

## $C^*$ -ALGEBRAS OF BERGMAN TYPE OPERATORS WITH CONTINUOUS COEFFICIENTS ON POLYGONAL DOMAINS

YURI I. KARLOVICH

*(Communicated by I. M. Spitkovsky)*

*Abstract.* Given  $\alpha \in (0, 2]$ , the  $C^*$ -algebra  $\mathfrak{A}_{\mathbb{K}_\alpha}$  generated by the identity operator and by the Bergman and anti-Bergman projections acting on the Lebesgue space  $L^2(\mathbb{K}_\alpha)$  over the open sector

$$\mathbb{K}_\alpha = \{z = re^{i\theta} : r > 0, \theta \in (0, \pi\alpha)\}$$

is studied. Then, for any bounded polygonal domain  $U$ , the  $C^*$ -algebra  $\mathfrak{B}_U$  generated by the operators of multiplication by continuous functions on the closure  $\bar{U}$  of  $U$  and by the Bergman and anti-Bergman projections acting on the Lebesgue space  $L^2(U)$  is investigated. Symbol calculi for the  $C^*$ -algebras  $\mathfrak{A}_{\mathbb{K}_\alpha}$  and  $\mathfrak{B}_U$  are constructed and an invertibility criterion for operators  $A \in \mathfrak{A}_{\mathbb{K}_\alpha}$  and a Fredholm criterion for the operators  $B \in \mathfrak{B}_U$  in terms of their symbols are established.

### 1. Introduction

Let  $\mathcal{B}(H)$  denote the  $C^*$ -algebra of all bounded linear operators acting on a Hilbert space  $H$ , and let  $\mathcal{K}(H)$  be the ideal of compact operators on  $H$ . Operator  $A \in \mathcal{B}(H)$  is called Fredholm if the coset  $A^\pi := A + \mathcal{K}(H)$  is invertible in the quotient  $C^*$ -algebra  $\mathcal{B}^\pi(H) := \mathcal{B}(H)/\mathcal{K}(H)$  (see, e.g., [2]).

Let  $U$  be a domain in  $\mathbb{C}$  equipped with the Lebesgue area measure  $dA(z) = dx dy$ , and let  $\mathcal{A}^2(U)$  and  $\widetilde{\mathcal{A}}^2(U)$  denote the Hilbert subspaces (see, e.g., [5], [9]) of  $L^2(U) = L^2(U, dA)$  that consist of differentiable functions such that, respectively,  $\partial_{\bar{z}}f = 0$  and  $\partial_z f = 0$ , where

$$\partial_{\bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad \partial_z := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$

These spaces are related by the anti-linear norm one operator

$$C : L^2(U) \rightarrow L^2(U), \quad Cf = \bar{f}. \tag{1.1}$$

Obviously,  $C(\mathcal{A}^2(U)) = \widetilde{\mathcal{A}}^2(U)$  because  $\partial_z f = \overline{\partial_{\bar{z}} \bar{f}}$ .

*Mathematics subject classification* (2010): Primary 47L15; Secondary 45E10, 47A53, 47G10, 47L30.

*Keywords and phrases:* Bergman and anti-Bergman projections, sector, polygon,  $C^*$ -algebra, local principle, symbol calculi, invertibility, Fredholmness.

The work was supported by the SEP-CONACYT Projects No. 168104 and No. 169496 (México).

The Bergman projection  $B_U$  and anti-Bergman projection  $\tilde{B}_U$  are the orthogonal projections of the Lebesgue space  $L^2(U)$  onto its subspaces  $\mathcal{A}^2(U)$  and  $\widetilde{\mathcal{A}}^2(U)$ , respectively. Clearly,  $\tilde{B}_U = CB_U C$ .

According to [5, Chapter 2], for a bounded multiply connected domain  $U \subset \mathbb{C}$  with sufficiently smooth boundary, the Bergman and anti-Bergman projections are represented in the form

$$B_U = I - S_U S_U^* + K, \quad \tilde{B}_U = I - S_U^* S_U + \tilde{K}, \quad (1.2)$$

where  $S_U$  and  $S_U^*$  are two-dimensional singular integral operators bounded on the space  $L^2(U)$  [17] and given for  $z \in U$  by

$$\begin{aligned} (S_U f)(z) &= -\frac{1}{\pi} \int_U \frac{f(w)}{(w-z)^2} dA(w), \\ (S_U^* f)(z) &= -\frac{1}{\pi} \int_U \frac{f(w)}{(\bar{w}-\bar{z})^2} dA(w), \end{aligned} \quad (1.3)$$

and  $K, \tilde{K}$  are compact operators on the space  $L^2(U)$ . Clearly,  $S_U^* = CS_U C$  is the adjoint operator for  $S_U$ .

If  $U$  is the upper half-plane  $\Pi = \{z \in \mathbb{C} : \text{Im } z > 0\}$ , then  $K = \tilde{K} = 0$  and hence (1.2) takes the form (see [28, Lemma 7.5]):

$$B_\Pi = I - S_\Pi S_\Pi^*, \quad \tilde{B}_\Pi = I - S_\Pi^* S_\Pi. \quad (1.4)$$

On the other hand, if the boundary of a domain  $U$  admit angles different of  $\pi$ , then formulas (1.2) and (1.4), in general, are violated. For example, this happens for the open sectors

$$\mathbb{K}_\alpha = \{z = re^{i\theta} : r > 0, \theta \in (0, \pi\alpha)\} \quad (\alpha \in (0, 2]) \quad (1.5)$$

if  $\alpha \in \{1/2, 1/3, \dots\}$  (see [10, Theorem 5.3]).

The Fredholmness for the  $C^*$ -algebra generated by the Bergman projection of a bounded multi-connected domain  $G$  with a smooth boundary  $\partial G$  and by piecewise continuous coefficients having one-sided limits at the points of the finite union of curves intersecting  $\partial G$  at distinct points was investigated in [25]. A generalization of this work to piecewise continuous coefficients admitting more than two one-sided limits at the points of  $\partial G$  was elaborated in [13] (also see [14] and [15]). The  $C^*$ -algebras generated by the Bergman and anti-Bergman projections (as well as by  $n$  poly-Bergman and  $m$  anti-poly-Bergman projections) with piecewise continuous coefficients admitting finite numbers of one-sided limits at the points of  $\partial G$  were studied in the papers [8]–[11].  $C^*$ -algebras of Toeplitz operators on the Bergman space were studied in [28], [29] (also see references therein). In all these works it was assumed that the boundary of a domain  $G$  is sufficiently smooth.

The present paper deals with studying the invertibility and Fredholmness in the  $C^*$ -algebras of Bergman type operators acting on the space  $L^2(U)$  over domains  $U$  whose boundaries admit angles. The invertibility in the  $C^*$ -algebra generated by the

operators of multiplication by piecewise constant functions and by the Bergman and anti-Bergman projections on the space  $L^2$  over sectors  $\mathbb{K}_{1/m}$  for  $m = 2, 3, \dots$  was recently studied in [6]. The obtained results essentially depend on the values of the sector angles.

In the present paper, making use of another approach, we study the invertibility of operators in the  $C^*$ -algebra

$$\mathfrak{A}_{\mathbb{K}_\alpha} := \text{alg} \{I, B_{\mathbb{K}_\alpha}, \widetilde{B}_{\mathbb{K}_\alpha}\} \subset \mathcal{B}(L^2(\mathbb{K}_\alpha)) \tag{1.6}$$

generated by the identity operator  $I$ , by the Bergman projection  $B_{\mathbb{K}_\alpha}$  and by the anti-Bergman projection  $\widetilde{B}_{\mathbb{K}_\alpha}$  for any  $\alpha \in (0, 2]$ . Further, applying the Allan-Douglas local principle (see [4, Theorem 7.47] and [2, Theorem 1.35]) and the limit operators techniques (see, e.g., [21, Chapter 1]), we study the Fredholmness of operators in the  $C^*$ -algebras

$$\mathfrak{B}_U := \text{alg} \{aI, B_U, \widetilde{B}_U : a \in C(\overline{U})\} \subset \mathcal{B}(L^2(U)) \tag{1.7}$$

generated by the operators  $aI$  of multiplication by all complex-valued functions  $a \in C(\overline{U})$ , by the Bergman projection  $B_U$  and by the anti-Bergman projection  $\widetilde{B}_U$ , where  $\overline{U}$  is the closure of a bounded polygonal domain  $U$  whose angles admit values  $\pi\alpha$  with  $\alpha \in (0, 2]$ .

The paper is organized as follows. In Section 2, following [7] and [6], we expound the Plamenevsky decomposition for the two-dimensional Fourier transform and give its application to convolution type operators with homogeneous data. In Section 3, for any  $\alpha \in (0, 2]$ , we construct a  $C^*$ -algebra isomorphism of the  $C^*$ -algebra  $\mathfrak{A}_{\mathbb{K}_\alpha}$  given by (1.6) into a  $C^*$ -algebra  $\widetilde{\Omega}_\alpha$  of bounded norm-continuous operator functions  $\mathbb{R} \rightarrow \mathcal{B}(L^2(\mathbb{T}))$ . In Section 4, for all  $\lambda \in \mathbb{R}$ , we calculate the images of the operators  $B_\alpha(\lambda)$  and  $\widetilde{B}_\alpha(\lambda)$  related to the projections  $B_{\mathbb{K}_\alpha}$  and  $\widetilde{B}_{\mathbb{K}_\alpha}$  according to Section 3.

In Section 5, modifying the symbol calculus constructed in [8, Theorem 8.1], we describe an abstract symbol calculus for a  $C^*$ -algebra generated by the identity  $I$  and by a finite number of one-dimensional self-adjoint idempotents that are not pairwise orthogonal. Given  $\alpha \in (0, 2]$  and  $\lambda \in \mathbb{R}$ , we check in Section 6 the fulfillment of all conditions of the abstract symbol calculus from Section 5 for the  $C^*$ -algebras

$$\mathcal{A}_{\alpha,\lambda} := \text{alg} \{I, B_\alpha(\lambda), \widetilde{B}_\alpha(\lambda)\} \tag{1.8}$$

generated by the operators  $I$ ,  $B_\alpha(\lambda)$  and  $\widetilde{B}_\alpha(\lambda)$ . Applying results of Sections 5 and 6, we construct in Section 7 symbol calculi for the  $C^*$ -algebras  $\mathcal{A}_{\alpha,\lambda}$  and  $\mathfrak{A}_{\mathbb{K}_\alpha}$  (see (1.6)) for all  $\alpha \in (0, 2]$  and all  $\lambda \in \mathbb{R}$ , and establish an invertibility criterion for the operators  $A \in \mathfrak{A}_{\mathbb{K}_\alpha}$  in terms of their symbols.

