol. It is well known that all composition operators on  $H^2$  are bounded. Actually, this fact is a principle of function theory called *Littlewood's Subordination Principle* [17, Theorem 1.7], which says that composition operators induced by symbols fixing the origin are contractions. The fact that all composition operators are bounded follows from Littlewood's Principle with little technical effort.

The space  $H^2$  is a *reproducing kernel Hilbert space*. This means that the special functions  $k_w(z) = 1/(1 - \bar{w}z)$ ,  $z, w \in \mathbb{U}$ , called *kernel functions*, have the property

$$\langle f, k_w \rangle = f(w) \quad w \in \mathbb{U} \quad (3)$$

called the *reproducing property*. Caughran and Schwartz [10] observed the following immediate consequence of that property

$$C_{\varphi}^* k_w = k_{\varphi(w)} \quad w \in \mathbb{U} \quad (4)$$

where  $C_{\varphi}^*$  denotes the adjoint of  $C_{\varphi}$ .

This introductory section is dedicated to setting up the notation and describing the content of the next sections.

The eigenvalue equation for composition operators is called “Schröder’s equation”. In Section 2 we briefly discuss G. Koenigs’s theorem on the aforementioned equation. The main reason for that is recording Remark 2, saying that if an operator induced by some non-automorphic  $\varphi$  fixing a point in  $\mathbb{U}$  has simply connected essential spectrum, then its spectrum is the union of the essential spectrum and the set consisting of the number 1 and the powers of the derivative of  $\varphi$  at the fixed point. This remark is repeatedly used in Sections 3–5, which contain the main results in this paper.

In Section 3 we consider the essential infimum  $\text{ess inf } \psi$ , where  $\psi$  denotes the Nikodym derivative of the pull-back measure induced by  $\varphi$ . We show that one case when composition operators have circular (and hence simply connected), essential spectra is that of composition operators induced by non-automorphic maps  $\varphi$  fixing a point, whose essential spectral radius satisfies the inequality  $r_e(C_\varphi) \leq \sqrt{\text{ess inf } \psi}$  (Theorem 6). Another result in Section 3 is the formula  $\|C_\varphi\|_e = \sqrt{\text{ess sup } \psi}$  (Theorem 6), for the essential norm of  $C_\varphi$ , which is valid in the case of particular symbols  $\varphi$ . We characterize the situation when  $C_\varphi^* C_\varphi$  is a compact perturbation of a scalar multiple of the identity operator, observing that only maps  $\varphi$  with a fixed point in  $\mathbb{U}$  can have that property (Theorem 8). The spectrum and essential spectrum of an operator with the previously described property are determined. Analytic selfmaps with orthogonal powers are examples of maps  $\varphi$  satisfying the assumptions of Theorem 8.

Section 4 contains properties of composition operators induced by symbols without fixed points in  $\mathbb{U}$ . The computations of the essential spectral radius and spectral radius of such operators are currently scattered in the literature and their proofs broken into many cases. We are able to write an elegant short proof covering all cases simultaneously (Theorem 10). Based on that proof, we are able to obtain a formula for the angular derivative of the symbols  $\varphi$  at the Denjoy-Wolff point in this setting (formula (35)). Section 4 also contains the computation of spectra of some hypercyclic composition operators (Theorem 11), that is composition operators with dense orbits. The same section contains results about compact perturbations of composition operators. They are applied to obtain as immediate consequences the descriptions of spectra of essentially linear-fractional composition operators (Corollary 2), which appeared initially in [2].

In Section 5, upper estimates of the spectra of the adjoints of composition operators are established (Proposition 4) using formula (35). As a consequence, the spectra of composition operators induced by selfmaps  $\varphi$  fixing some  $\omega \in \mathbb{U}$  and satisfying the dynamical attractiveness condition

$$\limsup_{n \rightarrow +\infty} \sqrt[n]{\|\varphi^{[n]} - \omega\|} = 0 \tag{5}$$

are determined (Corollary 3). Examples of classes of symbols  $\varphi$  satisfying (5) are given (Proposition 5 and Theorem 15).

### 2. Schröder’s equation and spectra

The main message in this section is that the spectrum of a composition operator whose symbol fixes a point in  $\mathbb{U}$  can be immediately found if the essential spectrum of that operator is determined and turns out to be a simply connected set. The following brief collection of known results, leads to that conclusion.

The functional equation

$$f \circ \varphi = \lambda f \tag{6}$$

is customarily called Schröder’s equation, after Ernst Schröder, the first mathematician known to have considered solving it [30]. If we seek nonzero solutions  $f \in H^2$ , then this is the eigenvalue equation of  $C_\varphi$ .

Denote by  $\{\varphi^{[n]}\}$  the sequence of iterates of  $\varphi$ . Recall the following ([15]):

**THEOREM 1. (Denjoy-Wolff)** *Let  $\varphi$  be an analytic selfmap of  $\mathbb{U}$  other than the identity or an elliptic disc automorphism. Then the sequence of iterates  $\{\varphi^{[n]}\}$  converges uniformly on compacts to a point  $\omega \in \overline{\mathbb{U}}$  called the Denjoy-Wolff point of  $\varphi$ .*

An immediate consequence is the fact that an analytic selfmap  $\varphi$  of  $\mathbb{U}$ , other than the identity map, can have at most one fixed point in  $\mathbb{U}$  so, whenever this happens we will speak of “the fixed point of  $\varphi$ ” rather than “a fixed point of  $\varphi$ ”. Now, the fact that the Denjoy-Wolff point of a non-automorphic selfmap with a fixed point in  $\mathbb{U}$  is exactly that fixed point is an easy consequence of Schwarz’s lemma in classical complex analysis. In their work, Denjoy and Wolff addressed the delicate case, that is the case of analytic selfmaps of  $\mathbb{U}$ , without fixed points in  $\mathbb{U}$ .

Solutions of (6) were constructed in the larger space  $\mathcal{H}(\mathbb{U})$  of all analytic functions on  $\mathbb{U}$ . This was done in the case when  $\varphi$  has a fixed point  $\omega \in \mathbb{U}$  but is not a conformal automorphism. In that case, a simple complex analysis argument shows that the only eigenvalues  $C_\varphi$  can have are 1 and  $(\varphi'(\omega))^n, n = 1, 2, 3, \dots$

Denote by  $\tilde{C}_\varphi$ , the composition operator induced by  $\varphi$  on  $\mathcal{H}(\mathbb{U})$ . Recall that G. Koenigs proved the following [22]:

**THEOREM 2. (Koenigs’s Theorem)** *Let  $\varphi$  be a nonconstant, non-automorphic, analytic selfmap of  $\mathbb{U}$  fixing  $\omega \in \mathbb{U}$ . If  $\varphi'(\omega) \neq 0$ , then there is an analytic function  $\sigma$  which satisfies equation (6) for  $\lambda = \varphi'(\omega)$ . Consequently, for all  $n = 1, 2, 3, \dots$ , the functions  $\sigma^n$  satisfy the same equation for  $\lambda = (\varphi'(\omega))^n$ , respectively. The eigenspaces of  $\tilde{C}_\varphi$  corresponding to the eigenvalues above are all 1-dimensional.*

Thus  $\sigma^n$  is an eigenfunction of  $C_\varphi$ , for some  $n = 1, 2, \dots$ , if and only if  $\sigma^n \in H^2$ . This is not always the case, and the mean growth of the function  $\sigma$  makes the object of deeper phenomena. For a good analysis of this topic, we refer the reader to [32].

We call the values  $(\varphi'(\omega))^n, n = 1, 2, \dots$  the Schröder eigenvalues of  $C_\varphi$ , even when they are not, technically speaking, eigenvalues. Whether they are or not, the values above are always in the spectrum  $\sigma(C_\varphi)$  (see [15, Theorem 7.32] for more details on this fact).

For any composition operator, the constant function 1 is obviously an eigenfunction associated to the eigenvalue 1, by the obvious equality  $1 \circ \varphi = 1$ . The eigenvalues problem is easy to treat in the case  $\varphi'(\omega) = 0$  (not covered by Koenigs’s theorem). Indeed, in that case, the only eigenvalue, other than 1,  $C_\varphi$  can possibly have is 0 (and 0 is an eigenvalue only if the symbol of the composition operator is a constant function). Thus, to summarize, denote by  $\sigma_p(C_\varphi)$  the point spectrum of  $C_\varphi$  and note that:

**REMARK 1.** Let  $\varphi$  be a nonconstant analytic selfmap of  $\mathbb{U}$  fixing  $\omega \in \mathbb{U}$  other than the identity or an elliptic automorphism. Then

$$\{1\} \subseteq \sigma_p(C_\varphi) \subseteq \{(\varphi'(\omega))^n : n = 1, 2, 3, \dots\} \cup \{1\} \subseteq \sigma(C_\varphi) \tag{7}$$

and each eigenvalue is simple (that is each eigenvalue has multiplicity 1).

Let  $\sigma_e(C_\varphi)$  denote the essential spectrum of  $C_\varphi$ . The important consequence of our previous considerations is the following:

REMARK 2. Let  $\varphi$  be a non-automorphic analytic selfmap of  $\mathbb{U}$  with a fixed point  $\omega \in \mathbb{U}$  and let  $\rho_{e,\infty}(C_\varphi)$  be the unbounded connected component of the essential resolvent  $\rho_e(C_\varphi)$  of  $C_\varphi$ . Then

$$\sigma(C_\varphi) \cap \rho_{e,\infty}(C_\varphi) \subseteq \{(\varphi'(\omega))^n : n = 1, 2, \dots\} \cup \{1\}.$$

Consequently, if  $\sigma_e(C_\varphi)$  is simply connected then

$$\sigma(C_\varphi) = \sigma_e(C_\varphi) \cup \{(\varphi'(\omega))^n : n = 1, 2, \dots\} \cup \{1\}. \tag{8}$$

Equation (8) also holds if  $\sigma_p(C_\varphi^*) \subseteq \{0, 1\}$ .

Indeed, if  $\lambda \in \rho_{e,\infty}(C_\varphi)$ , then  $C_\varphi - \lambda I$  is a Fredholm operator. The Fredholm index, that is the map  $i(\lambda) = \dim(\ker(C_\varphi - \lambda I)) - \dim(\ker(C_\varphi - \lambda I)^*)$ , is continuous on the arc wise-connected set  $A = \{C_\varphi - \lambda I : \lambda \in \rho_{e,\infty}(C_\varphi)\}$ . Since the map  $i$  is valued in a discrete set, this means that it is constant on  $A$ , namely null since, for  $|\lambda|$  large enough,  $C_\varphi - \lambda I$  is invertible. The conclusion is that, if  $\lambda \in \rho_{e,\infty}(C_\varphi) \cap \sigma(C_\varphi)$ , then  $\lambda$  is an eigenvalue of  $C_\varphi$ .

