

ON THE CLOSURE OF THE DISCRETE SPECTRUM OF NUCLEARLY PERTURBED OPERATORS

FRANZ HANAUSKA

(Communicated by H. Bercovici)

Abstract. Let Z_0 be a bounded operator in a Banach space X with purely essential spectrum and K a nuclear operator in X . Using methods of complex analysis we study the set of accumulation points of the discrete spectrum of the operator $Z := Z_0 + K$. We formulate conditions for Z to exclude certain points or subsets of the essential spectrum of Z to be accumulation points of the discrete spectrum. These results are applied to the operator of multiplication perturbed by integral operators with continuous kernel and to the discrete Laplacian perturbed by nuclear Jacobi operators.

1. Introduction and results

In this paper we study the discrete spectrum of a bounded operator $Z = Z_0 + K$ on a complex Banach space X , where Z_0 is a bounded operator with purely real essential¹ spectrum (for example $\sigma(Z_0) = [a, b]$ where $[a, b]$ is an interval) and K a nuclear operator. By Weyl we know that $\sigma_{\text{ess}}(Z) = \sigma_{\text{ess}}(Z_0)$ (since every nuclear operator is a compact operator, see e.g. Kato [5] p. 238). Moreover, in this case the spectrum of Z is the disjoint union of the essential and the discrete² part, i.e. $\sigma(Z) = \sigma_{\text{ess}}(Z) \cup \sigma_{\text{disc}}(Z)$ (see e.g. Gohberg, Goldberg and Kaashoek [3] p.373 and Davies [1] p.122) and the discrete spectrum can only accumulate at the essential spectrum. The aim of this note is to find conditions to exclude certain points or subsets of $\sigma_{\text{ess}}(Z)$ to be accumulation points of the discrete spectrum.

THEOREM 1.1. *Let $E \subseteq \sigma_{\text{ess}}(Z_0)$ be an open set (open in \mathbb{R}) with the property that the operator valued map $\lambda \mapsto KR_{Z_0}(\lambda)$ with $\lambda \in \rho(Z_0)$ can be continuously extended to $\rho(Z_0) \cup E$, then $E \cap \sigma_{\text{disc}}(Z) = \emptyset$.*

If $\lambda_0 \in \sigma_{\text{ess}}(Z_0)$ is a single point with the property that there is a continuous extension of $KR_{Z_0}(\cdot)$ namely $KR_{Z_0}(\lambda_0)$, and 1 is not a discrete eigenvalue of $KR_{Z_0}(\lambda_0)$, then λ_0 is not an accumulation point of $\sigma_{\text{disc}}(Z)$.

Mathematics subject classification (2010): 26D15, 26A51, 32F99, 41A17.

Keywords and phrases: Eigenvalues, discrete spectrum, nuclear perturbations.

¹The essential spectrum of a linear operator Z is defined by $\sigma_{\text{ess}}(Z) := \{\lambda \in \mathbb{C} : \lambda - Z \text{ is not a Fredholm operator}\}$, where an operator A is Fredholm if A has closed range and both, the kernel and the cokernel of A are finite dimensional.

²The discrete spectrum of a linear Operator Z is defined by $\sigma_{\text{disc}}(Z) := \{\lambda \in \mathbb{C} : \lambda \text{ is a discrete eigenvalue of } Z\}$, where an eigenvalue $\lambda \in \sigma(Z)$ is discrete if it is isolated and its corresponding Riesz projection $\frac{1}{2\pi i} \int_{\Gamma} (\lambda - A)^{-1} d\lambda$ is of finite rank, where Γ is a circle with center λ excluding all other elements of $\sigma(Z)$.

Before proving Theorem 1.1 we want to illustrate in Corollary 1.2 and Corollary 1.3, that the assumptions of Theorem 1.1 can be fulfilled such that Theorem 1.1 is applicable in several situations.