Finally, in Section 8, making use of the Allan-Douglas local principle and the limit operators techniques, for any polygonal domain  $U$  we construct a Fredholm symbol calculus for the  $C^*$ -algebra  $\mathfrak{B}_U$  given by (1.7) and obtain a Fredholm criterion for the operators  $A \in \mathfrak{B}_U$  in terms of their Fredholm symbols.

**2.  $C^*$ -algebras of convolution type operators with homogeneous data**

The results of this section are essentially due to Plamenevsky’s decomposition of the multidimensional Fourier transform [19]. Such technique was also applied in [7], where the Plamenevsky results were extended in the two-dimensional case. Here we specify some results of [7] on the basis of [6].

Let  $PC(\mathbb{T})$  be the  $C^*$ -algebra of all complex-valued piecewise continuous functions on the unit circle  $\mathbb{T}$ , and let  $H(PC(\mathbb{T}))$  denote the  $C^*$ -algebra of all homogeneous of order zero functions in  $L^\infty(\mathbb{C})$  whose restrictions on  $\mathbb{T}$  belong to  $PC(\mathbb{T})$  and, for all  $\tau \in \mathbb{T}$  and all  $t > 0$ ,

$$\lim_{\theta \rightarrow +0} a(te^{i\theta} \tau) = \lim_{\theta \rightarrow +0} a(e^{i\theta} \tau), \quad \lim_{\theta \rightarrow -0} a(te^{i\theta} \tau) = \lim_{\theta \rightarrow -0} a(e^{i\theta} \tau).$$

We also apply the  $C^*$ -algebra  $H(C(\mathbb{T}))$  consisting of functions  $a \in H(PC(\mathbb{T}))$  such that  $a|_{\mathbb{T}} \in C(\mathbb{T})$ . Let  $\mathcal{B}$  stand for the  $C^*$ -subalgebra of  $\mathcal{B}(L^2(\mathbb{R}^2))$  generated by the multiplication operators

$$A = aI \quad (a \in H(PC(\mathbb{T})))$$

and by the two-dimensional singular integral operators

$$F^{-1}bF \quad (b \in H(C(\mathbb{T}))).$$

Here  $F : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$  is the Fourier transform defined by

$$(Fu)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} u(t)e^{-ix \cdot t} dt, \quad x \in \mathbb{R}^2, \tag{2.1}$$

where  $x \cdot t$  is the scalar product of vectors  $x, t \in \mathbb{R}^2$ , and  $F^{-1}$  is the inverse Fourier transform.

We also consider the Mellin transform and its inverse given by

$$M : L^2(\mathbb{R}_+, r dr) \rightarrow L^2(\mathbb{R}), \quad (Mv)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} v(r)r^{-i\lambda} dr,$$

$$M^{-1} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}_+, r dr), \quad (M^{-1}u)(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(\lambda)r^{i\lambda-1} d\lambda.$$

Following [19] and [7], for  $\lambda \in \mathbb{C}$  such that  $\text{Im } \lambda > 0$  and  $\lambda \neq ik, k = 1, 2, \dots$ , we define the operators  $E(\lambda) \in \mathcal{B}(L^2(\mathbb{T}))$  on functions  $u \in C^\infty(\mathbb{T})$  by

$$(E(\lambda)u)(\tau) = \gamma(\lambda) \int_{\mathbb{T}} (-\tau \cdot \omega + i0)^{-i\lambda-1} u(\omega) d\omega, \quad \tau \in \mathbb{T}, \tag{2.2}$$

where  $d\omega$  is the length measure on  $\mathbb{T}$ ,

$$\gamma(\lambda) = \frac{1}{2\pi} \Gamma(1 + i\lambda) e^{\pi(i-\lambda)/2} \tag{2.3}$$

and the expression  $(t \pm i0)^\mu$  for  $t \in \mathbb{R}$  and  $\mu \in \mathbb{C}$  is understood in the sense of distributions:

$$(t \pm i0)^\mu = \begin{cases} t_+^\mu + e^{\pm i\pi\mu} t_-^\mu & \text{if } \mu \neq -1, -2, \dots, \\ t^\mu \pm (-1)^\mu \frac{i\pi}{(-\mu - 1)!} \delta^{(-\mu-1)}(t) & \text{if } \mu = -1, -2, \dots, \end{cases}$$

$t_+^\mu = 0$  for  $t \leq 0$ ,  $t_+^\mu = e^{\mu \log t}$  for  $t > 0$ , and  $t_-^\mu = (-t)_+^\mu$ .

For  $\text{Im } \lambda \leq 0$  the integral (2.2) is understood in the sense of analytic continuation, since for every  $u \in C^\infty(\mathbb{T})$  the function  $\lambda \mapsto E(\lambda)u(t)$  admits analytic continuation in the complex plane minus the poles  $\lambda = ik$  ( $k = 1, 2, \dots$ ) of the  $\Gamma$ -function in (2.3) (see [19]). The inverse operator  $E(\lambda)^{-1}$  is given by

$$(E(\lambda)^{-1}v)(\omega) = \gamma(-\lambda) \int_{\mathbb{T}} (\omega \cdot \tau + i0)^{i\lambda-1} v(\tau) d\tau, \quad \lambda \neq -ik, \quad k = 1, 2, \dots$$

By [19, Proposition 4.4], the operators  $E(\lambda)$  are unitary for all  $\lambda \in \mathbb{R}$ .

Consider the reflection operator

$$V : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad (Vf)(\lambda) = f(-\lambda), \quad \lambda \in \mathbb{R}. \tag{2.4}$$

Passing to polar coordinates in the plane, we obtain the decomposition

$$L^2(\mathbb{R}^2) = L^2(\mathbb{R}_+, r dr) \otimes L^2(\mathbb{T}). \tag{2.5}$$

The tensor product  $M \otimes I$  will be taken relatively to the decomposition (2.5). For an operator-valued function

$$\mathbb{R} \rightarrow \mathcal{B}(L^2(\mathbb{T})), \quad \lambda \mapsto L(\lambda),$$

we denote by  $I \otimes_\lambda L(\lambda)$  the operator in  $\mathcal{B}(L^2(\mathbb{R}) \otimes L^2(\mathbb{T}))$  given by the formula

$$[(I \otimes_\lambda L(\lambda))f](\lambda, t) = [L(\lambda)f(\lambda, \cdot)](t), \quad (\lambda, t) \in \mathbb{R} \times \mathbb{T}. \tag{2.6}$$

Given  $\lambda \in \mathbb{R}$ , we introduce the  $C^*$ -algebra  $\Omega_\lambda \subset \mathcal{B}(L^2(\mathbb{T}))$  generated by the operators

$$aI \text{ and } E(\lambda)^{-1}bE(\lambda) \quad (a \in PC(\mathbb{T}), b \in C(\mathbb{T})).$$

Consider the orthogonal basis in the space  $L^2(\mathbb{T})$ :

$$\{h^m / \sqrt{2\pi}\}_{m \in \mathbb{Z}}, \text{ where } h(t) = t \text{ for all } t \in \mathbb{T}. \tag{2.7}$$

The following result corrects [7, Proposition 2.2 and (2.15)].

LEMMA 2.1. [6, Lemma 2.1] *For every  $m \in \mathbb{Z}$ ,*

$$\begin{aligned} E(\lambda)h^m &= \mu(|m|, \lambda)h^m && \text{if } \lambda \in \mathbb{C} \setminus \{ik : k \in \mathbb{N}\}, \\ E(\lambda)^{-1}h^m &= (-1)^m \mu(|m|, -\lambda)h^m && \text{if } \lambda \in \mathbb{C} \setminus \{-ik : k \in \mathbb{N}\}, \end{aligned} \tag{2.8}$$

where  $h$  is given by (2.7) and, for  $\lambda \in \mathbb{C} \setminus \{ik : k \in \mathbb{N}\}$ ,

$$\mu(m, \lambda) = (-i)^m 2^{i\lambda} \frac{\Gamma(\frac{m+i\lambda+1}{2})}{\Gamma(\frac{m-i\lambda+1}{2})} \quad (m = 0, 1, 2, \dots). \tag{2.9}$$

Although the operator functions  $\lambda \mapsto E(\lambda)^{\pm 1}$  are not norm-continuous, we have the following result.

LEMMA 2.2. [6, Lemma 2.2] *For every  $b \in C(\mathbb{T})$ , the operator-valued function*

$$\mathbb{R} \rightarrow \mathcal{B}(L^2(\mathbb{T})), \quad \lambda \mapsto E(\lambda)^{-1}bE(\lambda)$$

*is bounded and norm-continuous.*

Let  $\Omega$  be the  $C^*$ -algebra of bounded norm-continuous operator-valued functions

$$U : \mathbb{R} \rightarrow \mathcal{B}(L^2(\mathbb{T})), \quad \lambda \mapsto U(\lambda) \in \Omega_\lambda, \tag{2.10}$$

with the norm  $\|U\| = \sup_{\lambda \in \mathbb{R}} \|U(\lambda)\|$ .

According to [7, Proposition 2.4] (see also [19, Proposition 2.1]) we have the decomposition

$$F = (M^{-1} \otimes I)(V \otimes I)(I \otimes_\lambda E(\lambda))(M \otimes I).$$

Let  $a \in H(PC(\mathbb{T}))$  and  $b \in H(C(\mathbb{T}))$ . Taking in account the equalities

$$\begin{aligned} (M \otimes I)(a(x)I)(M^{-1} \otimes I) &= I \otimes a(t)I, \\ (M \otimes I)(F^{-1}b(\xi)F)(M^{-1} \otimes I) &= I \otimes_\lambda (E(\lambda)^{-1}b(w)E(\lambda)), \end{aligned} \tag{2.11}$$

where  $t, w \in \mathbb{T}$  and the operator function  $\lambda \mapsto E(\lambda)^{-1}b(w)E(\lambda)$  is norm-continuous by Lemma 2.2, and using the notation  $(U(\lambda))_{\lambda \in \mathbb{R}}$  for the operator-valued function (2.10), one can obtain the following.