If  $\sigma_p(C_\varphi^*) \subseteq \{0, 1\}$  and  $\lambda \in \sigma(C_\varphi) \setminus \sigma_e(C_\varphi)$ , then  $\lambda \in \sigma_p(C_\varphi)$  or  $\bar{\lambda} \in \sigma_p(C_\varphi^*)$ . By Koenigs's theorem, we deduce that  $\lambda$  can only be 0, 1, or one of the Schröder eigenvalues. Given that the only Fredholm composition operators are the automorphic composition operators ([12, Theorem 1]), the conclusion of our remark follows.

One situation when  $\sigma_e(C_\varphi)$  is simply connected is, of course, when  $C_\varphi$  is essentially quasinilpotent. But are such composition operators necessarily induced by non-automorphic symbols fixing a point? The answer is affirmative. It has been obtained initially by Caughran and Schwartz [10] in the particular setting of power compact composition operators and in full generality by Bourdon and Shapiro [8]. This is the first particular case when the spectral description (8) appeared in the literature.

There are two well known cases when the spectrum of  $C_\varphi$  is the union of a closed disc centered at the origin, the set of Schröder eigenvalues, and 1. The first is when  $\varphi$  is non-automorphic, fixes a point, and has an analytic extension on an open neighborhood of the closed unit disc. The proof is due to Kamowitz [20]. The second case is when  $\varphi$  is univalent, non-automorphic, with a fixed point; this time the credit goes to Cowen and MacCluer [14]. According to the author of [2], the circular disc involved in the aforementioned description of spectra is not proved to be the essential spectrum of  $C_\varphi$  by either H. Kamowitz or Cowen and MacCluer.

Finally, this author proved that *symbols having orthogonal powers* (that is analytic selfmaps  $\varphi$  of  $\mathbb{U}$  with the property that  $\{1, \varphi, \varphi^2, \varphi^3, \dots\}$  is an orthogonal subset of  $H^2$ ) induce composition operators with circular essential spectrum and therefore their spectrum is given by (8) [26].

When discussing composition operators whose symbol  $\varphi$  fixes  $\omega \in \mathbb{U}$ , it is useful to observe that, the conformal automorphism  $\alpha_\omega(z) = (\omega - z)/(1 - \bar{\omega}z)$  is self inverse

and so, the conformally conjugated symbol  $\psi = \alpha_\omega \circ \varphi \circ \alpha_\omega$  induces a composition operator  $C_\psi$  whose symbol fixes the origin.

The operators  $C_\varphi$  and  $C_\psi$  are similar operators since

$$C_\psi = C_{\alpha_\omega} C_\varphi C_{\alpha_\omega}. \tag{9}$$

A last tool we will use in the sequel (Section 5, Theorem 14 and Proposition 5) and wish to record in this section is the following:

**THEOREM 3.** ([33]) *If  $\varphi$  is a non-inner analytic selfmap of  $\mathbb{U}$  fixing the origin, then*

$$\|C_\varphi|_{\mathbb{C}^\perp}\| < 1. \tag{10}$$

Above,  $\mathbb{C}^\perp$  is the orthogonal complement in  $H^2$  of the subspace  $\mathbb{C}$  of constant functions.

### 3. The Nikodym derivative $dm\varphi^{-1}/dm$

A notion we wish to review is that of Aleksandrov measure. For any analytic self-map  $\varphi$  of  $\mathbb{U}$  and for all  $u \in \mathbb{T}$ , the function  $f_u(z) = P(\varphi(z), u)$ ,  $z \in \mathbb{U}$ , is a nonnegative harmonic function. Therefore, by Herglotz’s well known theorem in harmonic analysis, there is a unique, Borel, finite, nonnegative measure  $\tau_u$ , so that  $P_{\tau_u} = f_u$ . That measure is called the *Aleksandrov measure of  $\varphi$  having index  $u$* . Let  $\sigma_u$  denote the singular part of  $\tau_u$  in its Lebesgue decomposition with respect to  $m$ . The main result in [11] is the following formula for the essential norm  $\|C_\varphi\|_e$  of  $C_\varphi$ :

$$\|C_\varphi\|_e = \sqrt{\sup_{u \in \mathbb{T}} \|\sigma_u\|}. \tag{11}$$

Denote by  $P$  the orthogonal projection of  $L^2_{\mathbb{T}}(dm)$  onto  $H^2$ . For any essentially bounded, measurable function  $\psi$  on  $\mathbb{T}$ , the operator

$$T_\psi f = P(\psi f) \quad f \in H^2$$

is called the *Toeplitz operator* with symbol  $\psi$  (or induced by  $\psi$ ). Let us consider the coordinate function  $z$ . The Toeplitz operator  $T_z$  is called the unilateral shift on  $H^2$  because of its obvious shift-action on the Maclaurin coefficients of  $H^2$ -functions. Actually, it is a unilateral forward shift of multiplicity 1 in the sense of [36]. For that reason, we will use the notation  $S = T_z$ . It is well known that a bounded operator  $A$  on  $H^2$  is a Toeplitz operator if and only if  $S^*AS = A$ . In that case, the operator sequence  $\{S^{*n}AS^n\}$  tends to  $A$  in all the topologies of  $\mathcal{L}(H^2)$ , since it is a constant sequence. For all operators  $A$ , if the sequence  $\{S^{*n}AS^n\}$  is convergent, weakly, strongly, or uniformly, then the limit  $T$  of that sequence satisfies the operator equation  $S^*TS = T$  and is therefore a Toeplitz operator. Whenever that happens,  $A$  is called a *weakly asymptotically Toeplitz operator* (WAT), respectively *strongly asymptotically Toeplitz operator* (SAT), or *uniformly asymptotically Toeplitz operator* (UAT). The symbol  $\psi$  of the Toeplitz operator  $T$  described above is called the *asymptotic symbol* of  $A$ . Recently, this author proved the following:

THEOREM 4. ([25, Theorem 5]) *Let  $\varphi$  be an analytic selfmap of  $\mathbb{U}$ . Then  $C_\varphi^*C_\varphi$  is always WAT. Its asymptotic symbol  $\psi$  has Fourier coefficients  $\{c_n\}$  given by*

$$c_n = \int_{E_\varphi} \overline{\varphi^n} dm \quad n = 0, \pm 1, \pm 2, \dots \tag{12}$$

where  $E_\varphi = \{u \in \mathbb{T} : |\varphi(u)| = 1\}$ . If  $\|\varphi|_{E_\varphi^c}\|_\infty < 1$ , then  $C_\varphi^*C_\varphi$  is UAT.

Above we used the notation  $\|\varphi|_{E_\varphi^c}\|_\infty$  for the essential supremum norm of the restriction  $\varphi|_{E_\varphi^c}$  of  $\varphi$  to the complement  $E_\varphi^c$  of  $E_\varphi$ . A characterization of UAT-operators is contained by the following theorem of Feintuch:

THEOREM 5. ([18, Theorem 4.1]) *A Hilbert space operator  $T$  is UAT if and only if it is a compact perturbation of a Toeplitz operator.*

Besides Feintuch’s original paper, the proof can also be found in [25, Theorem 3]. According to that proof, if  $T = T_\psi + K$ , where  $K$  is a compact operator, then  $T$  is UAT with asymptotic symbol  $\psi$ . Thus, by Theorem 4, if  $C_\varphi^*C_\varphi$  is UAT, then its asymptotic symbol is the function  $\psi$  whose Fourier coefficients are given by (12). We will prove in the sequel that, for all  $\varphi$ , the function  $\psi$  with Fourier coefficients described by (12) is necessarily an essentially bounded, essentially nonnegative function, namely the Nikodym derivative  $dm\varphi^{-1}/dm$  of the pull-back measure  $m\varphi^{-1}$  induced by  $\varphi$ .

Recall that the pull-back measure of  $m$  under  $\varphi$  is the Borel measure

$$m\varphi^{-1}(E) = m(\varphi^{-1}(E)) \quad E \subseteq E_\varphi. \tag{13}$$

In one of the earliest papers on composition operators [27], Nordgren proved that  $m\varphi^{-1} \ll m$ , if  $\varphi$  is inner. His result extends with little effort to any  $\varphi$ . The Nikodym derivative  $dm\varphi^{-1}/dm$  is related to the essential spectral radius  $r_e(C_\varphi)$  and the essential spectral norm  $\|C_\varphi\|_e$  of  $C_\varphi$  as follows.

THEOREM 6. *Let  $\varphi$  be a non-automorphic analytic selfmap of  $\mathbb{U}$  and  $\psi = \frac{dm\varphi^{-1}}{dm}$ . Then, the Fourier coefficients of  $\psi$  are given by (12) and the following inequality holds*

$$\sqrt{\text{ess sup } \psi} \leq \|C_\varphi\|_e. \tag{14}$$

If  $\varphi$  fixes a point in  $\mathbb{U}$ , then

$$(\sqrt{\text{ess inf } \psi})\mathbb{U} \subseteq \sigma_e(C_\varphi) \tag{15}$$

and hence

$$\sqrt{\text{ess inf } \psi} \leq r_e(C_\varphi). \tag{16}$$

If  $C_\varphi^*C_\varphi$  is UAT, then

$$\|C_\varphi\|_e = \sqrt{\text{ess sup } \psi}. \tag{17}$$

*Proof.* Using the well known change of measure formula [19, Ch. VIII, Section 39, Theorem C], one can write

$$\int_{\mathbb{T}} \overline{z}^n dm\varphi^{-1} = \int_{E_\varphi} \overline{\varphi}^n dm \quad n = 0, \pm 1, \pm 2, \dots,$$

thus showing that the Fourier coefficients of  $dm\varphi^{-1}/dm$  are given by (12). The consequence is that  $dm\varphi^{-1}/dm$  and the asymptotic symbol of the WAT-operator  $C_\varphi^*C_\varphi$  coincide a.e. Besides proving that the asymptotic symbol of  $C_\varphi^*C_\varphi$  is  $dm\varphi^{-1}/dm$ , this computation has an immediate consequence: it establishes that  $dm\varphi^{-1}/dm$  is essentially bounded (a known fact).

In the process of establishing formula (11), the authors of [11, relation (2.6)] prove that

$$\|\sigma_\alpha\| = \lim_{r \rightarrow 1^-} (1 - r^2) \int_{\mathbb{T}} \frac{dm(u)}{|\alpha - r\varphi(u)|^2}.$$

Keeping this in mind, note that

$$\begin{aligned} (1 - r^2) \int_{\mathbb{T}} \frac{dm(u)}{|\alpha - r\varphi(u)|^2} &\geq (1 - r^2) \int_{E_\varphi} \frac{dm(u)}{|\alpha - r\varphi(u)|^2} \\ &= \int_{\mathbb{T}} \frac{(1 - r^2)}{|\alpha - ru|^2} \psi(u) dm(u) = \int_{\mathbb{T}} \frac{(1 - r^2)}{|u - r\alpha|^2} \psi(u) dm(u) = P_\psi(r\alpha). \end{aligned}$$

Letting  $r \rightarrow 1 -$ , one gets  $\psi(\alpha) \leq \|\sigma_\alpha\|$  a.e. which, combined with (11), establishes (14).

