

## AUTOMATIC CONTINUITY FOR LINEAR SURJECTIVE MAPS COMPRESSING THE POINT SPECTRUM

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*Abstract.* We prove that if  $X$  is a complex Banach space,  $\mathcal{A} \subseteq \mathcal{B}(X)$  is a unital standard subalgebra of linear and continuous operators on  $X$  and  $\varphi : \mathcal{A} \rightarrow \mathcal{A}$  a surjective linear map such that for each  $T \in \mathcal{A}$  the point spectrum of  $\varphi(T)$  is a subset of the point spectrum of  $T$ , then  $\varphi$  is automatically continuous. As a corollary, we prove that a characterization of bilocal automorphisms of  $\mathcal{A}$  given by L. Molnár, P. Šemrl and A. R. Sourour can be obtained without any continuity assumption on them.

### 1. Introduction and statement of the main result

Let  $X$  be a complex Banach space and  $\mathcal{B}(X)$  the algebra of linear and bounded operators on  $X$ . For  $T \in \mathcal{B}(X)$  we denote by  $\sigma(T)$ ,  $\sigma_p(T)$  and  $r(T)$  respectively its classical spectrum, its point spectrum and its spectral radius.

A subalgebra  $\mathcal{A} \subseteq \mathcal{B}(X)$  is called *standard* if it is closed and contains all the finite rank operators and *unital* if it contains the identity  $I \in \mathcal{B}(X)$ . Following Zhu and Xiong [5], for a unital standard subalgebra  $\mathcal{A} \subseteq \mathcal{B}(X)$  we call a linear map  $\varphi : \mathcal{A} \rightarrow \mathcal{B}(X)$  a *bilocal derivation* if for every  $T$  in  $\mathcal{A}$  and  $x$  in  $X$  there exists a derivation  $\varphi_{T,x} : \mathcal{A} \rightarrow \mathcal{B}(X)$  (that is, depending on  $T$  and  $x$ ), such that

$$\varphi(T)(x) = (\varphi_{T,x}(T))(x). \quad (1)$$

It is proved in [5, Theorem 3] that any such bilocal derivation must be in fact a derivation.

Replacing derivations with automorphisms, a linear map  $\varphi : \mathcal{A} \rightarrow \mathcal{A}$  is called a *bilocal automorphism* if for every  $T$  in  $\mathcal{A}$  and  $x$  in  $X$  there exists an automorphism  $\varphi_{T,x} : \mathcal{A} \rightarrow \mathcal{A}$  such that (1) holds. Unlike for derivations, in the particular case when  $X$  is an infinite-dimensional separable Hilbert space,  $\mathcal{A} = \mathcal{B}(X)$  and  $\varphi : \mathcal{A} \rightarrow \mathcal{A}$  is a linear bilocal  $*$ -automorphism (that is, given any  $T$  in  $\mathcal{B}(X)$  and  $x$  in  $X$  there exists an algebra  $*$ -automorphism  $\varphi_{T,x}$  of  $\mathcal{B}(X)$  such that (1) holds), then  $\varphi$  is not necessarily a  $*$ -automorphism of  $\mathcal{B}(X)$  ([2, Theorem 1]). In the case of bilocal automorphisms of unital standard operator algebras over an infinite-dimensional separable Banach space, the following result holds.

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**THEOREM 1.** (See [3, Theorem 1.2].) *Let  $X$  be an infinite-dimensional separable complex Banach space and  $\mathcal{A} \subseteq \mathcal{B}(X)$  a unital standard operator algebra. Assume that a linear map  $\varphi : \mathcal{A} \rightarrow \mathcal{A}$  is bijective and continuous. Then  $\varphi$  is a bilocal automorphism if and only if it is an automorphism.*

The authors also provided examples which show that the surjectivity and the separability are indispensable in the statement of Theorem 1. They also conjectured that Theorem 1 holds without the continuity assumption on the map  $\varphi$ . In fact, it is proved at [3, Proposition 2.1] that any bilocal automorphism  $\varphi$  of  $\mathcal{A}$  compresses the point spectrum of operators, that is

$$\sigma_p(\varphi(T)) \subseteq \sigma_p(T) \quad (T \in \mathcal{A}). \quad (2)$$

In the last section of [3] the authors asked whether bijective linear maps on unital standard operator algebras compressing the point spectrum are automatically continuous. The next result asserts that this is true even in the case when  $\varphi$  is supposed to be only surjective.