LEMMA 2.3. [7, Proposition 2.5] *The  $C^*$ -algebra  $\mathcal{R}$  is  $*$ -isomorphic to a  $C^*$ -subalgebra of  $\Omega$ , and the isomorphism is given on the generators  $aI$  ( $a \in H(PC(\mathbb{T}))$ ) and  $F^{-1}bF$  ( $b \in H(C(\mathbb{T}))$ ) of  $\mathcal{R}$  by*

$$a(x)I \mapsto (a(t)I)_{\lambda \in \mathbb{R}}, \quad F^{-1}b(\xi)F \mapsto (E(\lambda)^{-1}b(w)E(\lambda))_{\lambda \in \mathbb{R}}.$$

### 3. Study of the $C^*$ -algebra $\mathfrak{A}_{\mathbb{K}_\alpha}$

Given  $\alpha \in (0, 2]$ , consider the representations of the Bergman projection  $B_{\mathbb{K}_\alpha}$  and the anti-Bergman projection  $\tilde{B}_{\mathbb{K}_\alpha}$  over the open sectors  $\mathbb{K}_\alpha = \{z = re^{i\theta} : r > 0, \theta \in (0, \pi\alpha)\}$  via the two-dimensional singular integral operators  $S_\Pi$  and  $S_\Pi^*$  given by (1.3) with  $U = \Pi$ . Obviously,  $\mathbb{K}_1$  coincides with the upper half-plane  $\Pi$ , and then such representations are given by (1.4).

We define the unitary shift operator

$$W_\alpha : L^2(\mathbb{K}_\alpha) \rightarrow L^2(\Pi), \quad (W_\alpha f)(z) = \alpha z^{\alpha-1} f(z^\alpha) \quad (z \in \Pi). \tag{3.1}$$

Identifying  $B_{\mathbb{K}_\alpha}$  and  $\tilde{B}_{\mathbb{K}_\alpha}$  with the operators  $\chi_{\mathbb{K}_\alpha} B_{\mathbb{K}_\alpha} \chi_{\mathbb{K}_\alpha} I$  and  $\chi_{\mathbb{K}_\alpha} \tilde{B}_{\mathbb{K}_\alpha} \chi_{\mathbb{K}_\alpha} I$  acting on the space  $L^2(\mathbb{C})$ , we immediately obtain the following assertion (cf. [10, Theorem 5.3]).

LEMMA 3.1. For every  $\alpha \in (0, 2]$ , the Bergman and anti-Bergman projections of the space  $L^2(\mathbb{K}_\alpha)$  are self-adjoint projections having, respectively, the form

$$B_{\mathbb{K}_\alpha} = W_\alpha^{-1} B_\Pi W_\alpha = I - W_\alpha^{-1} S_\Pi S_\Pi^* W_\alpha, \tag{3.2}$$

$$\tilde{B}_{\mathbb{K}_\alpha} = C W_\alpha^{-1} B_\Pi W_\alpha C = I - C W_\alpha^{-1} S_\Pi S_\Pi^* W_\alpha C. \tag{3.3}$$

where the unitary shift operator  $W_\alpha$  is given by (3.1) and the operator  $C$  is given by (1.1) for  $U = \mathbb{K}_\alpha$ .

According to [28, Theorem 3.14], we have the following.

LEMMA 3.2. For the upper half-plane  $\Pi$ ,  $\tilde{B}_\Pi B_\Pi = 0$  and  $B_\Pi \tilde{B}_\Pi = 0$ .

Note that such assertion fails for the sectors  $\mathbb{K}_\alpha$  with  $\alpha \in (0, 1) \cup (1, 2)$  (see Remark 4.5 below).

Consider the arc  $\gamma_\alpha := \{e^{i\tau} : \tau \in [0, \alpha\pi]\}$  of the unit circle  $\mathbb{T}$ , and the upper semicircle  $\mathbb{T}_+ := \gamma_1$ . Let  $\chi_B$  be the characteristic function of a set  $B \subset \mathbb{C}$ , and let  $\chi_+ := \chi_{\mathbb{T}_+}$ .

LEMMA 3.3. If  $\alpha \in (0, 2]$ , then

$$(M \otimes I)(\chi_\Pi W_\alpha \chi_{\mathbb{K}_\alpha} I)(M^{-1} \otimes I) = U_{1/\alpha} \otimes (\chi_+ \tilde{U}_\alpha \chi_{\gamma_\alpha} I), \tag{3.4}$$

where  $U_{1/\alpha} \in \mathcal{B}(L^2(\mathbb{R}))$  and  $\tilde{U}_\alpha : L^2(\gamma_\alpha) \rightarrow L^2(\mathbb{T}_+)$  are unitary operators given, respectively, by

$$\begin{aligned} [U_{1/\alpha} \psi](\lambda) &= (1/\alpha)^{1/2} \psi(\lambda/\alpha) \text{ for all } \lambda \in \mathbb{R}, \\ [\tilde{U}_\alpha \phi](t) &= \alpha^{1/2} t^{\alpha-1} \phi(t^\alpha) \text{ for all } t \in \mathbb{T}_+. \end{aligned} \tag{3.5}$$

*Proof.* Obviously, the operator equalities

$$\chi_\Pi W_\alpha \chi_{\mathbb{K}_\alpha} I = W_\alpha \chi_{\mathbb{K}_\alpha} I = \chi_\Pi W_\alpha, \quad \chi_+ \tilde{U}_\alpha \chi_{\gamma_\alpha} I = \tilde{U}_\alpha \chi_{\gamma_\alpha} I = \chi_+ \tilde{U}_\alpha, \tag{3.6}$$

are fulfilled on the spaces  $L^2(\mathbb{C})$  and  $L^2(\mathbb{T})$ , respectively, and

$$(M \otimes I) \chi_{\mathbb{K}_\alpha} (M^{-1} \otimes I) = I \otimes \chi_{\gamma_\alpha} I. \tag{3.7}$$

Given a function  $f \in L^2(\mathbb{R}) \otimes L^2(\mathbb{T})$ , for  $(\lambda, t) \in \mathbb{R} \times \mathbb{T}$  we then obtain

$$\begin{aligned} & [(M \otimes I)(\chi_\Pi W_\alpha \chi_{\mathbb{K}_\alpha} I)(M^{-1} \otimes I)f](\lambda, t) \\ &= \frac{\chi_\Pi(t)}{\sqrt{2\pi}} \int_{\mathbb{R}_+} [W_\alpha \chi_{\gamma_\alpha} (M^{-1} \otimes I)f](r, t) r^{-i\lambda} dr \\ &= \frac{\chi_+(t)}{\sqrt{2\pi}} \int_{\mathbb{R}_+} \alpha r^{\alpha-1} t^{\alpha-1} [(M^{-1} \otimes I)f](r^\alpha, t^\alpha) r^{-i\lambda} dr \\ &= \frac{\chi_+(t) t^{\alpha-1}}{\sqrt{2\pi}} \int_{\mathbb{R}_+} [(M^{-1} \otimes I)f](\rho, t^\alpha) \rho^{-i\lambda/\alpha} d\rho \\ &= \frac{\chi_+(t) t^{\alpha-1}}{\sqrt{2\pi}} \int_{\mathbb{R}_+} \left( \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\mu, t^\alpha) \rho^{i\mu-1} d\mu \right) \rho^{-i\lambda/\alpha} d\rho \\ &= \chi_+(t) t^{\alpha-1} f(\lambda/\alpha, t^\alpha) = [(U_{1/\alpha} \otimes (\chi_+ \tilde{U}_\alpha \chi_{\gamma_\alpha} I))f](\lambda, t), \end{aligned}$$

which gives (3.4).  $\square$

LEMMA 3.4. *If  $C : L^2(\mathbb{C}) \rightarrow L^2(\mathbb{C})$  is the anti-linear operator (1.1), then*

$$(M \otimes I)C(M^{-1} \otimes I) = \widehat{C}V \otimes \widetilde{C}, \tag{3.8}$$

where  $V \in \mathcal{B}(L^2(\mathbb{R}))$  is the reflection operator defined by (2.4), and the anti-linear operators  $\widehat{C} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  and  $\widetilde{C} : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$  act by  $\widehat{C}f = \overline{f}$  and  $\widetilde{C}f = \overline{f}$ .

*Proof.* For  $f \in L^2(\mathbb{R}) \otimes L^2(\mathbb{T})$  and  $(\lambda, t) \in \mathbb{R} \times \mathbb{T}$ , we obtain

$$\begin{aligned} [(M \otimes I)C(M^{-1} \otimes I)f](\lambda, t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} [C(M^{-1} \otimes I)f](r, t)r^{-i\lambda} dr \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} \overline{[(M^{-1} \otimes I)f](r, t)r^{i\lambda} dr} \\ &= f(-\lambda, t) = [(\widehat{C}V \otimes \widetilde{C})f](\lambda, t), \end{aligned}$$

which gives (3.8).  $\square$

From the formula for the Fourier transform of the kernels of multidimensional singular integral operators (see, e.g., [17, Chapter X, p. 249]) it follows that

$$S_{\mathbb{C}} = F^{-1}\widetilde{h}^{-1}F, \quad S_{\mathbb{C}}^* = F^{-1}\widetilde{h}F, \tag{3.9}$$

where  $F$  is the two-dimensional Fourier transform acting on  $L^2(\mathbb{C})$  by the formula (2.1), and  $\widetilde{h}$  is given by

$$\widetilde{h}(z) := z/\overline{z} \quad \text{for all } z \in \mathbb{C}. \tag{3.10}$$

Let  $\widetilde{\Omega}$  be the  $C^*$ -algebra of all bounded norm-continuous operator functions

$$Y : \mathbb{R} \rightarrow \mathcal{B}(L^2(\mathbb{T})), \quad \lambda \mapsto Y(\lambda),$$

with the norm  $\|Y\| = \sup_{\lambda \in \mathbb{R}} \|Y(\lambda)\|_{\mathcal{B}(L^2(\mathbb{T}))}$ .