By the measure theoretical formula already cited in this proof, one obtains

$$\|C_\varphi f\|^2 \geq \int_{E_\varphi} |f \circ \varphi|^2 dm = \int_{\mathbb{T}} |f|^2 \psi dm \geq (\text{ess inf } \psi) \|f\|^2 \quad f \in H^2. \quad (18)$$

Assume  $\text{ess inf } \psi > 0$ . We note that  $C_\varphi - \lambda I$  is bounded below if  $|\lambda| < \sqrt{\text{ess inf } \psi}$ . Indeed, by (18), one has

$$\|(C_\varphi - \lambda I)f\|^2 \geq (\|C_\varphi f\| - \|\lambda f\|)^2 \geq \left(\sqrt{\text{ess inf } \psi} - |\lambda|\right)^2 \|f\|^2 \quad f \in H^2.$$

As we noted before, the only Fredholm composition operators are the invertible ones [12, Theorem 1], that is the composition operators induced by disc automorphisms. Therefore, relation (15) is an immediate consequence of that fact if  $\text{ess inf } \psi = 0$ .

If  $\text{ess inf } \psi > 0$ , then note that  $C_\varphi$  is bounded below, since we established that  $C_\varphi - \lambda I$  is bounded below for all  $|\lambda| < \sqrt{\text{ess inf } \psi}$ . Given that, by Theorem 2, the kernel of  $C_\varphi - \lambda I$  has dimension at most 1 for all complex  $\lambda$ , one deduces that  $C_\varphi$  is a semi-Fredholm operator having index  $-\infty$ . The semi-Fredholm index being norm-continuous, it follows that  $\sigma_e(C_\varphi)$  contains open discs  $r\mathbb{U}$ ,  $r > 0$ , so that  $C_\varphi - \lambda I$  is a semi-Fredholm operator having index  $-\infty$  for all  $\lambda$  in  $r\mathbb{U}$ . The union of all those discs is also an open disc centered at the origin. Denote its radius  $\rho$ . The disc  $\rho\mathbb{U}$  is the circular disc centered at 0 of largest radius which is contained in  $\sigma_e(C_\varphi)$  and has the property that  $C_\varphi - \lambda I$  is a semi-Fredholm operator having index  $-\infty$  for all  $\lambda \in \rho\mathbb{U}$ .

It follows that  $C_\varphi - \lambda I$  has non-closed range for at least one  $\lambda$  on the boundary of  $\rho\mathbb{U}$ . To see that, argue by contradiction, assuming  $C_\varphi - \lambda I$  is a semi-Fredholm operator having index  $-\infty$  for all  $\lambda$  with property  $|\lambda| = \rho$ . By the continuity of the semi-Fredholm index, for each such  $\lambda$  there is an open disc about  $\lambda$  so that  $C_\varphi - \alpha I$  is a semi-Fredholm operator having index  $-\infty$  for all  $\alpha$  in that open disc. Using the compactness of the boundary  $\rho\mathbb{T}$  of  $\rho\mathbb{U}$ , one can cover  $\rho\mathbb{T}$  with finitely many such discs. The fact is contradictory because it produces an open disc centered at the origin, contained in  $\sigma_e(C_\varphi)$  with radius larger than  $\rho$  and the property that  $C_\varphi - \lambda I$  is a semi-Fredholm operator having index  $-\infty$  for all  $\lambda$  in that open disc.

We have established that there is some  $\lambda$ ,  $|\lambda| = \rho$ , with the property that  $C_\varphi - \lambda I$  has non-closed range, or the range is closed but the semi-Fredholm index is not  $-\infty$ . Note that the later situation is not possible since, if  $C_\varphi - \lambda I$  had closed range and semi-Fredholm index other than  $-\infty$ , then that index would have to be finite (by Theorem 2), and so,  $\lambda$  would belong to the essential resolvent-set of  $C_\varphi$ , contrary to fact.

Since for all  $\lambda$  with property  $|\lambda| < \sqrt{\text{ess inf } \psi}$ , the range of  $C_\varphi - \lambda I$  is closed, one deduces that (15) holds.

Assume now that  $C_\varphi^* C_\varphi$  is UAT. Then  $C_\varphi^* C_\varphi$  is a compact perturbation of  $T_\psi$ . By [16, Theorem 7.20],  $\sigma(T_\psi) = [\text{ess inf } \psi, \text{ess sup } \psi] = \sigma_e(T_\psi)$ . The cited theorem establishes only that the spectrum of a selfadjoint Toeplitz operator  $T_\psi$  equals the line-interval above. The fact that it coincides with the essential spectrum of the same operator is a consequence of the following argument. In the interesting case  $\text{ess inf } \psi < \text{ess sup } \psi$ , if any of the interior points in the line interval  $[\text{ess inf } \psi, \text{ess sup } \psi]$  belonged to the essential resolvent  $\rho_e(T_\psi)$ , then one could consider a Jordan loop having  $\text{ess inf } \psi$  in its interior and  $\text{ess sup } \psi$  in its exterior, thus separating the essential range of  $\psi$ . This would contradict [16, Theorem 7.42].

Equality (17) is a direct consequence of equality  $\sigma_e(T_\psi) = [\text{ess inf } \psi, \text{ess sup } \psi]$ , the fact that  $T_\psi$  and  $C_\varphi^* C_\varphi$  are essentially equal, and the equality  $r_e(T) = \|T\|_e$  valid for all selfadjoint operators  $T$ .  $\square$

The utility of formula (17) is being a substitute for (11) or the better known “Nevanlinna counting function essential norm formula” (the first known formula for the essential norm of a composition operator [31]), in select cases, when the aforementioned formulas might be hard to use, but (17) applies and works easier. Here is an example.

EXAMPLE 1. Let  $\varphi(e^{i\theta}) = e^{i\theta}$  if  $0 \leq \theta \leq \pi$  and  $\varphi(e^{i\theta}) = e^{i\theta}/2$  if  $\pi < \theta < 2\pi$ . The outer function having boundary function  $\varphi$  is also denoted by  $\varphi$  and has expression

$$\varphi(z) = e^{\int_{\mathbb{T}} \frac{u+z}{u-z} \log |\varphi(u)| dm(u)} \quad z \in \mathbb{U}.$$

Clearly, that outer function is an analytic selfmap of  $\mathbb{U}$  and the computation of the Fourier coefficients of  $\psi$  leads to the conclusion that  $\psi$  is the characteristic function of the upper semicircle of  $\mathbb{T}$ . Thus  $\text{ess inf } \psi = 0 < 1 = \text{ess sup } \psi$ . Since  $\|\varphi|E_\varphi^c\|_\infty = 1/2 < 1$ , it follows, by Theorem 4, that  $C_\varphi^* C_\varphi$  is UAT. Therefore, by Theorem 6,  $\|C_\varphi\|_e = 1$ .

An interesting consequence of Theorem 6 is:

COROLLARY 1. *Let  $\varphi$  be a non-automorphic analytic selfmap of  $\mathbb{U}$  with a fixed point  $\omega \in \mathbb{U}$ . If  $r_e(C_\varphi) \leq \sqrt{\text{ess inf } \psi}$ , then*

$$\sigma(C_\varphi) = (\sqrt{\text{ess inf } \psi})\overline{\mathbb{U}} \cup \{(\varphi'(\omega))^n : n = 1, 2, \dots\} \cup \{1\} \tag{19}$$

and

$$\sigma_e(C_\varphi) = (\sqrt{\text{ess inf } \psi})\overline{\mathbb{U}}. \tag{20}$$

Clearly, if  $r_e(C_\varphi) = 0$ , then  $r_e(C_\varphi) \leq \sqrt{\text{ess inf } \psi}$ , which leads to the already mentioned description of the spectra of essentially quasinilpotent composition operators. Let us consider the case  $r_e(C_\varphi) > 0$  now.

The spectral picture of an operator  $T$  is, according to [28], the structure consisting of the set  $\sigma_e(T)$  and the collection of holes (bounded connected components of the essential resolvent) and pseudoholes (connected components of the difference set of the essential spectrum and the right respectively left essential spectrum) and associated semi-Fredholm indices. It is easy to note that, if  $\varphi$  satisfies the hypothesis of Corollary 1 and  $r_e(C_\varphi) > 0$ , that structure contains only a circular pseudohole centered at the origin of radius  $\sqrt{\text{ess inf } \psi}$ . This fact is established in the proof of Theorem 6. The semi-Fredholm index (which should be constant on pseudoholes), is equal to  $-\infty$ , according to the aforementioned proof.

Actually, in that proof, it is shown that, if  $C_\varphi$  is a closed range operator induced by some non-automorphic  $\varphi$ , then  $\sigma_e(C_\varphi)$  contains an open disc centered at the origin, so that the semi-Fredholm index of  $C_\varphi - \lambda I$  is constantly  $-\infty$  for all  $\lambda$  in that disc. This fact will be used in the proof of Theorem 8.

For all notions related to Fredholm theory, we refer the reader to Chapter 1 of Carl Pearcy’s beautiful monograph [28]. Among other things, we will make use of the following theorem (due to Brown, Douglas, and Fillmore [9]) which can be also found in [28, Theorem 1.35]:

THEOREM 7. *Two essentially normal operators are compalent if and only if they have the same spectral picture.*

Let us recall that two operators  $A$  and  $B$  are called *compalent* (or essentially unitarily equivalent), if  $U^*AU - B$  is compact, for some unitary operator  $U$  and an operator  $T$  is called *essentially normal* if  $T^*T - TT^*$  is compact.

With these explanations, we prove:

THEOREM 8. *Let  $\varphi$  be a non-automorphic analytic selfmap of  $\mathbb{U}$ . Then, the following are equivalent:*

$$C_\varphi^*C_\varphi \text{ is UAT with constant asymptotic symbol.} \tag{21}$$

$$\sigma_e(C_\varphi^*C_\varphi) = \{|\lambda|^2\} \text{ for some } \lambda \in \mathbb{C}. \tag{22}$$

$$C_\varphi = \lambda V + K \tag{23}$$

for some  $\lambda \in \mathbb{C}$ , some isometry  $V$ , and some compact operator  $K$ .

In case  $\varphi$  has the above properties,  $\lambda = |\lambda| = r_e(C_\varphi) = \|C_\varphi\|_e = \lim_{n \rightarrow +\infty} \|\varphi^n\|$ , hence  $\lambda \leq 1$ . Also, equations (19) and (20) hold with  $\sqrt{\psi} = \lambda$  a.e.