We set $X = C[\alpha, \beta]$, the space of continuous functions on an interval $[\alpha, \beta]$, $Z_0 = M_f$ the operator of multiplication defined by $(M_f g)(t) := f(t)g(t)$, where $f \in C[\alpha, \beta]$ is a real-valued injective function, and K is an integral operator with continuous kernel, i.e. $(Kg)(t) := \int_{\alpha}^{\beta} k(t, s)g(s)ds$ with $k(\cdot, \cdot)$ continuous. In this case $\sigma_{ess}(M_f) = \{f(x)|x \in [\alpha, \beta]\}$.

The set $f(I) \subseteq \sigma_{ess}(M_f)$, where

$$I := \{x \in [\alpha, \beta] | k(t, x) = 0 \text{ for all } t \in [\alpha, \beta]\},$$

plays an important role in the next corollary.

COROLLARY 1.2. *Let $Z := M_f + K$ be as described above, then the discrete spectrum of Z does not accumulate to any point belonging to $\text{int} f(I)$ (the inner points of $f(I)$ according to \mathbb{R}).*

It is also possible to apply Theorem 1.1 to another interesting class of operators. We define $Z_0 := \Delta$ as the discrete Laplacian on $l^1(\mathbb{Z})$ given by the infinite matrix

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & 0 & 1 & & \\ & & & 1 & 0 & 1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1}$$

and $\sigma_{ess}(\Delta) = [-2, 2]$ (see section 2). The nuclear operator $K := J$ is a Jacobi operator given by the infinite matrix

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & \alpha_{-1} & \beta_{-1} & \alpha_{-1} & & & \\ & & \alpha_0 & \beta_0 & \alpha_0 & & \\ & & & \alpha_1 & \beta_1 & \alpha_1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2}$$

with $\alpha_k = x_k a$, $\beta_k = x_k b$, where $(x_k) \in l^1(\mathbb{C})$ and $a, b \in \mathbb{C}$.

COROLLARY 1.3. *Let $Z := \Delta + J$. Then the point $-\frac{b}{a}$ is not an accumulation point of $\sigma_{disc}(Z)$.*

REMARK 1.4. If $-\frac{b}{a} \notin \sigma_{ess}(Z)$ this assertion is trivial, since there is no accumulation point of the discrete spectrum outside the essential spectrum.

2. Proof of Theorem 1.1, Corollary 1.2 and Corollary 1.3

Before we prove the previous theorem and corollaries we summarize some facts about nuclear operators and discrete eigenvalues. One essential tool is to define a holomorphic function on $\rho(Z_0)$ the zeros of which coincide with the discrete eigenvalues of Z .

LEMMA 2.1. (see Demuth and Hanauska [2] Section 3) *Let $A(\cdot)$ be an analytic nuclear operator-valued function on a domain Ω , then the function $\det(\mathbb{1} - A(\cdot))$ is holomorphic on Ω and $|\det(\mathbb{1} - A(\lambda))| \leq \exp\left(\frac{1}{2}\|A(\lambda)\|_{\mathcal{N}}^2\right)$.³*

For Jacobi operators it would be possible to treat the object in some other l^p space. The advantage of l^1 is, that there is an explicit formula of the nuclear norm according to the matrix-representation of an arbitrary operator. However, there is also an estimate for the nuclear norm of an integral operator in $C[\alpha, \beta]$ with continuous kernel.

LEMMA 2.2. (see Gohberg, Goldberg and Krupnik [4] Chapter V Theorem 2.1 and Theorem 2.2) *For any nuclear operator $A \in \mathcal{N}(l^1(\mathbb{Z}))$ with matrix representation (a_{kj}) the nuclear norm is given by the formula*

$$\|A\|_{\mathcal{N}} = \sum_{k \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} |a_{kj}|.$$

Every integral operator $K : C[\alpha, \beta] \rightarrow C[\alpha, \beta]$ with continuous kernel $k(\cdot, \cdot)$ is nuclear and the nuclear norm can be estimated by

$$\|K\|_{\mathcal{N}} \leq \int_{\alpha}^{\beta} \sup_{t \in [\alpha, \beta]} |k(t, s)| ds.$$

Now we define for a bounded operator Z_0 and a nuclear operator K on an arbitrary Banach space X ,

$$d(\lambda) := \det(\mathbb{1} - KR_{Z_0}(\lambda)), \text{ with } \lambda \in \rho(Z_0).$$

Since $\mathcal{N}(X)$ is a Banach ideal (see [4] p. 92) d is well defined. We know that the resolvent is analytic in $\rho(Z_0)$ and hence with Lemma 2.1 also d is holomorphic on $\rho(Z_0)$.