THEOREM 3.5. *For every  $\alpha \in (0, 2]$ , the  $C^*$ -algebra  $\mathfrak{A}_{\mathbb{K}\alpha}$  given by (1.6) is  $*$ -isomorphic to the  $C^*$ -subalgebra  $\widetilde{\Omega}_\alpha$  of  $\widetilde{\Omega}$  generated by the bounded norm-continuous operator functions  $\mathbb{R} \rightarrow \mathcal{B}(L^2(\mathbb{T}))$  of the form*

$$\lambda \mapsto \chi_{\gamma\alpha}I, \quad \lambda \mapsto B_\alpha(\lambda), \quad \lambda \mapsto \widetilde{B}_\alpha(\lambda), \tag{3.11}$$

where for every  $\lambda \in \mathbb{R}$ ,

$$B_\alpha(\lambda) = \widetilde{U}_\alpha^{-1}B_1(\lambda\alpha)\widetilde{U}_\alpha, \tag{3.12}$$

$$\widetilde{B}_\alpha(\lambda) = (\widetilde{C}\widetilde{U}_\alpha^{-1}\widetilde{C})\widetilde{B}_1(\lambda\alpha)(\widetilde{C}\widetilde{U}_\alpha\widetilde{C}), \tag{3.13}$$

$$B_1(\lambda) = \chi_{+}I - \chi_{+}S(\lambda)\chi_{+}S^*(\lambda)\chi_{+}I, \tag{3.14}$$

$$\widetilde{B}_1(\lambda) = \chi_{+}I - \chi_{+}S^*(\lambda)\chi_{+}S(\lambda)\chi_{+}I, \tag{3.15}$$

$\chi_+$  is the characteristic function of  $\mathbb{T}_+$ , the operator  $\tilde{U}_\alpha : L^2(\gamma_\alpha) \rightarrow L^2(\mathbb{T}_+)$  is given by (3.5),  $\tilde{C}f = \overline{f}$  on the subspaces of  $L^2(\mathbb{T})$ , and the operators  $S(\lambda), S^*(\lambda) \in \mathcal{B}(L^2(\mathbb{T}))$  for  $\lambda \in \mathbb{R}$  are defined by

$$S(\lambda) := E(\lambda)^{-1}h^{-2}E(\lambda), \quad S^*(\lambda) := E(\lambda)^{-1}h^2E(\lambda), \tag{3.16}$$

with  $h(t) = t$  for all  $t \in \mathbb{T}$ .

*Proof.* Since  $\tilde{h}^{\pm 1} \in H(C(\mathbb{T}))$ , we infer from (3.9), (3.10) and (2.11) that

$$\begin{aligned} (M \otimes I)S_{\mathbb{C}}(M^{-1} \otimes I) &= I \otimes_\lambda S(\lambda), \\ (M \otimes I)S_{\mathbb{C}}^*(M^{-1} \otimes I) &= I \otimes_\lambda S^*(\lambda), \end{aligned} \tag{3.17}$$

where  $S(\lambda)$  and  $S^*(\lambda)$  are given by (3.16). Hence, applying the equality

$$(M \otimes I)\chi_{\mathbb{T}}(M^{-1} \otimes I) = I \otimes \chi_{+I},$$

we conclude from (1.4), (3.17) and (3.14)–(3.15) that

$$\begin{aligned} (M \otimes I)B_{\mathbb{T}}(M^{-1} \otimes I) &= I \otimes_\lambda (\chi_{+S(\lambda)}\chi_{+S^*(\lambda)}\chi_{+I}) = I \otimes_\lambda B_1(\lambda), \\ (M \otimes I)\tilde{B}_{\mathbb{T}}(M^{-1} \otimes I) &= I \otimes_\lambda (\chi_{+S^*(\lambda)}\chi_{+S(\lambda)}\chi_{+I}) = I \otimes_\lambda \tilde{B}_1(\lambda). \end{aligned} \tag{3.18}$$

Fix  $\alpha \in (0, 2]$ . Taking into account (3.6), (3.18) and the equality (3.7), we infer from (3.5) and (2.6) that

$$(U_{1/\alpha} \otimes \tilde{U}_\alpha)^{-1}(I \otimes_\lambda B_1(\lambda))(U_{1/\alpha} \otimes \tilde{U}_\alpha) = I \otimes_\lambda (\tilde{U}_\alpha^{-1}B_1(\lambda\alpha)\tilde{U}_\alpha). \tag{3.19}$$

Indeed, for  $f \in L^2(\mathbb{R}) \otimes L^2(\mathbb{T})$  and  $(\lambda, t) \in \mathbb{R} \times \mathbb{T}$ , we obtain

$$\begin{aligned} [(U_{1/\alpha} \otimes \tilde{U}_\alpha)^{-1}f](\lambda, t) &= \alpha^{1/2}[\tilde{U}_\alpha^{-1}f(\lambda\alpha, \cdot)](t), \\ [(I \otimes_\lambda B_1(\lambda))f](\lambda\alpha, t) &= [B_1(\lambda\alpha)f(\lambda\alpha, \cdot)](t), \\ [(U_{1/\alpha} \otimes \tilde{U}_\alpha)f](\lambda\alpha, t) &= \alpha^{-1/2}[\tilde{U}_\alpha f(\lambda, \cdot)](t), \end{aligned}$$

which gives (3.19). Hence, taking into account (3.6), we deduce from (3.2), (3.18), (3.4) and (3.19) that

$$\begin{aligned} I \otimes_\lambda B_\alpha(\lambda) &:= (M \otimes I)B_{\mathbb{R}\alpha}(M^{-1} \otimes I) \\ &= (M \otimes I)(W_\alpha^{-1}B_{\mathbb{T}}W_\alpha)(M^{-1} \otimes I) \\ &= (U_{1/\alpha} \otimes \tilde{U}_\alpha)^{-1}(I \otimes_\lambda B_1(\lambda))(U_{1/\alpha} \otimes \tilde{U}_\alpha) \\ &= I \otimes_\lambda (\tilde{U}_\alpha^{-1}B_1(\lambda\alpha)\tilde{U}_\alpha), \end{aligned} \tag{3.20}$$

which implies (3.12).

Since for  $f \in L^2(\mathbb{R}) \otimes L^2(\mathbb{T})$  and  $(\lambda, t) \in \mathbb{R} \times \mathbb{T}$  we have

$$\begin{aligned} [(V \otimes I)f](\lambda, t) &= f(-\lambda, t), \\ [(I \otimes_\lambda B_\alpha(\lambda))f](-\lambda, t) &= [B_\alpha(-\lambda)f(-\lambda, \cdot)](t), \\ [(V \otimes I)f](-\lambda, t) &= f(\lambda, t), \end{aligned}$$

it follows that

$$(V \otimes I)(I \otimes_\lambda B_\alpha(\lambda))(V \otimes I) = I \otimes_\lambda (B_\alpha(-\lambda)).$$

Hence

$$\begin{aligned} (\widehat{C}V \otimes \widetilde{C})(I \otimes_\lambda B_\alpha(\lambda))(\widehat{C}V \otimes \widetilde{C}) &= (\widehat{C} \otimes \widetilde{C})(I \otimes_\lambda B_\alpha(-\lambda))(\widehat{C} \otimes \widetilde{C}) \\ &= I \otimes_\lambda (\widetilde{C}B_\alpha(-\lambda)\widetilde{C}). \end{aligned} \tag{3.21}$$

Applying now (3.3), (3.8), (3.20) and (3.21), we obtain

$$\begin{aligned} I \otimes_\lambda \widetilde{B}_\alpha(\lambda) &:= (M \otimes I)\widetilde{B}_{\mathbb{K}_\alpha}(M^{-1} \otimes I) \\ &= (M \otimes I)(CB_{\mathbb{K}_\alpha}C)(M^{-1} \otimes I) \\ &= (\widehat{C}V \otimes \widetilde{C})(M \otimes I)B_{\mathbb{K}_\alpha}(M^{-1} \otimes I)(\widehat{C}V \otimes \widetilde{C}) \\ &= (\widehat{C}V \otimes \widetilde{C})(I \otimes_\lambda B_\alpha(\lambda))(\widehat{C}V \otimes \widetilde{C}) \\ &= I \otimes_\lambda (\widetilde{C}B_\alpha(-\lambda)\widetilde{C}), \end{aligned} \tag{3.22}$$

which means that

$$\widetilde{B}_\alpha(\lambda) = \widetilde{C}B_\alpha(-\lambda)\widetilde{C}. \tag{3.23}$$

Making use of (3.12), we deduce from (3.23) that

$$\widetilde{B}_\alpha(\lambda) = \widetilde{C}\widetilde{U}_\alpha^{-1}B_1(-\lambda\alpha)\widetilde{U}_\alpha\widetilde{C} = (\widetilde{C}\widetilde{U}_\alpha^{-1}\widetilde{C})\widetilde{B}_1(\lambda\alpha)(\widetilde{C}\widetilde{U}_\alpha\widetilde{C}),$$

which gives (3.13).

Thus, by the first equalities in (3.20) and (3.22), we conclude that

$$\begin{aligned} (M \otimes I)B_{\mathbb{K}_\alpha}(M^{-1} \otimes I) &= I \otimes_\lambda B_\alpha(\lambda), \\ (M \otimes I)\widetilde{B}_{\mathbb{K}_\alpha}(M^{-1} \otimes I) &= I \otimes_\lambda \widetilde{B}_\alpha(\lambda), \end{aligned} \tag{3.24}$$

where the operators  $B_\alpha(\lambda), \widetilde{B}_\alpha(\lambda) \in \mathcal{B}(L^2(\mathbb{T}))$  for every  $\lambda \in \mathbb{R}$  are given by (3.12)–(3.15). From Lemma 2.2 and (3.12)–(3.15) it follows that the operator functions  $\lambda \mapsto B_\alpha(\lambda)$  and  $\lambda \mapsto \widetilde{B}_\alpha(\lambda)$  are norm-continuous, which completes the proof according to (3.24).  $\square$

Fix  $\alpha \in (0, 2]$ . By Theorem 3.5, the  $C^*$ -algebra

$$\mathfrak{A}_{\mathbb{K}_\alpha} := \text{alg} \{I, B_{\mathbb{K}_\alpha}, \widetilde{B}_{\mathbb{K}_\alpha}\} \subset \mathcal{B}(L^2(\mathbb{K}_\alpha))$$

is  $*$ -isomorphic to the  $C^*$ -algebra  $\widetilde{\Omega}_\alpha$  generated by the bounded norm-continuous operator functions  $\mathbb{R} \rightarrow \mathcal{B}(L^2(\gamma_\alpha))$  given by (3.11). On the other hand, the  $C^*$ -algebra  $\widetilde{\Omega}_\alpha$  can be considered as

$$\widetilde{\Omega}_\alpha \subset \bigoplus_{\lambda \in \mathbb{R}} \mathcal{A}_{\alpha, \lambda}, \tag{3.25}$$

where the  $C^*$ -algebras  $\mathcal{A}_{\alpha, \lambda} \subset \mathcal{B}(L^2(\gamma_\alpha))$  are defined by (1.8). By (3.7) and (3.24), for every  $A \in \mathfrak{A}_{\mathbb{K}_\alpha}$  and every  $\lambda \in \mathbb{R}$  there exists an operator  $A_\alpha(\lambda) \in \mathcal{A}_{\alpha, \lambda}$  such that

$$(M \otimes I)A(M^{-1} \otimes I) = I \otimes_\lambda A_\alpha(\lambda) \tag{3.26}$$

and the operator function  $\lambda \mapsto A_\alpha(\lambda)$  is norm-continuous. Theorem 3.5 in view of (3.25) and (3.26) immediately imply the following invertibility criterion.