*Proof.* Clearly (21)  $\implies$  (22), because if (21) holds, then

$$C_\varphi^* C_\varphi = |\lambda|^2 I + K$$

for some  $\lambda \in \mathbb{C}$  and compact  $K$ , hence (22) holds.

Conversely, (22)  $\implies$  (21). Indeed, if (22) holds, then the nonnegative operators  $C_\varphi^* C_\varphi$  and  $|\lambda|^2 I$  are compalent (by Theorem 7), and hence there are some operators  $U$  unitary and  $K$  compact so that

$$U^* |\lambda|^2 I U = |\lambda|^2 I = C_\varphi^* C_\varphi + K$$

that is, (21) holds.

The fact that (23)  $\implies$  (21) is equally easy to prove. Indeed, if  $C_\varphi$  has representation (23), then

$$C_\varphi^* C_\varphi = |\lambda|^2 I + K_1$$

where  $K_1$  is the (necessarily compact operator)  $K_1 = (\lambda V + K)^* K + \lambda K^* V$ . Thus,  $C_\varphi^* C_\varphi$  is UAT with asymptotic symbol  $\psi = |\lambda|^2$  a.e.

To prove (21)  $\implies$  (23), assume  $C_\varphi^* C_\varphi$  is UAT with constant asymptotic symbol  $\psi = |\lambda|^2$  a.e. that is assume

$$C_\varphi^* C_\varphi = |\lambda|^2 I + K \tag{24}$$

for some compact operator  $K$ . The implication is trivially true if  $\lambda = 0$ , since  $C_\varphi$  is compact if  $C_\varphi^* C_\varphi$  is compact.

In case  $\lambda \neq 0$ , the operator  $\sqrt{C_\varphi^* C_\varphi}$  is bounded below and hence it is an injective selfadjoint operator, for which reason its range must be dense. Thus,  $\sqrt{C_\varphi^* C_\varphi}$  is actually invertible.

By the polar decomposition theorem, there is a partial isometry  $V$  so that

$$C_\varphi = V \sqrt{C_\varphi^* C_\varphi}. \tag{25}$$

Given that  $\sqrt{C_\varphi^* C_\varphi}$  is invertible,  $V$  is actually an isometry, not just a partial isometry.

On the other hand, the compact operator  $K = C_\varphi^* C_\varphi - |\lambda|^2 I$  is normal hence diagonal (with respect to some complete orthonormal basis), with diagonal entries tending to 0. Therefore, the nonnegative operator  $C_\varphi^* C_\varphi$  is a diagonal operator with nonnegative entries tending to  $|\lambda|^2$ . Therefore the operator  $\sqrt{C_\varphi^* C_\varphi}$  is the diagonal operator having entries  $\{\lambda_n\}$  equal to the square roots of the entries of  $C_\varphi^* C_\varphi$ .

One gets that

$$\sqrt{C_\varphi^* C_\varphi} = |\lambda| I + K_2$$

where  $K_2$  is the diagonal compact operator having diagonal entries  $\{\lambda_n - |\lambda|\}$ . Thus, by (25),

$$C_\varphi = |\lambda| V + V K_2$$

that is  $C_\varphi$  has representation (23), and hence  $\lambda = |\lambda|$ , that is,  $\lambda$  must be nonnegative. The case  $\lambda = 0$  is that of compact composition operators, when necessarily  $|\varphi| < 1$  a.e. and hence  $\lim_{n \rightarrow +\infty} \|\varphi^n\| = 0 = r_e(C_\varphi) = \|C_\varphi\|_e$ . As we noted before, if  $\varphi$  induces a compact composition operator, then  $\varphi$  must have a fixed point in  $\mathbb{U}$ . Let us treat the case  $\lambda > 0$  now.

Given that  $\psi = \lambda > 0$  a.e., one has that  $\sigma_e(C_\varphi)$  contains an open disc centered at the origin (by (15)), and hence, the isometry  $V$  cannot be unitary. Indeed, recall the famous Wold decomposition theorem (see [36]), which says that any isometry is representable in a unique way as the (possibly degenerate), direct sum of a unitary operator and a forward shift. In our case, the forward shift must act on a nonzero space and have infinite multiplicity. Thus

$$\sigma_e(C_\varphi) = \sigma_e(V) = \lambda \overline{\mathbb{U}}$$

and so  $\lambda = r_e(C_\varphi) = \|C_\varphi\|_e$  by (24).

The sequence  $\{\|\varphi^n\|\}$  is a decreasing sequence of nonnegative numbers hence a convergent sequence. Given representation (23), the fact that  $V$  is isometric,  $K$  compact, and the sequence  $\{z^n\}$  tends weakly to 0, one can write

$$|\lambda - \|\varphi^n\|| = |\|\lambda V(z^n)\| - \|C_\varphi(z^n)\|| \leq \|(C_\varphi - \lambda V)(z^n)\| = \|K(z^n)\| \rightarrow 0.$$

Hence  $\lambda = \lim_{n \rightarrow +\infty} \|\varphi^n\| \leq 1$ .

The last thing to prove is that  $\varphi$  must fix a point in  $\mathbb{U}$ . Given that the sequence  $\{\|\varphi^n\|\}$  is decreasing, the case  $\lambda = 1$  occurs if and only if  $\|\varphi\| = 1$ . This happens if and only if  $\varphi$  is an inner function fixing the origin. Indeed,  $|\varphi| \leq 1$  a.e. so  $1 - |\varphi|^2 \geq 0$  a.e. and so, if  $\|\varphi\| = 1$ , one gets

$$\int_{\mathbb{T}} (1 - |\varphi(u)|^2) dm(u) = 0$$

hence  $|\varphi| = 1$  a.e., that is  $\varphi$  is inner. The converse, that is the fact that  $\|\varphi\| = 1$  if  $\varphi$  is inner is evident. Now, if  $\varphi$  is inner, then according to [4],  $C_\varphi^* C_\varphi$  is the Toeplitz operator  $T_\psi$  with symbol

$$\psi(u) = P(\varphi(0), u) \quad \text{a.e.}$$

(see also [25, Theorem 4]). That symbol is constant a.e. if and only if  $\varphi(0) = 0$ . If  $0 < \lambda < 1$ , one has that  $r_e(C_\varphi) < 1$ , hence  $\varphi$  needs to be a non-inner function fixing some  $\omega \in \mathbb{U}$  by [26, Theorem 3.3] (see also Theorem 9 in Section 4 of the current paper).

In conclusion,  $\varphi$  has a fixed point, hence it satisfies the hypothesis of Corollary 1, and so, equations (19) and (20) hold with  $\sqrt{\psi} = \lambda$  a.e..

Here is an example of analytic selfmaps of  $\mathbb{U}$  with the properties in Theorem 8:

EXAMPLE 2. If  $\varphi$  has orthogonal powers and  $\lambda := \lim_{n \rightarrow +\infty} \|\varphi^n\|$ , then  $\sigma_e(C_\varphi^* C_\varphi) = \{\lambda\}$ .

Indeed, one can easily see that

$$C_\varphi^* C_\varphi = \lambda^2 I + K$$

where  $\lambda = \lim_{n \rightarrow +\infty} \|\varphi^n\|$  and  $K$  is the compact diagonal operator, having diagonal entries  $\{\lambda^2 - \|\varphi^n\|^2\}$  (see [26, Proof of Theorem 2.1] for full details).

The analytic selfmaps with orthogonal powers made the object of a noted problem raised by Rudin and solved recently by Bishop and Sundberg. In 1988 Rudin asked if the only analytic selfmaps with orthogonal powers are the scalar multiples of inner functions fixing the origin. The answer is negative (see [1] and [35]).

### 4. Boundary fixed point case

In the case when the Denjoy-Wolff point  $\omega$  is on the unit circle, it is known that the angular derivative  $\varphi'(\omega)$  at that point exists and is a real number with the property  $0 < \varphi'(\omega) \leq 1$ . Actually,  $\omega$  is the only *boundary fixed point* of  $\varphi$  with the afore mentioned properties. (Boundary fixed point means point  $\omega \in \mathbb{T}$  with property  $\lim_{r \rightarrow 1^-} \varphi(r\omega) = \omega$ .)

In that case,  $\varphi$  is called a selfmap of *hyperbolic type* if  $\varphi'(\omega) < 1$ , respectively a selfmap of *parabolic type* if  $\varphi'(\omega) = 1$ .

Analytic selfmaps of parabolic type are classified into two categories. The first is selfmaps of *parabolic automorphic type*. This means that the selfmap  $\varphi$  of parabolic type has the property

$$\lim_{n \rightarrow +\infty} \rho(\varphi^{[n+1]}(z), \varphi^{[n]}(z)) > 0 \quad z \in \mathbb{U}, \tag{26}$$

where  $\rho$  is the pseudohyperbolic metric  $\rho(z, w) = |(w - z)/(1 - \bar{w}z)|$ ,  $z, w \in \mathbb{U}$ . If

$$\lim_{n \rightarrow +\infty} \rho(\varphi^{[n+1]}(z), \varphi^{[n]}(z)) = 0 \quad z \in \mathbb{U}, \tag{27}$$

then  $\varphi$  is called a selfmap of *parabolic non-automorphic type*. The limits in (26) or (27) necessarily exist because the sequence under scrutiny is decreasing, by the Schwarz-Pick lemma [31, Section 4.3].