**THEOREM 2.** *Let  $\mathcal{A}$  be a unital standard algebra on  $X$  and  $\varphi : \mathcal{A} \rightarrow \mathcal{A}$  a surjective linear map such that (2) holds. Then  $\varphi$  is continuous.*

A classical result of B. E. Johnson in the theory of semisimple (unital, complex) Banach algebras states that all the Banach algebra norms on such an algebra are equivalent. The original proof uses mainly representation theory, but a much shorter proof using subharmonicity methods for the spectral radius may be found in [1, Section 5.5]. The norm-equivalence result is obtained as a direct consequence of [1, Theorem 5.5.2], where it is proved that if  $\mathcal{A}$  and  $\mathcal{B}$  are Banach algebras, with  $\mathcal{B}$  semisimple, and  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  linear and onto has the property that it decreases the spectral radius, then  $\varphi$  is automatically continuous. In particular, the same is true by supposing  $\sigma(\varphi(x)) \subseteq \sigma(x)$  for each  $a \in \mathcal{A}$ . Theorem 2 shows that in the case of a unital standard algebra on a Banach space  $X$  the result also holds by replacing the spectrum with the point spectrum which, unlike its classical counterpart, might be empty!

As an application of Theorem 2 we can eliminate the continuity assumption in the statement of Theorem 1.

**THEOREM 3.** *Let  $\mathcal{A}$  be a unital standard operator algebra on  $X$  and  $\varphi : \mathcal{A} \rightarrow \mathcal{A}$  a surjective linear map which is a bilocal automorphism. Then  $\varphi$  is automatically continuous. Furthermore, if  $X$  is an infinite-dimensional separable Banach space and  $\varphi$  is also supposed injective, then  $\varphi$  is an automorphism.*

## 2. Proofs

The following lemma is the key fact in the proof of our main result. It allows us to obtain elements in the point spectrum of a rank-one perturbation of an element in  $\mathcal{B}(X)$ . For  $x \in X$  and  $f \in X'$  (the dual of  $X$ ), by  $x \otimes f \in \mathcal{B}(X)$  we denote the rank one operator given by  $y \mapsto f(y)x$ .

LEMMA 1. (See [4, Lemma 3.2].) Let  $T \in \mathcal{B}(X)$ ,  $x \in X$  and  $f \in X'$  and suppose that  $\lambda \in \mathbb{C} \setminus \sigma(T)$  and  $\mu := f((\lambda I - T)^{-1}x)$  is not zero. Then

$$\lambda \in \sigma_p(T + x \otimes f/\mu).$$

*Proof.* We have

$$\begin{aligned} (\lambda I - T - x \otimes f/\mu)((\lambda I - T)^{-1}x) &= x - \frac{f((\lambda I - T)^{-1}x)}{f((\lambda I - T)^{-1}x)}x \\ &= 0. \end{aligned}$$

Since  $f((\lambda I - T)^{-1}x) \neq 0$  then  $(\lambda I - T)^{-1}x \neq 0$  in  $X$ , and therefore  $\lambda \in \sigma_p(T + x \otimes f/\mu)$ .  $\square$

The next result gives semisimplicity for the algebras considered in the statement of Theorem 2.

LEMMA 2. Let  $\mathcal{A} \subseteq \mathcal{B}(X)$  be a unital standard algebra. Then  $\mathcal{A}$  is semisimple.

*Proof.* Let us recall that  $\mathcal{A}$  is semisimple if its Jacobson radical is zero. By [1, Theorem 3.1.3], for  $Q \in \mathcal{A}$  we have that  $Q \in \text{Rad}(\mathcal{A})$  if and only if  $I - TQ \in \mathcal{A}$  is invertible in  $\mathcal{A}$ , for all  $T \in \mathcal{A}$ . So if  $Q \in \text{Rad}(\mathcal{A})$  then  $\lambda I - TQ \in \mathcal{A}$  is invertible in  $\mathcal{A}$  for all  $T \in \mathcal{A}$  and every nonzero  $\lambda \in \mathbb{C}$ . Therefore,  $r(TQ) = 0$  for all  $T \in \mathcal{A}$ . In particular, this holds for every rank one operator  $T$ .

Suppose there exists  $x \in X$  such that  $Q(x) \neq 0$  in  $X$ . Let then  $f \in X'$  such that  $f(Q(x)) = 1$ . Putting  $T = x \otimes f$ , then  $(TQ)(x) = x$ . Therefore  $1 \in \sigma_p(TQ) \subseteq \sigma(TQ)$ , which implies  $r(TQ) \geq 1$ , arriving to a contradiction. Therefore  $Q(x) = 0$  for all  $x \in X$ .  $\square$

We have used in the above proof (and we shall also use the same fact in the final part of the next proof) the fact that for  $W \in \mathcal{A}$  the spectral radius of  $W$  computed with respect to the subalgebra  $\mathcal{A} \subseteq \mathcal{B}(X)$  equals  $r(W)$ , the spectral radius of  $W \in \mathcal{B}(X)$  computed with respect to  $\mathcal{B}(X)$ .

We are now ready for the proof of our main result.