**THEOREM 3.6.** *Given  $\alpha \in (0, 2]$ , an operator  $A \in \mathfrak{A}_{\mathbb{K}\alpha}$  is invertible on the space  $L^2(\mathbb{K}\alpha)$  if and only if the operators  $A_\alpha(\lambda) \in \mathfrak{A}_{\alpha,\lambda}$  are invertible on the space  $L^2(\gamma_\alpha)$  for all  $\lambda \in \mathbb{R}$  and*

$$\sup_{\lambda \in \mathbb{R}} \|(A_\alpha(\lambda))^{-1}\|_{\mathcal{B}(L^2(\gamma_\alpha))} < \infty. \tag{3.27}$$

#### 4. The images of the operators $B_\alpha(\lambda)$ and $\tilde{B}_\alpha(\lambda)$

To calculate the images of the operators  $B_\alpha(\lambda)$  and  $\tilde{B}_\alpha(\lambda)$  for any  $\alpha \in (0, 2]$ , we partially follow the scheme of [13] and [8], [9]. By [12, Chapter 1, Lemma 4.10], if  $P(t)$  is a family of projections on a Hilbert space  $H$  continuously depending, in the norm topology, on the real parameter  $t$  running through a connected set of  $\mathbb{R}$ , then all the spaces  $P(t)H$  are isomorphic; in particular, all the images of  $P(t)$  have the same dimensions. Thus, taking into account that the projections  $B_\alpha(\lambda)$  and  $\tilde{B}_\alpha(\lambda)$  norm-continuously depend on  $\lambda \in \mathbb{R}$  in view of Theorem 3.5, we get

$$\dim B_\alpha(\lambda) = \dim B_\alpha(0), \quad \dim \tilde{B}_\alpha(\lambda) = \dim \tilde{B}_\alpha(0).$$

We are going to show that  $B_\alpha(0)$  and  $\tilde{B}_\alpha(0)$  are one-dimensional projections and then all the projections  $B_\alpha(\lambda)$  and  $\tilde{B}_\alpha(\lambda)$  are one-dimensional.

**LEMMA 4.1.** *Let  $\alpha \in (0, 2]$  and let  $h(t) = t$  for all  $t \in \mathbb{T}$ . Then*

$$\text{Im } B_\alpha(0) = \text{span} \left\{ \frac{\chi_{\gamma_\alpha} h^{-1}}{\sqrt{\pi\alpha}} \right\}, \quad \text{Im } \tilde{B}_\alpha(0) = \text{span} \left\{ \frac{\chi_{\gamma_\alpha} h}{\sqrt{\pi\alpha}} \right\},$$

where  $\chi_{\gamma_\alpha}$  is the characteristic function of the arc  $\gamma_\alpha \subset \mathbb{T}$ .

*Proof.* Since the function  $\Gamma$  is analytic on the set  $\mathbb{C} \setminus \{0, -1, -2, \dots\}$  and has poles of order one at the points  $0, -1, -2, \dots$ , we deduce from (2.8) and (2.9) that

$$E(0)h^n = (-i)^{|n|}h^n, \quad E(0)^{-1}h^n = i^{|n|}h^n \quad (n \in \mathbb{Z}), \tag{4.1}$$

where  $h(t) = t$  for all  $t \in \mathbb{T}$ . Then, in view of (3.16) and (4.1), we obtain

$$\begin{aligned} S(0)h^n &= E(0)^{-1}h^{-2}E(0)h^n = (-i)^{|n|}i^{|n-2|}h^{n-2}, \\ S^*(0)h^n &= E(0)^{-1}h^2E(0)h^n = (-i)^{|n|}i^{|n+2|}h^{n+2}, \end{aligned}$$

which implies that, for every  $n \in \mathbb{Z}$ ,

$$S(0)h^n = \begin{cases} -h^{n-2}, & n \neq 1, \\ h^{n-2}, & n = 1, \end{cases} \quad S^*(0)h^n = \begin{cases} -h^{n+2}, & n \neq -1, \\ h^{n+2}, & n = -1. \end{cases} \tag{4.2}$$

Given  $f \in L^2(\mathbb{T})$ , let us calculate  $B_\alpha(0)f$ , where, by (3.12) and (3.14),

$$B_\alpha(0) = \chi_{\gamma_\alpha} I - \chi_{\gamma_\alpha} \tilde{U}_\alpha^{-1} \chi_+ S(0) \chi_+ S^*(0) \chi_+ \tilde{U}_\alpha \chi_{\gamma_\alpha} I.$$

Representing a function  $\chi_+ \psi \in L^2(\mathbb{T})$  by the convergent in  $L^2(\mathbb{T})$  series

$$\chi_+ \psi = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \langle \chi_+ \psi, h^k \rangle h^k, \tag{4.3}$$

we infer from (4.2) and (4.3) that

$$\begin{aligned} S^*(0)(\chi_+ \psi) &= -\frac{1}{2\pi} \sum_{k \in \mathbb{Z} \setminus \{-1\}} \langle \chi_+ \psi, h^k \rangle h^{k+2} + \frac{1}{2\pi} \langle \chi_+ \psi, h^{-1} \rangle h \\ &= -\frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \langle \chi_+ \psi h^2, h^{k+2} \rangle h^{k+2} + \frac{2}{2\pi} \langle \chi_+ \psi, h^{-1} \rangle h \\ &= -\chi_+ \psi h^2 + \frac{1}{\pi} \langle \chi_+ \psi, h^{-1} \rangle h \end{aligned} \tag{4.4}$$

and, analogously,

$$\begin{aligned} S(0)(\chi_+ \psi) &= -\frac{1}{2\pi} \sum_{k \in \mathbb{Z} \setminus \{1\}} \langle \chi_+ \psi, h^k \rangle h^{k-2} + \frac{1}{2\pi} \langle \chi_+ \psi, h \rangle h^{-1} \\ &= -\frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \langle \chi_+ \psi h^{-2}, h^{k-2} \rangle h^{k-2} + \frac{2}{2\pi} \langle \chi_+ \psi, h \rangle h^{-1} \\ &= -\chi_+ \psi h^{-2} + \frac{1}{\pi} \langle \chi_+ \psi, h \rangle h^{-1}. \end{aligned} \tag{4.5}$$

Applying (4.4), (4.5) and the equality  $\langle \chi_+ h, h \rangle = \pi$ , we infer from (3.14) that

$$\begin{aligned} B_1(0)\psi &= \chi_+ \psi - \chi_+ S(0) \chi_+ S^*(0) \chi_+ \psi \\ &= \chi_+ \psi - [\chi_+ S(0)] \left( -\chi_+ \psi h^2 + \frac{1}{\pi} \langle \chi_+ \psi, h^{-1} \rangle \chi_+ h \right) \\ &= \chi_+ \psi - \chi_+ \psi + \frac{1}{\pi} \langle \chi_+ \psi h^2, h \rangle \chi_+ h^{-1} \\ &\quad + \frac{1}{\pi} \langle \chi_+ \psi, h^{-1} \rangle \left( \chi_+ h^{-1} - \frac{1}{\pi} \langle \chi_+ h, h \rangle \chi_+ h^{-1} \right) \\ &= \frac{1}{\pi} \langle \chi_+ \psi, h^{-1} \rangle \chi_+ h^{-1}. \end{aligned} \tag{4.6}$$

Finally, taking  $\chi_{\gamma_\alpha} f \in L^2(\mathbb{T})$  and applying (3.12), we infer from (4.6) and (3.5) that

$$\begin{aligned} [B_\alpha(0)](\chi_{\gamma_\alpha} f) &= \tilde{U}_\alpha^{-1} B_1(0) \tilde{U}_\alpha (\chi_{\gamma_\alpha} f) \\ &= \frac{1}{\pi} \langle \chi_+ \tilde{U}_\alpha f, h^{-1} \rangle \tilde{U}_\alpha^{-1} (\chi_+ h^{-1}) \\ &= \frac{1}{\pi} \langle f, \tilde{U}_\alpha^{-1} (\chi_+ h^{-1}) \rangle \tilde{U}_\alpha^{-1} (\chi_+ h^{-1}) \\ &= \frac{1}{\pi \alpha} \langle f, \chi_{\gamma_\alpha} h^{-1} \rangle \chi_{\gamma_\alpha} h^{-1}. \end{aligned} \tag{4.7}$$

This means that the linear space  $\text{Im} B_\alpha(0)$  is one-dimensional, and the norm one function  $\chi_{\gamma_\alpha} h^{-1} / \sqrt{\pi\alpha} \in L^2(\gamma_\alpha)$  is its generator.