Cowen proved the formula  $r(C_\varphi) = 1/\sqrt{\varphi'(\omega)}$  [13, Theorem 2.1], valid for symbols  $\varphi$  of parabolic or hyperbolic type. In addition, if  $\varphi$  is of hyperbolic type, Cowen showed that the point spectrum  $\sigma_p(C_\varphi)$  contains the annulus  $\{z \in \mathbb{C} : \sqrt{\varphi'(\omega)} < |z| < 1/\sqrt{\varphi'(\omega)}\}$  and all eigenvalues in that annulus have infinite multiplicity [13, Theorem 4.5]. The immediate consequence is that the essential spectral radius  $r_e(C_\varphi)$  can be calculated with the formula  $r(C_\varphi) = r_e(C_\varphi) = 1/\sqrt{\varphi'(\omega)}$  in the case of symbols of hyperbolic type. The same is true for symbols of parabolic type, given the following:

**THEOREM 9.** ([26, Theorem 3.3]) *The inequality*

$$r_e(C_\varphi) < 1 \tag{28}$$

*holds if and only if  $\varphi$  is a non-inner selfmap of  $\mathbb{U}$  fixing a point in  $\mathbb{U}$ .*

To review:

**THEOREM 10.** *Let  $\varphi$  denote an analytic selfmap of  $\mathbb{U}$  of parabolic or hyperbolic type with Denjoy-Wolff point  $\omega \in \mathbb{T}$ . Then*

$$r(C_\varphi) = r_e(C_\varphi) = 1/\sqrt{\varphi'(\omega)}. \tag{29}$$

We found a short elegant proof for Theorem 10 which we present in the following for multiple purposes. Among other things, we are able to establish all equalities simultaneously for symbols of both parabolic and hyperbolic type:

*Proof of Theorem 10.* We begin by recalling a well known norm estimate satisfied by any composition operator on  $H^2$ , namely:

$$\|C_\varphi\| \leq \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}. \tag{30}$$

Since, by the Denjoy-Wolff theorem,  $\varphi^{[n]}(0) \rightarrow \omega$ , one can write

$$\varphi'(\omega) = \liminf_{z \rightarrow \omega} \frac{1 - |\varphi(z)|}{1 - |z|} \leq \liminf_{n \rightarrow +\infty} \frac{1 - |\varphi^{[n+1]}(0)|}{1 - |\varphi^{[n]}(0)|}.$$

It is well known (and rather easy to prove) that, for any sequence  $\{c_n\}$  of positive numbers, one has

$$\liminf_{n \rightarrow +\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n \rightarrow +\infty} \sqrt[n]{c_n} \leq \limsup_{n \rightarrow +\infty} \sqrt[n]{c_n} \leq \limsup_{n \rightarrow +\infty} \frac{c_{n+1}}{c_n}. \tag{31}$$

We draw the conclusion

$$\sqrt{\varphi'(\omega)} \leq \liminf_{n \rightarrow +\infty} \sqrt{2^n} \sqrt{1 - |\varphi^{[n]}(0)|}. \tag{32}$$

As the authors of [11, (3.1)] observe, one always has

$$1/\sqrt{\varphi'(\omega)} \leq \|C_\varphi\|_e. \tag{33}$$

Let us apply the estimate above to an arbitrary iterate  $\varphi^{[n]}$  of  $\varphi$ . One gets

$$1/(\varphi'(\omega))^{n/2} \leq \|C_\varphi^n\|_e. \tag{34}$$

Indeed, by the Denjoy-Wolff theorem, the Denjoy-Wolff point of any iterate  $\varphi^{[n]}$  of  $\varphi$  is also  $\omega$ . The chain rule for angular derivatives and the fact that  $\omega$  is a boundary fixed point of  $\varphi$  combine into showing that (34) holds.

By (30)–(34), one can write:

$$\begin{aligned}
 1/\sqrt{\varphi'(\omega)} &\leq r_e(C_\varphi) = \lim_{n \rightarrow +\infty} \sqrt[n]{\|C_\varphi^n\|_e} \leq \lim_{n \rightarrow +\infty} \sqrt[n]{\|C_\varphi^n\|} = r(C_\varphi) \\
 &\leq \liminf_{n \rightarrow +\infty} \sqrt[2n]{\frac{1 + |\varphi^{[n]}(0)|}{1 - |\varphi^{[n]}(0)|}} = \frac{1}{\limsup_{n \rightarrow +\infty} \sqrt[2n]{1 - |\varphi^{[n]}(0)|}} \\
 &\leq \limsup_{n \rightarrow +\infty} \sqrt[2n]{\frac{1 + |\varphi^{[n]}(0)|}{1 - |\varphi^{[n]}(0)|}} = \frac{1}{\liminf_{n \rightarrow +\infty} \sqrt[2n]{1 - |\varphi^{[n]}(0)|}} \\
 &\leq 1/\sqrt{\varphi'(\omega)}. \quad \square
 \end{aligned}$$

Besides elegance and completeness, the other reason for including the proof of Theorem 10 is the following:

REMARK 3. Let  $\varphi$  denote an analytic selfmap of  $\mathbb{U}$  of parabolic or hyperbolic type with Denjoy-Wolff point  $\omega \in \mathbb{T}$ . By the proof of Theorem 10, one can see that

$$\varphi'(\omega) = \lim_{n \rightarrow +\infty} \sqrt[n]{1 - |\varphi^{[n]}(0)|}. \tag{35}$$

Formula (35) is particularly useful because it helps us prove the following proposition, which will be used in Section 5 of this paper to obtain estimates of the spectra of adjoints of composition operators.

PROPOSITION 1. *If  $\varphi$  is an analytic selfmap of  $\mathbb{U}$  with Denjoy-Wolff point  $\omega \in \mathbb{T}$ , then*

$$\varphi'(\omega) = \lim_{n \rightarrow +\infty} \sqrt[n]{\|\varphi^{[n]} - \omega\|} \tag{36}$$

*if  $\varphi$  is of parabolic type. If  $\varphi$  is of hyperbolic type, then*

$$\varphi'(\omega) \leq \liminf_{n \rightarrow +\infty} \sqrt[n]{\|\varphi^{[n]} - \omega\|} \leq \limsup_{n \rightarrow +\infty} \sqrt[n]{\|\varphi^{[n]} - \omega\|} \leq \sqrt{\varphi'(\omega)}. \tag{37}$$

*Moreover, the inequalities (37) are sharp.*

*Proof.* If  $|\omega| = 1$ , one can write

$$1 - |\varphi^{[n]}(0)| \leq |\varphi^{[n]}(0) - \omega| \leq \|\varphi^{[n]} - \omega\| \leq 2 \quad n = 1, 2, 3, \dots$$

and, if  $\varphi$  is of parabolic type, then one gets that (36) holds as a consequence of (35) (given that  $2^{1/n} \rightarrow 1$ ).

Assume that  $\varphi$  is of hyperbolic type. In that case, it is known that  $\varphi^{[n]}(0) \rightarrow \omega$  non-tangentially, that is the aforementioned sequence is contained (with the possible exception of finitely many indices) in any non-tangential approach region with vertex at  $\omega$ . For that reason, given  $M > 1$ , one can write

$$\frac{|\omega - \varphi^{[n]}(0)|}{1 - |\varphi^{[n]}(0)|} < M \quad n \geq n_0 \tag{38}$$

for some positive integer  $n_0$ .

Keeping this in mind, one has

$$\begin{aligned} \|\varphi^{[n]} - \omega\|^2 &= \langle \varphi^{[n]} - \omega, \varphi^{[n]} - \omega \rangle = 1 + \|\varphi^{[n]}\|^2 - 2\operatorname{Re}\langle \varphi^{[n]}, \omega \rangle \quad (39) \\ &\leq 2(1 - \operatorname{Re}(\overline{\omega}\varphi^{[n]}(0))) \\ &= 2\frac{\operatorname{Re}(\overline{\omega}(\omega - \varphi^{[n]}(0)))}{|\omega - \varphi^{[n]}(0)|} \frac{|\omega - \varphi^{[n]}(0)|}{1 - |\varphi^{[n]}(0)|} (1 - |\varphi^{[n]}(0)|) \\ &\leq 2M(1 - |\varphi^{[n]}(0)|) \quad n \geq n_0. \end{aligned}$$

So, given that

$$\sqrt[n]{1 - |\varphi^{[n]}(0)|} \leq \sqrt[n]{\|\varphi^{[n]} - \omega\|} \leq (2M)^{1/2n} \sqrt[n]{1 - |\varphi^{[n]}(0)|} \quad n \geq n_0,$$

condition (37) follows by (35). To see that the inequalities (37) are sharp, consider the hyperbolic disc automorphism  $\varphi(z) = (2z + 1)/(z + 2)$  with Denjoy-Wolff point  $\omega = 1$  and note that  $\varphi^{[n]}(0) > 0$  and  $\|\varphi^{[n]}\| = 1, n = 1, 2, 3, \dots$ . Thus, by the proof of Proposition 1,

$$\|\varphi^{[n]} - \omega\| = \sqrt{2(1 - |\varphi^{[n]}(0)|)} \quad n = 1, 2, 3, \dots$$

By (35) it follows that  $\lim_{n \rightarrow +\infty} \sqrt[n]{\|\varphi^{[n]} - \omega\|} = \sqrt{\varphi'(\omega)}$ . Now consider the linear-fractional symbol  $\varphi(z) = (z + 1)/2$  with Denjoy-Wolff point  $\omega = 1$ . Clearly  $\varphi'(\omega) = 1/2 < 1$ . A straightforward argument leads to  $\lim_{n \rightarrow +\infty} \sqrt[n]{\|\varphi^{[n]} - \omega\|} = \varphi'(\omega)$ .  $\square$

We mentioned earlier that certain subsets of the point spectrum of some composition operators consisted of eigenvalues of infinite multiplicity. Actually:

**PROPOSITION 2.** *If a composition operator has a nonconstant, bounded, invariant function, then all its eigenvalues have infinite multiplicity. In particular, composition operators with symbols of hyperbolic or parabolic automorphic type have that property.*

*Proof.* Let  $u \in H^\infty$  be such that  $C_\varphi u = u$ . Then  $C_\varphi u^n = u^n, n = 1, 2, \dots$ . The set of all powers of  $u$  is linearly independent because, if arguing by contradiction, one assumes otherwise, then there is some nonconstant polynomial  $p$  with complex coefficients and the property  $p \circ u = 0$ . This is a contradiction, given that  $u$  is nonconstant and  $p$  can have only finitely many zeros.

If  $f$  is any eigenfunction of  $C_\varphi$ , then  $\{u^n f : n = 1, 2, 3, \dots\}$  is a linearly independent set of eigenfunctions associated to the same eigenvalue. Thus all eigenvalues of  $C_\varphi$  have infinite multiplicity.

According to [29, Theorem 1], if  $\varphi$  is of parabolic automorphic type there is a real number  $b \neq 0$  and an analytic map  $\sigma$  of  $\mathbb{U}$  into the right half-plane so that

$$\sigma \circ \varphi = \sigma + ib.$$

Clearly  $\sigma$  is not constant since  $b \neq 0$ . In that case

$$u = e^{(-2\pi/b)\sigma} \text{ if } b > 0 \text{ respectively } u = e^{(2\pi/b)\sigma} \text{ if } b < 0$$

is a nonconstant, bounded, invariant function of  $C_\varphi$ .

Again, by [29, Theorem 1], when  $\varphi$  is of hyperbolic type then there exist  $K > 1$  and an analytic map  $\sigma$  of  $\mathbb{U}$  into the right half-plane so that

$$\sigma \circ \varphi = K\sigma.$$

In that case

$$u = e^{(2\pi i \log \sigma / \log K)}$$

is a nonconstant, bounded, invariant function of  $C_\varphi$ .  $\square$

The fact that the eigenvalues of composition operators with symbol of hyperbolic type have infinite multiplicity was first proved in [13]. Considerations in that paper can also be used to prove the same property for symbols of parabolic automorphic type.

We preferred to prove Proposition 2 instead of just citing [13] both for the sake of completeness and because the principle contained by it works in any space of analytic functions where bounded analytic functions are multipliers (that is the given space is closed under multiplication by bounded analytic functions).