*Proof of Theorem 2.* Let  $T \in \mathcal{A}$  and let  $\lambda_0 \in \sigma(\varphi(T))$  such that  $|\lambda_0| = r(\varphi(T))$ . Let  $\{\lambda_n\}_{n \geq 1} \subseteq \mathbb{C}$  be a sequence such that  $|\lambda_n| > r(\varphi(T))$  for  $n \geq 1$  and  $\lambda_n \rightarrow \lambda_0$ . Since  $\lambda_n I - \varphi(T)$  is invertible in  $\mathcal{B}(X)$  for each  $n \geq 1$  and  $\lambda_0 I - \varphi(T)$  is not, by [1, Theorem 3.2.11] we have

$$\lim_{n \rightarrow \infty} \|(\lambda_n I - \varphi(T))^{-1}\| \rightarrow +\infty.$$

Denote  $T_n = (\lambda_n I - \varphi(T))^{-1} \in \mathcal{B}(X)$ ,  $n \geq 1$ . Since  $(\|T_n\|)_n$  is not bounded, using the Uniform Boundedness Principle we find  $x \in X$  such that  $(T_n(x))_n$  is not bounded in  $X$ . Then  $(T_n(x))_n$  is not bounded in the bidual of  $X$ , so using once more the Uniform Boundedness Principle we find  $f \in X'$  such that  $(f(T_n(x)))_n$  is

not bounded in  $\mathbb{C}$ . By passing to a subsequence, we may therefore suppose that  $f((\lambda_n I - \varphi(T))^{-1}(x)) \neq 0$  for each  $n \geq 1$  and that

$$\lim_{n \rightarrow \infty} |f((\lambda_n I - \varphi(T))^{-1}(x))| = +\infty.$$

Since  $\varphi$  is surjective and  $x \otimes f \in \mathcal{A}$ , there exists  $R \in \mathcal{A}$  such that  $\varphi(R) = x \otimes f$ . Denoting

$$\mu_n = f((\lambda_n I - \varphi(T))^{-1}(x)) \quad (n = 1, 2, \dots),$$

then  $|\mu_n| \rightarrow \infty$  and by (2) we have that

$$\sigma_p(\varphi(T) + x \otimes f / \mu_n) \subseteq \sigma_p(T + R / \mu_n) \quad (n = 1, 2, \dots).$$

By Lemma 1 we have that  $\lambda_n \in \sigma_p(\varphi(T) + x \otimes f / \mu_n)$  for each integer  $n \geq 1$ , and therefore

$$\lambda_n \in \sigma_p(T + R / \mu_n) \subseteq \sigma(T + R / \mu_n) \quad (n = 1, 2, \dots).$$

Using the upper semicontinuity of the spectrum [1, Theorem 3.4.2], that  $\lambda_n \rightarrow \lambda_0$  in  $\mathbb{C}$  and  $R / \mu_n \rightarrow 0$  in  $\mathcal{B}(X)$  imply  $\lambda_0 \in \sigma(T)$ . Thus  $|\lambda_0| \leq r(T)$ , and therefore

$$r(\varphi(T)) \leq r(T) \quad (T \in \mathcal{A}). \quad (3)$$

Using (3), the surjectivity of  $\varphi$ , the fact that by Lemma 2 the algebra  $\mathcal{A}$  is semisimple and [1, Theorem 5.5.2], we infer that  $\varphi$  is continuous.  $\square$

A subalgebra  $\mathcal{A} \subseteq \mathcal{B}(X)$  is called a *regular operator algebra* on  $X$  if the following conditions are fulfilled:

- $\mathcal{A}$  contains the identity  $I$ ;
- every automorphism  $\psi$  of  $\mathcal{A}$  is spatial, that is there exists  $A \in \mathcal{B}(X)$  invertible such that

$$\psi(T) = ATA^{-1} \quad (T \in \mathcal{A}).$$

- for every pair of linearly independent vectors  $u, v \in X$  and every pair of linearly independent vectors  $x, y \in X$ , there exists  $A \in \mathcal{A}$  invertible such that  $Ax = u$  and  $Ay = v$ .

It is proved in the beginning of [3, Proof of Theorem 1.2] that if  $\mathcal{A}$  is a unital standard operator algebra on  $X$ , then  $\mathcal{A}$  is a regular operator algebra. This fact and [3, Proposition 2.1] allow us to use Theorem 2 to obtain the automatic continuity result for bilocal automorphisms also.

*Proof of Theorem 3.* By [3, Proposition 2.1] we have that  $\varphi$  is unital and that (2) holds. We use then Theorem 2 and the surjectivity of  $\varphi$  to obtain the first part of the statement.

For the second part, if  $\varphi$  is supposed bijective, since  $\varphi$  is proved to be continuous, by [3, Theorem 1.2] we have that it is necessarily an automorphism.  $\square$

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