Taking into account (3.23), we infer from (4.7) that

$$\begin{aligned} [\tilde{B}_\alpha(0)](\chi_{\gamma_\alpha} f) &= [\tilde{C}B_\alpha(0)\tilde{C}](\chi_{\gamma_\alpha} f) \\ &= \frac{1}{\pi\alpha} \tilde{C}(\langle \bar{f}, \chi_{\gamma_\alpha} h^{-1} \rangle \chi_{\gamma_\alpha} h^{-1}) \\ &= \frac{1}{\pi\alpha} \langle f, \chi_{\gamma_\alpha} h \rangle \chi_{\gamma_\alpha} h. \end{aligned}$$

Thus, the linear space  $\text{Im} \tilde{B}_\alpha(0)$  also is one-dimensional and it is generated by the norm one function  $\chi_{\gamma_\alpha} h / \sqrt{\pi\alpha} \in L^2(\gamma_\alpha)$ .  $\square$

LEMMA 4.2. For each  $\alpha \in (0, 2]$  and each  $\lambda \in \mathbb{R}$ , the spaces  $\text{Im} B_\alpha(\lambda)$  and  $\text{Im} \tilde{B}_\alpha(\lambda)$  are one-dimensional and their generators of norm one on the space  $L^2(\gamma_\alpha)$  are given for  $t \in \gamma_\alpha$ , respectively, by

$$g_{\alpha,\lambda}(t) = G_\alpha(\lambda) t^{i\lambda-1}, \tag{4.8}$$

$$\tilde{g}_{\alpha,\lambda}(t) = \overline{g_{\alpha,-\lambda}}(t) = G_\alpha(-\lambda) t^{1-i\lambda}, \tag{4.9}$$

where

$$G_\alpha(\lambda) := \begin{cases} \left( \frac{2\lambda}{1 - e^{-2\pi\lambda\alpha}} \right)^{1/2} & \text{if } \lambda \in \mathbb{R} \setminus \{0\}, \\ \lim_{\lambda \rightarrow 0} G_\alpha(\lambda) = (\pi\alpha)^{-1/2} & \text{if } \lambda = 0. \end{cases} \tag{4.10}$$

Proof. We deduce from the equality  $(M \otimes I)B_{\mathbb{K}_\alpha}(M^{-1} \otimes I) = I \otimes_\lambda B_\alpha(\lambda)$  that

$$(I \otimes_\lambda B_\alpha(\lambda))(L^2(\mathbb{R}) \otimes L^2(\mathbb{T})) = (M \otimes I)(\mathcal{A}^2(\mathbb{K}_\alpha)), \tag{4.11}$$

where the functions in  $\mathcal{A}^2(\mathbb{K}_\alpha)$  are considered as elements of the space  $L^2(\mathbb{R}_+, r dr) \otimes L^2(\mathbb{T})$  after their extension by zero to the set  $\mathbb{C} \setminus \mathbb{K}_\alpha$ . So, if  $f \in \mathcal{A}^2(\mathbb{K}_\alpha)$ , then by (4.11) there exists a function  $g \in L^2(\mathbb{R}) \otimes L^2(\mathbb{T})$  such that

$$[(M \otimes I)f](\lambda, t) = [(I \otimes_\lambda B_\alpha(\lambda))g](\lambda, t) = [B_\alpha(\lambda)g(\lambda, \cdot)](t), \quad t \in \mathbb{T}.$$

Consequently,  $[(M \otimes I)f](\lambda, \cdot) \in \text{Im} B_\alpha(\lambda)$  for every  $f \in \mathcal{A}^2(\mathbb{K}_\alpha)$ . Taking the function  $h_0(z) = \chi_\Pi(z)(z+i)^{-2} \in \mathcal{A}^2(\Pi)$ , we conclude that

$$(W_\alpha^{-1}h_0)(z) = \chi_{\mathbb{K}_\alpha}(z)(1/\alpha)z^{1/\alpha-1}(z^{1/\alpha} + i)^{-2} \in \mathcal{A}^2(\mathbb{K}_\alpha). \tag{4.12}$$

Applying then [18, formula 2.19], we infer by analogy with [13, Proposition 3.10] that for  $(\lambda, t) \in \mathbb{R} \times \mathbb{T}$ ,

$$\begin{aligned} [(M \otimes I)(W_\alpha^{-1}h_0)](\lambda, t) &= \frac{\chi_{\gamma_\alpha}(t)}{\sqrt{2\pi}} \int_{\mathbb{R}_+} (1/\alpha)(rt)^{1/\alpha-1}(r^{1/\alpha}t^{1/\alpha} + i)^{-2} r^{-i\lambda} dr \\ &= \frac{\chi_{\gamma_\alpha}(t)t^{1/\alpha-1}}{\sqrt{2\pi}} \int_{\mathbb{R}_+} (\rho t^{1/\alpha} + i)^{-2} \rho^{-i\lambda\alpha} d\rho \\ &= \frac{\chi_{\gamma_\alpha}(t)}{\sqrt{2\pi}} \frac{B(1 - i\lambda\alpha, 1 + i\lambda\alpha)}{i^{1+i\lambda\alpha}} t^{i\lambda-1}, \end{aligned} \tag{4.13}$$

where  $B(\cdot, \cdot)$  is the Beta function. Thus, the function  $t \mapsto \chi_{\gamma_\alpha}(t)t^{i\lambda-1}$  belongs to  $\text{Im}B_\alpha(\lambda)$ . Since the space  $\text{Im}B_\alpha(\lambda)$  is one-dimensional, we conclude that

$$\text{Im}B_\alpha(\lambda) = \text{span}\{\chi_{\gamma_\alpha}h^{i\lambda-1}\},$$

where  $h(t) = t$  for all  $t \in \mathbb{T}$ . Simple calculations give

$$\left\| \chi_{\gamma_\alpha}h^{i\lambda-1} \right\|_{L^2(\gamma_\alpha)} = \begin{cases} \sqrt{(1 - e^{-2\pi\alpha\lambda})/(2\lambda)}, & \lambda \in \mathbb{R} \setminus \{0\}, \\ \sqrt{\pi\alpha}, & \lambda = 0. \end{cases}$$

Hence, the function (4.8) is a generator of norm one of the one-dimensional space  $\text{Im}B_\alpha(\lambda)$ .

In the same way, from the relation  $(M \otimes I)\tilde{B}_{\mathbb{K}_\alpha}(M^{-1} \otimes I) = I \otimes_\lambda \tilde{B}_\alpha(\lambda)$  we get

$$(I \otimes_\lambda \tilde{B}_\alpha(\lambda))(L^2(\mathbb{K}_\alpha)) = (M \otimes I)(\mathcal{A}^2(\mathbb{K}_\alpha)).$$

Hence, if  $f \in \mathcal{A}^2(\mathbb{K}_\alpha)$ , then  $[(M \otimes I)f](\lambda, \cdot) \in \text{Im}\tilde{B}_\alpha(\lambda)$ . Since the function  $W_\alpha^{-1}h_0$  given by (4.12) belongs to the space  $\mathcal{A}^2(\mathbb{K}_\alpha)$ , we conclude that  $\overline{W_\alpha^{-1}h_0} \in \mathcal{A}^2(\mathbb{K}_\alpha)$ . Therefore, we deduce from (4.13) that for  $(\lambda, t) \in \mathbb{R} \times \mathbb{T}$ ,

$$\begin{aligned} [(M \otimes I)(\overline{W_\alpha^{-1}h_0})](\lambda, t) &= \frac{\chi_{\gamma_\alpha}(t)}{\sqrt{2\pi}} \int_{\mathbb{R}_+} \overline{(1/\alpha)(rt)^{1/\alpha-1}(r^{1/\alpha}t^{1/\alpha} + i)^{-2}} r^{-i\lambda} dr \\ &= \overline{[(M \otimes I)(W_\alpha^{-1}h_0)](-\lambda, t)}. \end{aligned} \tag{4.14}$$

As we already proved, there exists  $c \in \mathbb{C} \setminus \{0\}$  such that

$$[(M \otimes I)(W_\alpha^{-1}h_0)](-\lambda, t) = c g_{\alpha, -\lambda}(t).$$

Then we infer from (4.14) that  $\overline{g_{\alpha, -\lambda}} \in \text{Im}\tilde{B}_\alpha(\lambda)$ . So, the function  $\tilde{g}_{\alpha, \lambda} := \overline{g_{\alpha, -\lambda}}$  given by (4.9) is the generator of norm one of  $\text{Im}\tilde{B}_\alpha(\lambda)$ .  $\square$

REMARK 4.3. It follows from (3.12) and (3.23) that

$$\begin{aligned} g_{\alpha, \lambda}(t) &= [\tilde{U}_\alpha^{-1}g_{1, \lambda\alpha}](t) = \alpha^{-1/2}t^{1/\alpha-1}g_{1, \lambda\alpha}(t^{1/\alpha}) \\ &= \alpha^{-1/2}G_1(\lambda\alpha)t^{1/\alpha-1}t^{(1/\alpha)(i\lambda\alpha-1)} = G_\alpha(\lambda)t^{i\lambda-1}, \\ \tilde{g}_{\alpha, \lambda}(t) &= [\tilde{C}g_{\alpha, -\lambda}](t) = \overline{G_\alpha(-\lambda)t^{-i\lambda-1}} = G_\alpha(-\lambda)t^{1-i\lambda}. \end{aligned}$$

LEMMA 4.4. For every  $\alpha \in (0, 2]$  and every  $\lambda \in \mathbb{R}$ ,

$$\langle g_{\alpha, \lambda}, \tilde{g}_{\alpha, \lambda} \rangle = \zeta_{\alpha, \lambda}, \tag{4.15}$$

where

$$\zeta_{\alpha, \lambda} := e^{-\pi\alpha i}\beta_\alpha(\lambda)\sin(\pi\alpha), \tag{4.16}$$

$$\beta_\alpha(\lambda) := G_\alpha(\lambda)G_\alpha(-\lambda) = \begin{cases} \frac{\lambda}{\sinh(\pi\alpha\lambda)} & \text{if } \lambda \in \mathbb{R} \setminus \{0\}, \\ (\pi\alpha)^{-1} & \text{if } \lambda = 0, \end{cases} \tag{4.17}$$

and

$$\lim_{\lambda \rightarrow \pm\infty} \zeta_{\alpha,\lambda} = 0. \tag{4.18}$$

*Proof.* Obviously, (4.17) follows from (4.10). Further, by (4.9), for all  $t \in \mathbb{T}$  we obtain

$$\overline{\tilde{g}_{\alpha,\lambda}(t)} = g_{\alpha,-\lambda}(t) = G_\alpha(-\lambda)\chi_{\gamma_\alpha}(t)t^{-1-i\lambda}. \tag{4.19}$$

Applying (4.8), (4.19), (4.17) and (4.16), we infer that

$$\begin{aligned} \langle g_{\alpha,\lambda}, \tilde{g}_{\alpha,\lambda} \rangle &= \int_0^{\pi\alpha} g_{\alpha,\lambda}(t) g_{\alpha,-\lambda}(t) |dt| = \beta_\alpha(\lambda) \int_0^{\pi\alpha} e^{-2i\theta} d\theta \\ &= \beta_\alpha(\lambda) (1 - e^{-2\pi\alpha i}) / (2i) = \zeta_{\alpha,\lambda}, \end{aligned}$$

which gives (4.15).