Operators with dense orbits are called *hypercyclic*. It is rather well known that the adjoints of such operators have empty point spectra. We refer the reader to [24] for the basic properties of hypercyclic operators. Only univalent symbols of parabolic or hyperbolic type can induce hypercyclic composition operators. According to [7], many such symbols induce hypercyclic operators. An exact description of the set of analytic selfmaps of  $\mathbb{U}$  inducing hypercyclic composition operators on  $H^2$  is not known, though.

All that makes interesting the following application of our results:

**THEOREM 11.** *If  $\varphi$  is a symbol of either parabolic automorphic or hyperbolic type inducing a hypercyclic composition operator, then  $\sigma(C_\varphi) = \sigma_e(C_\varphi)$ . If  $\varphi$  is a non-automorphic symbol of hyperbolic type inducing a hypercyclic composition operator, then  $\sigma(C_\varphi) = \sigma_e(C_\varphi) = (1/\sqrt{\varphi'(\omega)})\overline{\mathbb{U}}$ .*

*Proof.* Indeed, given that all eigenvalues of  $C_\varphi$  have infinite multiplicity, the set  $\sigma(C_\varphi) \setminus \sigma_e(C_\varphi)$ , if nonempty, must consist of eigenvalues of  $C_\varphi^*$  having finite multiplicity. The conclusion is  $\sigma(C_\varphi) = \sigma_e(C_\varphi)$ .

In the case when  $\varphi$  is of hyperbolic type, Cowen proved  $\sigma(C_\varphi)$  has circular symmetry, that is if  $\lambda$  belongs to  $\sigma(C_\varphi)$ , the whole circle centered at the origin and containing  $\lambda$  must be contained in  $\sigma(C_\varphi)$  [13, Theorem 4.3].

If  $\varphi$  is not an automorphism, then  $\sigma(C_\varphi)$  contains both the origin and  $\{z \in \mathbb{C} : \sqrt{\varphi'(\omega)} < |z| < 1/\sqrt{\varphi'(\omega)}\}$  [13, Theorem 4.5].

Arguing by contradiction, assume there is some  $0 < r < \sqrt{\varphi'(\omega)}$  so that  $\sigma(C_\varphi) \cap r\mathbb{T} = \emptyset$ .

If the spectrum of a hypercyclic operator can be represented as the union of two disjoint, nonempty compact sets  $K_1$  and  $K_2$ , then both  $K_1$  and  $K_2$  must meet the unit circle [24, Theorem 2.5]. This property was originally proved in [21], an unpublished Ph. D. thesis.

In our case,  $\sigma(C_\varphi)$  is the union of the set  $K_1$  of all points in  $\sigma(C_\varphi)$  interior to the circle  $r\mathbb{T}$  and the set  $K_2$  of all points in  $\sigma(C_\varphi)$  exterior to the circle  $r\mathbb{T}$ . Clearly  $K_1$  misses the unit circle which is a contradiction.

The conclusion is that  $\sigma(C_\varphi)$  must meet each circle  $r\mathbb{T}$ ,  $0 < r < \sqrt{\varphi'(\omega)}$ , which, given the circular symmetry of  $\sigma(C_\varphi)$ , ends the proof.  $\square$

Another application of our results is finding the spectra of composition operators that are compact perturbations of other composition operators. More explicitly, we say  $C_\varphi$  is a compact perturbation of another composition operator if there is some analytic selfmap  $\psi \neq \varphi$  of  $\mathbb{U}$  so that

$$C_\varphi - C_\psi \quad \text{is a compact operator.} \tag{40}$$

Essentially linear fractional composition operators have the aforementioned property and, in their case,  $C_\psi$  is induced by a linear fractional selfmap of  $\mathbb{U}$  [3, Theorem 7.6].

Recall the definition of essentially linear fractional composition operators [3]. They are the composition operators  $C_\varphi$  induced by symbols  $\varphi$  with the following properties:

- $\varphi(\mathbb{U})$  is a subset of an open horocycle (that is of a proper subdisc of  $\mathbb{U}$ , tangent to the unit circle) tangent to  $\mathbb{T}$  at some point  $\eta \in \mathbb{T}$ .
- The set  $\varphi^{-1}(\eta)$  of all points on  $\mathbb{T}$  whose cluster sets contain  $\eta$  is a singleton.
- The map  $\varphi'''$  extends continuously at that singleton.

Relative to composition operators who are compact perturbations of other composition operators recall the following theorem in [34]:

**THEOREM 12.** *For any pair of distinct composition operators  $C_\varphi$  and  $C_\psi$ , one has the following essential norm estimate*

$$\|C_\varphi - C_\psi\|_e \geq \sqrt{m(E_\varphi) + m(E_\psi)}. \tag{41}$$

Thus, only composition operators whose symbols  $\varphi$  satisfy the condition  $|\varphi| < 1$  a.e. have the chance of being compact perturbations of other composition operators. Note that operators with essential linear fractional symbols  $\varphi$  obviously have property  $|\varphi| < 1$  a.e. because the ranges of their symbols are contained by horocycles.

Along these lines, a useful result is:

**PROPOSITION 3.** *If  $\varphi$  and  $\psi$  are analytic selfmaps of  $\mathbb{U}$  satisfying condition (40), then exactly one of the following is true:*

- (i)  $\varphi$  and  $\psi$  are maps having a fixed point in  $\mathbb{U}$ ,
- (ii)  $\varphi$  and  $\psi$  are maps of hyperbolic type,
- (iii)  $\varphi$  and  $\psi$  are maps of parabolic type.

*Proof.* We begin by noting that  $\varphi$  and  $\psi$  cannot be inner maps by (41). Therefore, given the evident equality  $r_e(C_\varphi) = r_e(C_\psi)$ , Theorem 9, and Theorem 10, situation (i) occurs if and only if  $r_e(C_\varphi) < 1$ , situation (ii) if and only if  $r_e(C_\varphi) > 1$ , and situation (iii) if and only if  $r_e(C_\varphi) = 1$ .  $\square$

Proposition 3 can also be proved by combining the fact that the Denjoy-Wolff point of some  $\varphi$ , if situated on  $\mathbb{T}$ , is the only boundary fixed point having the property  $\varphi'(\omega) \leq 1$ , with the following interesting theorem proved by MacCluer in [23]:

**THEOREM 13.** *If  $C_\varphi$  and  $C_\psi$  satisfy (40), then for all  $\omega \in \mathbb{T}$ ,  $\varphi$  has an angular derivative at  $\omega$  if and only if  $\psi$  has the same property and, in that case, one has that:*

$$\varphi(\omega) = \psi(\omega) \quad \text{and} \quad \varphi'(\omega) = \psi'(\omega).$$

Our next application is to obtain as a consequence of the results above [2, Theorems 3.1 and 3.2 ]:

**COROLLARY 2.** *If  $\varphi$  is an essentially linear-fractional symbol fixing a point in  $\mathbb{U}$ , then  $\sigma(C_\varphi)$  is a disc centered at the origin plus the Schröder eigenvalues and 1. The radius of that disc is  $r_e(C_\varphi)$ , a quantity computable with the following formulas:*

$$r_e(C_\varphi) = 0 \text{ if } \varphi \text{ has no boundary fixed points.} \tag{42}$$

$$r_e(C_\varphi) = \frac{1}{\sqrt{\varphi'(\eta)}} \text{ if } \varphi \text{ has a (necessarily unique) boundary fixed point } \eta. \tag{43}$$

*If  $\varphi$  is an essentially linear-fractional symbol of hyperbolic type, then  $\sigma_e(C_\varphi) = \sigma(C_\varphi) = (1/\sqrt{\varphi'(\omega)})\overline{\mathbb{U}}$ , where  $\omega \in \mathbb{T}$  is the Denjoy-Wolff point of  $\varphi$ .*

Indeed, if  $\varphi$  is an essentially linear fractional symbol, then  $\sigma_e(C_\varphi) = \sigma_e(C_\psi)$  where  $\psi$  is a linear fractional map satisfying condition (40). If  $\varphi$  has a fixed point in  $\mathbb{U}$ , then so does  $\psi$ , and  $\psi$  is not a disc automorphism (since  $|\psi| < 1$  a.e). In that case,  $\sigma(C_\psi)$  contains the disc centered at the origin having radius  $r_e(C_\varphi)$  ([15, Theorem 7.30]) and so, the desired description of the spectrum follows by Remark 2.

By Theorem 13,  $\psi$  and  $\varphi$  have the same boundary fixed points, if any. If there aren't any, then  $\|\psi \circ \psi\|_\infty < 1$ , hence  $C_\varphi^2$  and  $C_\psi^2$  are compact, so  $r_e(C_\varphi) = 0$ . If there is a boundary fixed point  $\eta$ , that point is unique (since  $\psi$  is univalent and has circular range internally tangent to  $\mathbb{T}$ ). All iterates  $\psi^{[n]}$  share this unique fixed point and, since  $\psi^{[-n]}(u)$  is a finite set for all  $u \in \mathbb{T}$ , one has that

$$\|C_\psi^n\|_e = \|C_{\psi^{[n]}}\|_e = \frac{1}{(\psi'(\eta))^{n/2}} \quad n = 1, 2, 3, \dots$$

(see [11, page 63]), and hence

$$r_e(C_\varphi) = r_e(C_\psi) = \lim_{n \rightarrow +\infty} \sqrt[n]{\|C_\psi^n\|_e} = \frac{1}{\sqrt{\psi'(\eta)}} = \frac{1}{\sqrt{\varphi'(\eta)}}.$$

If  $\varphi$  is of hyperbolic type, then so is  $\psi$ . It follows that  $C_\psi$  is hypercyclic [6], so Theorems 10 and 11 combine to show that  $\sigma_e(C_\varphi) = \sigma(C_\varphi) = (1/\sqrt{\varphi'(\omega)})\overline{\mathbb{U}}$ .

REMARK 4. Our considerations providing a shortcut for [2, Theorems 3.1 and 3.2 ], miss one fact: in the case when  $\varphi$  fixes a point and  $C_\varphi$  has positive essential spectral radius, the disc involved in the “disc plus Schröder eigenvalues” representation of spectra is not proved to equal  $\sigma_e(C_\varphi)$ , a fact known to hold [2].

### 5. On the spectrum of $C_\varphi^*$

The following stronger version of the Denjoy-Wolff theorem is our tool to get information on  $\sigma(C_\varphi^*)$ .