If we have a look on the following equation

$$(\lambda - Z)R_{Z_0}(\lambda) = \mathbb{1} - KR_{Z_0}(\lambda), \text{ with } \lambda \in \rho(Z_0),$$

we see that the righthand side is not invertible iff $\lambda \in \sigma_{disc}(Z_0)$. And this is equivalent to $\mathbb{1} - KR_{Z_0}(\lambda)$ is not invertible. In this case 1 is an eigenvalue of $\mathbb{1} - KR_{Z_0}(\lambda)$. Comparing this and the definition of \det we have

$$\lambda \in \sigma_{disc}(Z) \Leftrightarrow \det(\mathbb{1} - KR_{Z_0}(\lambda)) = 0.$$

³ $\det(\mathbb{1} - K) := \prod_{\lambda \in \sigma(K)} (1 - \lambda) \exp(\lambda)$ for all $K \in \mathcal{N}(X)$, where $\mathcal{N}(X)$ denotes the Banach ideal of all nuclear operators.

Now it is not hard to prove Theorem 1.1.

Proof of Theorem 1.1. Let d be defined as before.
 Let $\lambda_0 \in \sigma_{\text{ess}}(Z_0)$ and as assumed in Theorem 1.1

$$\|KR_{Z_0}(\lambda) - KR_{Z_0}(\lambda_0)\|_{\mathcal{N}} \xrightarrow{\lambda \rightarrow \lambda_0} 0.$$

Combining the inequality in Lemma 2.1 and [4] Theorem II.4.1 we can derive the Lipschitz-type inequality

$$\begin{aligned} \|d(\lambda) - d(\lambda_0)\|_{\mathcal{N}} &\leq \|KR_{Z_0}(\lambda) - KR_{Z_0}(\lambda_0)\|_{\mathcal{N}} \\ &\quad \times \exp\left(\frac{1}{2}(\|KR_{Z_0}(\lambda)\|_{\mathcal{N}}^2 + KR_{Z_0}(\lambda_0)\|_{\mathcal{N}}^2 + 1)\right), \end{aligned}$$

and obtain, that d can be extended continuously.

Now, if the set of points in E , for which d is continuously extendable, is an open set (open according to \mathbb{R}) the Theorem of Morera (see e.g. [6]) tells us that even d is holomorphically extendable to $\rho(Z_0) \cup E$. Since the zeros of every non-zero function do not accumulate in its domain, we know that the zeros of d cannot accumulate in E . But this is equivalent to the assertion, that the discrete spectrum of Z does not accumulate to E .

Now, if there is only a single point $\lambda_0 \in \sigma_{\text{ess}}(Z_0)$, with the property that there is a continuous extension of $KR_{Z_0}(\cdot)$, we obtain that there is also a continuous extension of d . If $1 \notin \sigma(KR_{Z_0}(\lambda_0))$, $d(\lambda_0) \neq 0$. Hence it is not possible for the zeros of d to accumulate at λ_0 and also not for the discrete spectrum of Z . \square

Now we come back to Corollary 1.2 and Corollary 1.3.

Proof of Corollary 1.2. We assume $\text{int}(f(I)) \neq \emptyset$ and we take a $\lambda_0 \in \text{int}(f(I))$. We have to show, that there is a continuous extension of $KR_{M_f}(\cdot)$ from $\rho(M_f)$ to λ_0 . Obviously, since in this case the function $(t, x) \mapsto k(t, x)(f(x) - \lambda_0)^{-1}$ is a continuous function, there is a nuclear extension of $KR_{M_f}(\cdot)$ from $\rho(M_f)$ to the point λ_0 (lets call it $KR_{M_f}(\lambda_0)$). Now we have to show that this extension is also continuous. For this let $\lambda \in \rho(M_f)$:

$$\begin{aligned} &\|KR_{M_f}(\lambda) - KR_{M_f}(\lambda_0)\|_{\mathcal{N}} \\ &\leq \int_{\alpha}^{\beta} \sup_{t \in [\alpha, \beta]} |k(t, x)(f(x) - \lambda)^{-1} - k(t, x)(f(x) - \lambda_0)^{-1}| dx \\ &= \int_{\alpha}^{\beta} \sup_{t \in [\alpha, \beta]} |k(t, x)((f(x) - \lambda)^{-1} - (f(x) - \lambda_0)^{-1})| dx \\ &= \int_{\alpha}^{\beta} |(f(x) - \lambda)^{-1} - (f(x) - \lambda_0)^{-1}| \chi_{[\alpha, \beta] \setminus I} \sup_{t \in [\alpha, \beta]} |k(t, x)| dx \\ &\leq \sup_{\xi \in [\alpha, \beta] \setminus I} |(f(\xi) - \lambda)^{-1} - (f(\xi) - \lambda_0)^{-1}| \int_{\alpha}^{\beta} \sup_{t \in [\alpha, \beta]} |k(t, x)| dx \xrightarrow{\lambda \rightarrow \lambda_0} 0. \end{aligned}$$

The function χ_M defines the characteristic function on the set M , i.e. $\chi(x)_M = 1$ if $x \in M$ and else $\chi_M(x) = 0$.

Hence, the map $KR_{Mf}(\cdot)$ is continuously extendable to $\text{int}(f(I))$. \square

Proof of Corollary 1.3. The resolvent set of Δ is $\rho(\Delta) = \mathbb{C} \setminus [-2, 2]$. The resolvent (with $\lambda \in \rho(\Delta)$) is given by the matrix (see [2] Section 4)

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \\ \dots d_{-1}(\lambda) & d_0(\lambda) & d_1(\lambda) & d_2(\lambda) & \dots \\ \dots d_{-2}(\lambda) & d_{-1}(\lambda) & d_0(\lambda) & d_1(\lambda) & \dots \\ \dots d_{-3}(\lambda) & d_{-2}(\lambda) & d_{-1}(\lambda) & d_0(\lambda) & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{3}$$

with $d_k = \frac{1}{\sqrt{\lambda^2 - 4}} \left(\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2} \right)^{|k|}$, where the sign of the square-root has to be taken, such that $\left| \left(\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2} \right) \right| \leq 1$. We can compute the matrix representation of $JR_\Delta(\lambda)$:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \\ \dots \alpha_{-1}d_{-1}(\lambda) + \beta_{-1}d_{-2}(\lambda) + \alpha_{-1}d_{-3}(\lambda) & \alpha_{-1}d_0(\lambda) + \beta_{-1}d_{-1}(\lambda) + \alpha_{-1}d_{-2}(\lambda) & \dots & \dots & \\ \dots \alpha_0d_{-2}(\lambda) + \beta_0d_{-3}(\lambda) + \alpha_0d_{-4}(\lambda) & \alpha_0d_{-1}(\lambda) + \beta_0d_{-2}(\lambda) + \alpha_0d_{-3}(\lambda) & \dots & \dots & \\ \dots \alpha_1d_{-3}(\lambda) + \beta_1d_{-4}(\lambda) + \alpha_0d_{-5}(\lambda) & \alpha_1d_{-2}(\lambda) + \beta_1d_{-3}(\lambda) + \alpha_0d_{-4}(\lambda) & \dots & \dots & \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