Finally, we infer from (4.17) that, for every  $\alpha \in (0, 2]$ ,

$$\lim_{\lambda \rightarrow \pm\infty} \beta_\alpha(\lambda) = \lim_{\lambda \rightarrow \pm\infty} \frac{\lambda}{\sinh(\pi\alpha\lambda)} = 0,$$

which in view of (4.15) implies (4.18) and completes the proof.  $\square$

REMARK 4.5. One can see from (4.15) that for every  $\lambda \in \mathbb{R}$  the one-dimensional subspaces  $\text{Im}B_\alpha(\lambda)$  and  $\text{Im}\tilde{B}_\alpha(\lambda)$  of  $L^2(\gamma_\alpha)$  are not orthogonal if  $\alpha \in (0, 1) \cup (1, 2)$ , while  $B_\alpha(\lambda)\tilde{B}_\alpha(\lambda) = 0$  for all  $\lambda \in \mathbb{R}$  and  $\alpha = 1, 2$ .

### 5. A $C^*$ -algebra generated by projections

In this section we modify the symbol calculus constructed in [8] for a  $C^*$ -algebra  $\mathcal{A} \subset \mathcal{B}(H)$  generated by  $n$  orthogonal projections  $Q_i$  giving the identity operator  $I$  in sum and by  $m$  pairwise orthogonal one-dimensional self-adjoint projections  $P_k$  on a Hilbert space  $H$ . In the case  $n \in \mathbb{N}$  and  $m = 1$  such isomorphism was constructed in [22] (also see [26]). We establish a  $C^*$ -algebra isomorphism between the  $C^*$ -algebra  $\mathcal{A}$  and a  $C^*$ -algebra of finite matrices if  $n = 1$ ,  $m \in \mathbb{N}$  and the self-adjoint projections  $P_k$  are not pairwise orthogonal.

Let  $\langle x, y \rangle$  mean the inner product in a Hilbert space  $H$ , let  $\delta_{i,j}$  be the Kronecker symbol, and let  $I_k$  stand for the  $k \times k$  identity matrix. We will denote by  $H_1 \dot{+} H_2 \dot{+} \dots \dot{+} H_n$  the direct sum of Hilbert spaces  $H_1, H_2, \dots, H_n$  that consists of elements  $x_1 + x_2 + \dots + x_n$  with  $x_k \in H_k$  ( $k = 1, 2, \dots, n$ ) such that if  $\sum_{k=1}^n x_k = 0$ , then  $x_k = 0$  for all  $k = 1, 2, \dots, n$ .

THEOREM 5.1. *Let  $H$  be a Hilbert space and let  $P_k$  ( $k = 1, 2, \dots, m$ ) be self-adjoint projections in  $\mathcal{B}(H)$  satisfying the conditions:*

(i)  $P_k$  ( $k = 1, 2, \dots, m$ ) are one-dimensional projections,

(ii)  $\bigcap_{k=1}^m (\text{Im}P_k)^\perp \neq \{0\}$ ,

(iii) the norm one generators  $v_1, \dots, v_m$  of the spaces  $\text{Im}P_1, \dots, \text{Im}P_m$ , respectively, are linearly independent.

Let  $\mathcal{A}$  be the  $C^*$ -subalgebra of  $\mathcal{B}(H)$  generated by the identity operator  $I$  and by the projections  $P_k$  ( $k = 1, 2, \dots, m$ ), let  $S$  be the invertible matrix in  $\mathbb{C}^{m \times m}$  that transforms the system  $v = \{v_1, v_2, \dots, v_m\}$  of linearly independent vectors in  $H$  onto an orthonormal system  $v_0 = \{e_1, e_2, \dots, e_m\}$ , and let  $\mathfrak{S}$  be the  $C^*$ -subalgebra of  $\mathbb{C}^{m \times m}$  generated by the  $m \times m$  identity matrix  $I_m$  and by the  $m \times m$  matrices

$$\tilde{P}_k = \begin{bmatrix} \overline{\langle e_1, v_k \rangle} \langle e_1, v_k \rangle & \overline{\langle e_1, v_k \rangle} \langle e_2, v_k \rangle & \cdots & \overline{\langle e_1, v_k \rangle} \langle e_m, v_k \rangle \\ \langle e_2, v_k \rangle \langle e_1, v_k \rangle & \langle e_2, v_k \rangle \langle e_2, v_k \rangle & \cdots & \langle e_2, v_k \rangle \langle e_m, v_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle e_m, v_k \rangle \langle e_1, v_k \rangle & \langle e_m, v_k \rangle \langle e_2, v_k \rangle & \cdots & \langle e_m, v_k \rangle \langle e_m, v_k \rangle \end{bmatrix} \tag{5.1}$$

( $k = 1, 2, \dots, m$ ). Then the map  $\sigma$ , defined on generators of  $\mathcal{A}$  by

$$I \mapsto 1 \oplus I_m, \quad P_k \mapsto 0 \oplus \tilde{P}_k \quad (k = 1, 2, \dots, m), \tag{5.2}$$

extends to a  $C^*$ -algebra isomorphism of the  $C^*$ -algebra  $\mathcal{A}$  onto the  $C^*$ -algebra  $\mathbb{C} \oplus \mathfrak{S}$ .

*Proof.* Let  $L_k := \text{Im}P_k$  ( $k = 1, 2, \dots, m$ ). For every  $k$ , fix a norm one generator  $v_k$  of  $L_k$ . We divide the proof in several steps.

1) Since the projections  $P_1, \dots, P_m$  are self-adjoint, the closed subspaces

$$H_0 := L_1^\perp \cap \dots \cap L_m^\perp \quad \text{and} \quad \mathfrak{M} := H_0^\perp$$

of  $H$  are invariant with respect to these projections:

$$P_k H_0 = \{0\}, \quad P_k \mathfrak{M} \subset \mathfrak{M} \quad (k = 1, \dots, m). \tag{5.3}$$

The first equality is evident. Let us show that  $L_k \subset \mathfrak{M}$  for all  $k = 1, \dots, m$ . Indeed, representing every element  $l_k \in L_k$  in the form  $l_k = x_k + y_k$  where  $y_k \in \mathfrak{M}$  and  $x_k \in L_1^\perp \cap \dots \cap L_m^\perp$ , we obtain

$$0 = \langle l_k, x_k \rangle = \|x_k\|^2 + \langle y_k, x_k \rangle = \|x_k\|^2.$$

Thus, all  $x_k = 0$  and hence  $l_k \in \mathfrak{M}$ , which gives (5.3).

2) Consider the operator

$$\Pi := (P_1 + \dots + P_m)|_{\mathfrak{M}}.$$

If  $g \in \mathfrak{M}$  and  $\Pi g = 0$ , then  $P_1 g + \dots + P_m g = 0$ . By (i),  $P_k g = c_k v_k$  for every  $k = 1, \dots, m$ , where  $v_k$  is a norm one generator of the space  $\text{Im}P_k$  and  $c_k \in \mathbb{C}$ . Hence

$$c_1 v_1 + \dots + c_m v_m = 0. \tag{5.4}$$

Since the vectors  $v_1, \dots, v_m$  are linearly independent due to (iii), we deduce from (5.4) that  $c_k = 0$  and hence  $P_k g = c_k v_k = 0$  for every  $k = 1, \dots, m$ . Consequently,

$$\Pi : \mathfrak{M} \rightarrow L_1 \dot{+} \dots \dot{+} L_m.$$

On the other hand, since  $P_k g = 0$  for all  $k = 1, \dots, m$ , we conclude that  $g \in L_1^\perp \cap \dots \cap L_m^\perp$ . Thus,

$$g \in (L_1^\perp \cap \dots \cap L_m^\perp) \cap \mathfrak{M} = \{0\},$$

and hence the operator  $\Pi$  is injective.

3) We claim that  $\dim \mathfrak{M} = m$ . Indeed, since  $v_k \in L_k \subset \mathfrak{M}$  for all  $k = 1, \dots, m$  and since the vectors  $v_1, \dots, v_m \in \mathfrak{M}$  are linearly independent in view of (iii), we conclude that  $\dim \mathfrak{M} \geq m$ . On the other hand, because  $\Pi : \mathfrak{M} \rightarrow L_1 + \dots + L_m$  is an injective operator and  $\dim(L_1 + \dots + L_m) = m$  (see (iii)), it follows that  $\dim \mathfrak{M} \leq m$ , which proves the claim.

4) Since  $\dim \mathfrak{M} = m$ , we infer that  $v = \{v_1, v_2, \dots, v_m\}$  is an ordered basis of  $\mathfrak{M}$ . Applying the Gram-Schmidt orthogonalization process to the basis  $v$ , we obtain the orthonormal basis  $v_0 = \{e_1, e_2, \dots, e_m\}$  of  $\mathfrak{M}$ , where

$$e_1 := v_1, \quad e_k := \frac{f_k}{\|f_k\|}, \quad f_k := v_k - \sum_{s=1}^{k-1} \langle v_k, e_s \rangle e_s \neq 0 \quad (k = 2, 3, \dots, m). \tag{5.5}$$

Because for the self-adjoint projections  $P'_k := P_k|_{\mathfrak{M}}$  and all  $k, j, s \in \{1, 2, \dots, m\}$  we have

$$P'_k e_j = \langle P'_k e_j, v_k \rangle v_k = \langle e_j, P'_k v_k \rangle v_k = \langle e_j, v_k \rangle v_k,$$

and hence

$$\langle P'_k e_j, e_s \rangle = \langle \langle e_j, v_k \rangle v_k, e_s \rangle = \overline{\langle e_s, v_k \rangle} \langle e_j, v_k \rangle,$$

we conclude that

$$[P'_k e_1 \ P'_k e_2 \ \dots \ P'_k e_m] = [e_1 \ e_2 \ \dots \ e_m] \tilde{P}_k \quad (k = 1, 2, \dots, m),$$

where the matrices  $\tilde{P}_k \in \mathbb{C}^{m \times m}$  given by (5.1) are matrix representations of the projections  $P'_k \in \mathcal{B}(\mathfrak{M})$  in the basis  $v_0$ .

5) Finally, according to the decomposition

$$H = (L_1^\perp \cap \dots \cap L_m^\perp) \oplus \mathfrak{M}$$

where, by (ii),  $\bigcap_{k=1}^m (\text{Im } P_k)^\perp \neq \{0\}$  and  $\mathfrak{M}$  is taken with the basis  $v_0$ , we obtain the representations (5.2) for the generators  $I$  and  $P_k$  of the  $C^*$ -algebra  $\mathcal{A}$  in the  $C^*$ -algebra  $\mathbb{C} \oplus \mathfrak{S}$ . Thus, there exists a  $C^*$ -algebra isomorphism of  $\mathcal{A}$  onto the  $C^*$ -algebra  $\mathcal{D}$  of  $\mathbb{C} \oplus \mathfrak{S}$  generated by the elements (5.2).