THEOREM 14. ([5, Theorem 3.1]) *Let  $\varphi$  be an analytic selfmap of  $\mathbb{U}$  with Denjoy-Wolff point  $\omega$ . If  $|\omega| = 1$ , then  $\|\varphi^{[n]} - \omega\| \rightarrow 0$  when  $n \rightarrow +\infty$ . If  $\varphi$  is a non-inner analytic selfmap of  $\mathbb{U}$  fixing a point  $\omega \in \mathbb{U}$ , then  $\|\varphi^{[n]} - \omega\| \rightarrow 0$  when  $n \rightarrow +\infty$ .*

*Proof.* For the sake of completeness again, here is a very short proof. If  $|\omega| = 1$ , the statement in Theorem 14 is an immediate consequence of the Denjoy-Wolff theorem itself and estimate

$$\|\varphi^{[n]} - \omega\|^2 \leq 2(1 - \operatorname{Re}(\overline{\omega}\varphi^{[n]}(0))) \quad n = 1, 2, 3, \dots$$

which holds for parabolic or hyperbolic symbols and was established in (39). If  $|\omega| < 1$ , the conclusion of Theorem 14 is an immediate consequence of (9) and (10).  $\square$

Based on Theorem 14, one can prove the following technical lemma:

LEMMA 1. *Let  $\mathbb{C}[Z]$  denote the set of analytic polynomials with complex coefficients. If  $\varphi$  is an analytic selfmap of  $\mathbb{U}$  with Denjoy-Wolff point  $\omega$  and  $|\omega| = 1$ , then  $\|p \circ \varphi^{[n]} - p(\omega)\| \rightarrow 0$  when  $n \rightarrow +\infty$  for all  $p \in \mathbb{C}[Z]$ . If  $\varphi$  is not an inner function and  $|\omega| < 1$ , then again  $\|p \circ \varphi^{[n]} - p(\omega)\| \rightarrow 0$  when  $n \rightarrow +\infty$  for all  $p \in \mathbb{C}[Z]$ .*

*Proof.* Choose  $p \in \mathbb{C}[Z]$  arbitrary and fixed. Note that  $q = p - p(\omega)$  is an analytic polynomial with property  $q(\omega) = 0$  for which reason  $q$  can be written  $q(z) = (z - \omega)f(z)$  for some  $f \in \mathbb{C}[Z]$ . Note also that for all positive integers  $n$ , one has that

$$p \circ \varphi^{[n]} - p(\omega) = q \circ \varphi^{[n]} = (\varphi^{[n]} - \omega)f \circ \varphi^{[n]}.$$

Therefore

$$\|p \circ \varphi^{[n]} - p(\omega)\| \leq \|f \circ \varphi^{[n]}\|_\infty \|\varphi^{[n]} - \omega\| \leq \|f\|_\infty \|\varphi^{[n]} - \omega\| \rightarrow 0. \quad \square$$

Based on Lemma 1, one obtains the following estimates of the point spectrum  $\sigma_p(C_\varphi^*)$  of  $C_\varphi^*$ :

PROPOSITION 4. *Let  $\varphi$  be an analytic selfmap of  $\mathbb{U}$  with Denjoy-Wolff point  $\omega$ . If  $|\omega| = 1$ , then*

$$\sigma_p(C_\varphi^*) \subseteq \limsup_{n \rightarrow +\infty} \sqrt[n]{\|\varphi^{[n]} - \omega\|} \overline{\mathbb{U}} \subseteq \sqrt{\varphi'(\omega)} \overline{\mathbb{U}}. \tag{44}$$

If  $\varphi$  is non-inner and  $|\omega| < 1$ , then

$$\sigma_p(C_\varphi^*) \subseteq \limsup_{n \rightarrow +\infty} \sqrt[n]{\|\varphi^{[n]} - \omega\|} \overline{\mathbb{U}} \cup \{1\}. \tag{45}$$

If  $\varphi$  is of parabolic type, then

$$\sigma_p(C_\varphi^*) \subseteq \mathbb{U}. \tag{46}$$

*Proof.* Assume  $|\omega| = 1$ . The set  $\mathbb{C}[Z]_\omega$  of all analytic polynomials null at  $\omega$  is a dense subset of  $H^2$  [6]. Thus, consider  $\lambda \in \mathbb{C}$ ,  $|\lambda| > \limsup_{n \rightarrow +\infty} \sqrt[n]{\|\varphi^{[n]} - \omega\|}$  with property  $C_\varphi^* f = \lambda f$  for some  $f \in H^2$ . We will prove that  $f = 0$  by showing  $f$  is orthogonal to  $\mathbb{C}[Z]_\omega$ . Take any  $p(z) = (z - \omega)q(z) \in \mathbb{C}[Z]_\omega$  and note one can use the Cauchy-Schwarz inequality and write

$$|\langle f, p \rangle| = \frac{|\langle f, (\varphi^{[n]} - \omega)q \circ \varphi^{[n]} \rangle|}{|\lambda|^n} \leq \|f\| \|q\|_\infty \frac{\|\varphi^{[n]} - \omega\|}{|\lambda|^n} \rightarrow 0$$

because, under our assumptions

$$\sum_{n=1}^{+\infty} \frac{\|\varphi^{[n]} - \omega\|}{|\lambda|^n} < +\infty.$$

This establishes the relation (44), given Proposition 1.

If  $\varphi$  is not inner and  $|\omega| < 1$  then, the set  $\mathbb{C}[Z]_\omega$  of all analytic polynomials null at  $\omega$  is a dense subset of the orthocomplement of the 1-dimensional subspace spanned by the kernel function  $k_\omega$  and so, if one assumes  $C_\varphi^* f = \lambda f$  for some  $f \in H^2$ , the same argument as above leads to the fact that  $f$  is a scalar multiple of  $k_\omega$ . By (4), this means  $f$  is either null or an eigenfunction associated to the eigenvalue 1, which establishes (45).

If  $\varphi$  is of parabolic type, we must prove no unimodular number is an eigenvalue of  $C_\varphi^*$ . Choose  $\lambda \in \mathbb{C}$ ,  $|\lambda| = 1$ ,  $\lambda \neq 1$  and assume  $C_\varphi^* f = \lambda f$  for some  $f \in H^2$ . We want to prove  $f = 0$ . Arguing by contradiction, assume  $f$  is not the null function. Therefore  $\langle f, p \rangle \neq 0$  for some  $p \in \mathbb{C}[Z]$ , since  $\mathbb{C}[Z]$  is a dense subset of  $H^2$ . Observe that

$$\langle f, p \rangle = \frac{1}{\lambda^n} \langle (C_\varphi^*)^n f, p \rangle = \frac{\langle f, p \circ \varphi^{[n]} \rangle}{\lambda^n} \quad n = 1, 2, 3, \dots$$

hence

$$\lambda^n = \frac{\langle f, p \circ \varphi^{[n]} \rangle}{\langle f, p \rangle} \rightarrow \frac{\langle f, p(\omega) \rangle}{\langle f, p \rangle}.$$

This is a contradiction because, under our assumptions, the sequence  $\{\lambda^n\}$  is divergent. Thus  $f = 0$ .

The case  $\lambda = 1$  needs to be treated separately. Assume  $C_\varphi^* f = f$  for some  $f \in H^2$ . Then

$$\langle f, p \rangle = \langle (C_\varphi^*)^n f, p \rangle = \langle f, p \circ \varphi^{[n]} \rangle \rightarrow f(0) \overline{p(\omega)} \quad p \in \mathbb{C}[Z]. \tag{47}$$

Now take  $p(z) = z^k$ ,  $k = 0, 1, 2, \dots$  in (47). One obtains that the sequence of Maclaurin coefficients of  $f$  is  $\{f(0)\overline{\omega}^k\}$ . This sequence must be square-summable. Given that  $|\omega| = 1$ , it follows that the aforementioned sequence is the null sequence and hence  $f$  is the null function.  $\square$

COROLLARY 3. *If  $\varphi$  is a non-inner analytic selfmap of  $\mathbb{U}$ , fixing a point  $\omega \in \mathbb{U}$ , and*

$$\limsup_{n \rightarrow +\infty} \sqrt[n]{\|\varphi^{[n]} - \omega\|} = 0, \tag{48}$$

then  $\sigma(C_\varphi)$  satisfies equality (8).

Indeed, this is an immediate consequence of Proposition 4 and Remark 2.

In complex dynamics theory, a fixed point  $\omega$  of an analytic map  $\varphi$  is called a *super-attracting fixed point* if  $\varphi'(\omega) = 0$ . If condition (48) is satisfied, then  $\omega$  is such a fixed point and so, what Corollary 3 says is that  $\sigma(C_\varphi) = \sigma_e(C_\varphi) \cup \{1\}$ . Indeed:

PROPOSITION 5. *If  $\varphi$  is a non-inner analytic selfmap of  $\mathbb{U}$ , fixing  $\omega \in \mathbb{U}$ , and equation (48) holds, then  $\varphi'(\omega) = 0$ . If  $\varphi(\omega) = \omega$ ,  $\varphi'(\omega) = 0$ , and  $\varphi$  induces an essentially quasinilpotent composition operator, then equation (48) holds.*

*Proof.* To begin, assume  $\omega = 0$ . An elementary computation shows that, if  $\varphi(0) = 0$  then  $C_\varphi^*(z) = \varphi'(0)z$ . If one assumes  $\varphi'(0) \neq 0$ , then  $\sigma(C_\varphi^*)$ , contains values different from 0 and 1 and so, by Proposition 4, condition (48) fails for such  $\varphi$ . For arbitrary  $\omega \in \mathbb{U}$ , consider the operator similarity (9), and note that  $\psi$  fixes the origin.

If  $\varphi(0) = \varphi'(0) = 0$ , then the subspace  $\mathbb{C}$  of constant functions is a reducing subspace of  $C_\varphi$ . Denote by  $I$  the identity operator acting on that space. If  $r_e(C_\varphi) = 0$ , then, given that  $C_\varphi = I \oplus (C_\varphi|zH^2)$ , it follows that the restriction  $C_\varphi|zH^2$  of  $C_\varphi$  to the orthocomplement  $zH^2$  of  $\mathbb{C}$  is a quasinilpotent operator. Indeed,  $\sigma(C_\varphi) = \{0, 1\}$  and  $\sigma(C_\varphi|zH^2)$  cannot contain 1 given that  $\|C_\varphi|zH^2\| < 1$  (by Theorem 3). Thus

$$\limsup_{n \rightarrow +\infty} \sqrt[n]{\|\varphi^{[n]}\|} = \limsup_{n \rightarrow +\infty} \sqrt[n]{\|(C_\varphi|zH^2)^n(z)\|} \leq r((C_\varphi|zH^2)) = 0,$$

that is (48) holds.