It is obvious that this matrix also defines a nuclear operator for $\lambda \in (-2, 2)$. Now we have to show that $JR_\Delta(\lambda) \xrightarrow{\lambda \rightarrow -\frac{a}{b}} JR_\Delta(-\frac{a}{b})$. To do this it would be helpful to compute the single entries $e_{kj}(\lambda)$ of the matrix representaion for $j - k \neq 0$ (we assume $|j - k - 1| > |j - k| > |j - k + 1|$, the other case can be treated in the same way):

$$\begin{aligned} e_{k,j}(\lambda) &:= \alpha_k d_{j-k+1}(\lambda) + \beta_k d_{j-k}(\lambda) + \alpha_k d_{j-k-1}(\lambda) \\ &= \frac{1}{\sqrt{\lambda^2 - 4}} \left(\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2} \right)^{|j-k-1|} \\ &\quad \times \left(\alpha_k \left(\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2} \right)^2 + \beta_k \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2} + \alpha_k \right) \\ &= \frac{1}{\sqrt{\lambda^2 - 4}} \left(\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2} \right)^{|j-k-1|+1} (\alpha_k \lambda + \beta_k) \\ &= \frac{1}{\sqrt{\lambda^2 - 4}} \left(\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2} \right)^{|j-k-1|+1} x_k(a\lambda + b) \end{aligned}$$

So we know that for $|j-k| \neq 0$ the value $-\frac{b}{a}$ is a zero of $e_{k,j}$. Using the nuclear norm formula of Lemma 2.2 we obtain

$$\begin{aligned}
 & \left\| JR_{\Delta}(\lambda) - JR_{\Delta}\left(-\frac{b}{a}\right) \right\|_{\mathcal{N}} \\
 &= \sum_{k \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} \left| e_{k,j}(\lambda) - e_{k,j}\left(-\frac{b}{a}\right) \right| \\
 &= \sum_{k \in \mathbb{Z}} \sup \left\{ \left| \frac{1}{\sqrt{\lambda^2 - 4}} \left(\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2} \right)^{|j-k-1|+1} x_k(a\lambda + b) \right| \right\}_{j \in \mathbb{Z} \setminus \{k\}, j < k} \\
 & \quad \cup \left\{ \left| \frac{1}{\sqrt{\lambda^2 - 4}} \left(\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2} \right)^{|j-k+1|+1} x_k(a\lambda + b) \right| \right\}_{j \in \mathbb{Z} \setminus \{k\}, j > k} \\
 & \quad \cup \left\{ \left| e_{k,k}(\lambda) - e_{k,k}\left(-\frac{b}{a}\right) \right| \right\} \\
 & \leq \sum_{k \in \mathbb{Z}} \sup \left\{ \frac{|x_k(a\lambda + b)|}{\sqrt{\lambda^2 - 4}} \right\} \cup \left\{ \left| e_{k,k}(\lambda) - e_{k,k}\left(-\frac{b}{a}\right) \right| \right\} \\
 & \xrightarrow[\rightarrow^a]{\lambda \rightarrow -\frac{b}{a}} 0. \quad \square
 \end{aligned}$$

Acknowledgement. At the end I want to thank Prof. Dr. Michael Demuth, Dr. habil. Johannes Brasche, Dr. Marcel Hansman and Dr. habil. Thomas Kalmes for very valuable discussions. Once again I want to thank Prof. Dr. Michael Demuth for his helpful suggestions and corrections of the text.

REFERENCES

- [1] E. B. DAVIES, *Linear operators and their spectra*, Cambridge University Press, Cambridge 2008.
- [2] M. DEMUTH & F. HANAUSKA, *On the distribution of the discrete spectrum of nuclearly perturbed operators in Banach spaces*, to be published in Indian J. Pure Appl. Math.
- [3] I. C. GOHBERG, S. GOLDBERG & M. A. KAASHOEK, *Classes of linear operators*, Vol. 1. Birkhäuser Verlag, Basel 1990.
- [4] I. C. GOHBERG, S. GOLDBERG & N. KRUPNIK, *Traces and determinants of linear operators*, Birkhäuser Verlag, Basel 2000.
- [5] T. KATO, *Perturbation theory for linear operators*, Springer-Verlag, Berlin 1995.
- [6] R. REMMERT & G. SCHUMACHER, *Funktionentheorie I*, Springer, Berlin – Heidelberg, 6. edition, 2002.

(Received March 5, 2014)

Franz Hanauska
 Institut für Matheamtaik TU Clausthal
 Erzstr. 1, 38678 Clausthal-Zellerfeld
 e-mail: hanauska@math.tu-clausthal.de