For arbitrary  $\omega \in \mathbb{U}$  and  $z \in \mathbb{U}$ , note first that

$$|\omega - \alpha_\omega(z)| = \frac{(1 - |\omega|^2)|z|}{|1 - \overline{\omega}z|}$$

and hence

$$(1 - |\omega|)|z| \leq |\omega - \alpha_\omega(z)| \leq (1 + |\omega|)|z|. \tag{49}$$

Consider the operator similarity (9), note that  $r_e(C_\psi) = r_e(C_\varphi) = 0$ ,  $\psi(0) = 0$ , and using (49), one can write for all  $n = 1, 2, 3, \dots$  and  $z \in \mathbb{U}$

$$(1 - |\omega|)|\psi^{[n]}(\alpha_\omega(z))| \leq \left| \omega - \alpha_\omega(\psi^{[n]}(\alpha_\omega(z))) \right| \leq (1 + |\omega|)|\psi^{[n]}(\alpha_\omega(z))| \tag{50}$$

which leads to

$$(1 - |\omega|) \|C_{\alpha_\omega}(\psi^{[n]})\| \leq \| \omega - \varphi^{[n]} \| \leq (1 + |\omega|) \|C_{\alpha_\omega}(\psi^{[n]})\| \tag{51}$$

and hence to

$$c_1 \|\psi^{[n]}\| \leq \| \omega - \varphi^{[n]} \| \leq c_2 \|\psi^{[n]}\| \quad n = 1, 2, 3, \dots \tag{52}$$

for some positive constants  $c_1$  and  $c_2$ , because the bounded operator  $C_{\alpha_\omega}$  is invertible and hence, bounded below. Therefore

$$\sqrt[n]{\| \omega - \varphi^{[n]} \|} \leq \sqrt[n]{c_2 \|\psi^{[n]}\|} \rightarrow 0. \quad \square$$

The only closed range, essentially quasinilpotent composition operators are those induced by constant symbols. Actually:

REMARK 5. If  $\varphi$  is an analytic selfmap of  $\mathbb{U}$ , fixing  $\omega \in \mathbb{U}$ , and equation (48) holds, then  $C_\varphi$  is a closed range operator only if  $\varphi$  is constant.

Indeed, if  $\varphi$  is not constant and  $C_\varphi$  is a closed range operator, then  $C_\varphi$  is bounded below (since it is an injective operator), that is there is some  $c > 0$  such that

$$\|C_\varphi f\| \geq c \|f\| \quad f \in H^2.$$

Therefore, for all constants  $\omega$ ,

$$\|\varphi^{[n]} - \omega\| = \|C_\varphi^n(z - \omega)\| \geq c^n \|z - \omega\| \quad n = 1, 2, 3, \dots$$

and hence

$$\liminf_{n \rightarrow +\infty} \sqrt[n]{\|\varphi^{[n]} - \omega\|} \geq c > 0.$$

The Hilbert Hardy space is the Hardy space of index 2. This means that  $H^p$ , the Hardy spaces of index  $0 < p < +\infty$ , consist of all analytic functions  $f$  on  $\mathbb{U}$  with property

$$\|f\|_p := \sup_{0 < r < 1} \left( \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) \right)^{1/p} < +\infty. \tag{53}$$

As observed in [5], a fast consequence of the fact that the fixed point  $\omega \in \mathbb{U}$  of some non-inner analytic selfmap  $\varphi$  is  $\|\cdot\|_p$ -attractive (that is of property  $\|\varphi^{[n]} - \omega\| \rightarrow 0$ ) is the property

$$\|\varphi^{[n]} - \omega\|_p \rightarrow 0 \quad 0 < p < +\infty$$

satisfied by all non-inner  $\varphi$  fixing  $\omega$ . We will show that, if the  $\|\cdot\|_1$ -attractiveness of  $\omega$  is compatible to its  $\|\cdot\|_p$ -attractiveness (note that we continue to denote  $\|\cdot\|_2$  simply by  $\|\cdot\|$ ), then property (48) holds, provided  $\varphi'(\omega) = 0$ .

More formally, we say that the  $\|\cdot\|_1$ -attractiveness of  $\omega$  is *compatible* to its  $\|\cdot\|$ -attractiveness and denote

$$\{\|\varphi^{[n]} - \omega\|\} \sim \{\|\varphi^{[n]} - \omega\|_1\}$$

if there are positive constants  $c_1, c_2$  and some positive integer  $n_0$  so that

$$c_1\|\varphi^{[n]} - \omega\|_1 \leq \|\varphi^{[n]} - \omega\| \leq c_2\|\varphi^{[n]} - \omega\|_1 \quad n \geq n_0$$

a fact that is equivalent to

$$\limsup_{n \rightarrow +\infty} \frac{\|\varphi^{[n]} - \omega\|}{\|\varphi^{[n]} - \omega\|_1} < +\infty$$

given that  $\|\cdot\|_1 \leq \|\cdot\|$ .

It is important to make the following:

REMARK 6. If  $\varphi$  and  $\psi$  are related as in (9), then

$$\{\|\varphi^{[n]} - \omega\|_p\} \sim \{\|\psi^{[n]}\|_p\} \tag{54}$$

for all fixed  $0 < p \leq +\infty$ .

Indeed, the argument linking (49)–(52) can be repeated, using  $\|\cdot\|_p$  instead of  $\|\cdot\|$ . Keeping that in mind, we prove

THEOREM 15. *Let  $\varphi$  be a non-inner analytic selfmap of  $\mathbb{U}$  fixing  $\omega \in \mathbb{U}$  and having property  $\varphi'(\omega) = 0$ . Then  $\sqrt[n]{\|\varphi^{[n]} - \omega\|} \rightarrow 0$  if any of the conditions below hold:*

$$\lim_{n \rightarrow +\infty} \frac{\|\varphi^{[n]} - \omega\|}{\sqrt{\|\varphi^{[n]} - \omega\|_1}} = 0 \tag{55}$$

$$\{\|\varphi^{[n]} - \omega\|\} \sim \{\|\varphi^{[n]} - \omega\|_1\} \tag{56}$$

$$\|(\varphi - \omega)(\varphi^{[2]} - \omega) \dots (\varphi^{[n]} - \omega)\|_\infty \rightarrow 0. \tag{57}$$

If  $\omega = 0$ , then condition (57) is equivalent to the following

$$\|\varphi\varphi^{[2]} \dots \varphi^{[k]}\|_\infty < 1 \quad \text{for some } k. \tag{58}$$

*Proof.* Assume first that  $\omega = 0$ . Let  $\phi(z) = \varphi(z)/z$ ,  $z \neq 0$  and  $\phi(0) = 0$ . The map  $\phi$  satisfies the assumptions of the Schwarz lemma in classical complex analysis, for which reason, one can write for all  $z \in \mathbb{U}$  and  $n = 1, 2, 3, \dots$

$$|\phi(\varphi^{[n]}(z))| \leq |\varphi^{[n]}(z)|$$

that is

$$|\varphi^{[n+1]}(z)| \leq |\varphi^{[n]}(z)|^2, \tag{59}$$

which leads immediately to

$$\sqrt{\|\varphi^{n+1}\|_1} \leq \|\varphi^{[n]}\| \quad n = 1, 2, 3, \dots$$

On the other hand,  $C_\varphi$  is a contraction and hence

$$\|\varphi^{[n+1]}\| \leq \|\varphi^{[n]}\| \quad n = 1, 2, 3, \dots$$

Therefore, one has that

$$\left| \sqrt{\|\varphi^{[n+1]}\|_1} - \|\varphi^{[n+1]}\| \right| \leq \|\varphi^{[n]}\| \quad n = 1, 2, 3, \dots$$

In other words:

$$\frac{\|\varphi^{[n]}\|}{\|\varphi^{[n+1]}\|} \geq \left| \frac{\sqrt{\|\varphi^{[n+1]}\|_1}}{\|\varphi^{[n+1]}\|} - 1 \right| \rightarrow +\infty$$

which implies  $\frac{\|\varphi^{[n+1]}\|}{\|\varphi^{[n]}\|} \rightarrow 0$  and hence  $\sqrt[n]{\|\varphi^{[n]}\|} \rightarrow 0$ , by (31).

If  $\{\|\varphi^{[n]}\|\} \sim \{\|\varphi^{[n]}\|_1\}$ , there is some  $c > 0$  so that

$$\frac{\|\varphi^{[n]}\|}{\sqrt{\|\varphi^{[n]}\|_1}} \leq c \sqrt{\|\varphi^{[n]}\|_1} \rightarrow 0.$$

Let us write inequality (59) for successive values of  $n$ , that is, let us write

$$|\varphi^{[k+1]}(z)| \leq |\varphi^{[k]}(z)|^2, \quad k = 0, \dots, n$$

then multiply those inequalities, and let  $|z| \rightarrow 1$ . One gets

$$|\varphi^{[n+1]}| |\varphi^{[n]} \varphi^{[n-1]} \dots \varphi^{[2]} \varphi| \leq |\varphi^{[n]} \varphi^{[n-1]} \dots \varphi^{[2]} \varphi|^2 \quad \text{a.e.}$$

Given that

$$\varphi^{[n]} \varphi^{[n-1]} \dots \varphi^{[2]} \varphi \neq 0 \quad \text{a.e.,}$$

it follows that

$$|\varphi^{[n+1]}| \leq |\varphi^{[n]} \varphi^{[n-1]} \dots \varphi^{[2]} \varphi| \quad \text{a.e.}$$

and hence

$$\frac{\|\varphi^{[n+1]}\|}{\|\varphi^{[n]}\|} \leq \|\varphi^{[n-1]} \dots \varphi^{[2]} \varphi\|_\infty.$$

Therefore, if  $\|\varphi^{[n]} \dots \varphi^{[2]} \varphi\|_\infty \rightarrow 0$ , then  $\sqrt[n]{\|\varphi^{[n]}\|} \rightarrow 0$ .

If that is the case, then (58) holds. Conversely, if condition (58) holds, then that fact implies that  $\|\varphi^{[n]} \dots \varphi^{[2]} \varphi\|_\infty \rightarrow 0$ . Indeed, if one denotes  $\|\varphi \varphi^{[2]} \dots \varphi^{[k]}\|_\infty := \lambda < 1$  and  $S_n := \|\varphi^{[n]} \dots \varphi^{[2]} \varphi\|_\infty$ , then clearly  $S_n$  is a decreasing sequence of nonnegative numbers which contains the subsequence  $S_{nk}$  convergent to 0. Indeed, denote  $F := \varphi \varphi^{[2]} \dots \varphi^{[k]}$  and observe that:

$$S_{nk} = \|\varphi \varphi^{[2]} \dots \varphi^{[nk]}\|_\infty = \|F(F \circ \varphi^{[k]})(F \circ \varphi^{[2k]}) \dots (F \circ \varphi^{[(n-1)k]})\|_\infty \leq$$

$$\|F\|_{\infty}^n = \lambda^n \rightarrow 0.$$

For arbitrary  $\omega \in \mathbb{U}$ , consider  $\psi$  related to  $\varphi$  by (9) and note that the statements in this theorem are consequences of the case  $\omega = 0$  and Remark 6.  $\square$

In conclusion, the dynamical behavior of sequences of iterates of a selfmap influences the spectra of the composition operator induced by that map, and vice versa, spectral properties of composition operators lead to strong attractiveness properties of the point fixed by their symbol (such as (48) for instance), thus exhibiting a nice bilateral interaction of complex dynamics and the spectral theory of composition operators.

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