6) Since, by (5.5),  $v_k \in \text{span}\{e_1, e_2, \dots, e_k\}$ , from the orthogonality of the basis  $v_0$  it follows that

$$\langle e_j, v_k \rangle = 0 \quad \text{for all } k, j = 1, 2, \dots, m; \ j > k. \tag{5.6}$$

Moreover, (5.5) implies that

$$\|f_1\|^2 = 1, \quad \|f_k\|^2 = \left\| v_k - \sum_{s=1}^{k-1} \langle v_k, e_s \rangle e_s \right\|^2 = 1 - \sum_{s=1}^{k-1} |\langle v_k, e_s \rangle|^2 \neq 0 \tag{5.7}$$

for  $k = 2, 3, \dots, m$ . Hence, for all  $k = 1, 2, \dots, m$  we infer from (5.5) and (5.7) that

$$\langle e_k, v_k \rangle = \frac{1}{\|f_k\|} \left( 1 - \sum_{s=1}^{k-1} \overline{\langle e_s, v_k \rangle} \langle e_s, v_k \rangle \right) = \|f_k\| \neq 0. \tag{5.8}$$

Thus, if the orthogonal basis  $v_0$  is given by (5.5), we conclude from (5.1), (5.6) and (5.8) that

$$\tilde{P}_k = \begin{bmatrix} B_k & 0_{m-k} \\ 0_{m-k} & 0_{m-k} \end{bmatrix} \quad (k = 1, 2, \dots, m), \tag{5.9}$$

where

$$B_k = \left[ \overline{\langle e_s, v_k \rangle} \langle e_j, v_k \rangle \right]_{s,j=1}^k \tag{5.10}$$

and the  $(k, k)$ -entry  $\beta_k := |\langle e_k, v_k \rangle|^2$  of  $B_k$  is non-zero.

7) Then similarly to the proof of [8, Lemma 8.3] we can show that the  $C^*$ -algebra contains the operators  $U_k$  ( $k = 1, 2, \dots, m$ ) for which

$$\sigma(U_k) = 0 \oplus \text{diag}\{\delta_{k,s}\}_{s=1}^m.$$

Indeed, since  $\beta_k \neq 0$ , we can define by induction the operators

$$U_k := \beta_k^{-1} (I - U_1 - \dots - U_{k-1}) P_k (I - U_1 - \dots - U_{k-1}) \in \mathcal{B}(H),$$

where  $k = 1, 2, \dots, m$  and  $I - U_1 - \dots - U_{k-1} = I$  for  $k = 1$ . Then we infer, again by induction, that

$$\begin{aligned} \tilde{U}_k &:= \beta_k^{-1} (I_m - \tilde{U}_1 - \dots - \tilde{U}_{k-1}) \tilde{P}_k (I_m - \tilde{U}_1 - \dots - \tilde{U}_{k-1}) \\ &= \text{diag}\{\delta_{k,s}\}_{s=1}^m \in \mathcal{B}(\mathfrak{M}), \end{aligned}$$

which leads to the desired equalities

$$\sigma(U_k) = 0 \oplus \tilde{U}_k = 0 \oplus \text{diag}\{\delta_{k,s}\}_{s=1}^m \quad (k = 1, 2, \dots, m).$$

This immediately implies that the  $C^*$ -subalgebra  $\mathcal{D}$  of the  $C^*$ -algebra  $\mathbb{C} \oplus \mathfrak{S}$  coincides with  $\mathbb{C} \oplus \mathfrak{S}$ .  $\square$

It is natural to call the matrices  $\sigma(A) \in \mathbb{C} \oplus \mathfrak{S}$  the *symbols* of operators  $A \in \mathcal{A}$ .

**COROLLARY 5.2.** *Any operator  $A \in \mathcal{A}$  is invertible on the Hilbert space  $H$  if and only if its symbol  $\sigma(A) \in \mathbb{C} \oplus \mathfrak{S}$  is invertible in the  $C^*$ -algebra  $\mathbb{C} \oplus \mathbb{C}^{m \times m}$ .*

### 6. Fulfillment of conditions of Theorem 5.1

Given  $\alpha \in (0, 2]$  and  $\lambda \in \mathbb{R}$ , let us check the fulfillment of all conditions of Theorem 5.1 for the  $C^*$ -algebra  $\mathcal{A}_{\alpha,\lambda}$  given by (1.8).

Let  $m := 2$ . For every  $\lambda \in \mathbb{R}$  and all  $j, l = 1, 2$ , we introduce the inner products  $\langle v_{\alpha,\lambda,j}, v_{\alpha,\lambda,l} \rangle$  in  $L^2(\gamma_\alpha)$  of the norm one functions in  $L^2(\gamma_\alpha)$  given by

$$v_{\alpha,\lambda,1} := g_{\alpha,\lambda}, \quad v_{\alpha,\lambda,2} := \tilde{g}_{\alpha,\lambda}, \tag{6.1}$$

where the functions  $g_{\alpha,\lambda}$  and  $\tilde{g}_{\alpha,\lambda}$  are defined by (4.8) and (4.9), respectively.

LEMMA 6.1. For every  $\lambda \in \mathbb{R}$  and every  $\alpha \in (0, 2]$ , we have

$$\langle g_{\alpha,\lambda}, g_{\alpha,\lambda} \rangle_{L^2(\mathbb{T}_+)} = [G_\alpha(\lambda)/G_1(\lambda)]^2 \neq 0, \tag{6.2}$$

$$\langle \tilde{g}_{\alpha,\lambda}, \tilde{g}_{\alpha,\lambda} \rangle_{L^2(\mathbb{T}_+)} = [G_\alpha(-\lambda)/G_1(-\lambda)]^2 \neq 0, \tag{6.3}$$

$$\langle g_{\alpha,\lambda}, \tilde{g}_{\alpha,\lambda} \rangle_{L^2(\mathbb{T}_+)} = 0. \tag{6.4}$$

*Proof.* By (4.8) and (4.10), for every  $\lambda \in \mathbb{R}$ , we obtain

$$\begin{aligned} \langle g_{\alpha,\lambda}, g_{\alpha,\lambda} \rangle_{L^2(\mathbb{T}_+)} &= \int_{\mathbb{T}_+} g_{\alpha,\lambda}(t) \overline{g_{\alpha,\lambda}(t)} |dt| = G_\alpha^2(\lambda) \int_0^\pi e^{-2\lambda\theta} d\theta \\ &= G_\alpha^2(\lambda) (1 - e^{-2\lambda\pi}) / (2\lambda) = [G_\alpha(\lambda)/G_1(\lambda)]^2 \neq 0, \end{aligned} \tag{6.5}$$

which gives (6.2). Further, for every  $\lambda \in \mathbb{R}$ , we deduce from (6.5) that

$$\langle \tilde{g}_{m,\lambda}, \tilde{g}_{m,\lambda} \rangle_{L^2(\mathbb{T}_+)} = \langle g_{m,-\lambda}, g_{m,-\lambda} \rangle_{L^2(\mathbb{T}_+)} = [G_\alpha(-\lambda)/G_1(-\lambda)]^2 \neq 0,$$

which gives (6.3). Finally, applying (4.8) and (4.9), we get

$$\langle g_{\alpha,\lambda}, \tilde{g}_{\alpha,\lambda} \rangle_{L^2(\mathbb{T}_+)} = \int_{\mathbb{T}_+} g_{\alpha,\lambda}(t) g_{\alpha,-\lambda}(t) |dt| = \beta_\alpha(\lambda) \int_0^\pi e^{-2i\theta} d\theta = 0,$$

which gives (6.4).  $\square$

LEMMA 6.2. For every  $\lambda \in \mathbb{R}$  and every  $\alpha \in (0, 2]$ , the functions  $v_{\alpha,\lambda,k}$  ( $k = 1, 2$ ) given by (6.1) are linearly independent on the arc  $\gamma_\alpha$ .

*Proof.* Suppose that, for some constants  $c_1, c_2 \in \mathbb{C}$ ,

$$c_1 g_{\alpha,\lambda}(t) + c_2 \tilde{g}_{\alpha,\lambda} = 0 \quad \text{for all } t \in \gamma_\alpha.$$

By (4.8) and (4.9), the function

$$\varphi(t) := c_1 g_{\alpha,\lambda}(t) + c_2 \tilde{g}_{\alpha,\lambda}(t) \tag{6.6}$$

admits an analytic extension to the whole upper half-plane  $\Pi$ . Since this function identically equals zero on the arc  $\gamma_\alpha \subset \mathbb{T}_+$ , it equals zero for all  $t \in \mathbb{T}_+$  in view of its analyticity on  $\Pi$ . But by Lemma 6.1 the functions  $g_{\alpha,\lambda}$  and  $\tilde{g}_{\alpha,\lambda}$  are orthogonal on the space  $L^2(\mathbb{T}_+)$ , while their norms in  $L^2(\mathbb{T}_+)$  are different of 0. In that case it follows from the equality (6.6) fulfilled for all  $t \in \mathbb{T}_+$  that  $c_j = 0$  for all  $j = 1, 2$ , which means that the functions  $v_{\alpha,\lambda,k}$  ( $k = 1, 2$ ) are linearly independent on the arc  $\gamma_\alpha$ .  $\square$

Given  $\alpha \in (0, 2]$  and  $\lambda \in \mathbb{R}$ , we now check conditions (i)–(iii) of Theorem 5.1 for  $P_1 = P_1(\alpha, \lambda) = B_\alpha(\lambda)$  and  $P_2 = P_2(\alpha, \lambda) = \tilde{B}_\alpha(\lambda)$ . By Lemma 4.2, the spaces  $\text{Im}P_k(\alpha, \lambda)$  are one-dimensional for all  $\lambda \in \mathbb{R}$  and all  $k = 1, 2$ , whence condition (i)

of Theorem 5.1 holds. Because the space  $L^2(\gamma_\alpha)$  is infinite dimensional and the spaces  $\text{Im} P_k(\alpha, \lambda)$  are one-dimensional for all  $k = 1, 2$ , there is a  $g \in L^2(\gamma_\alpha)$  such that

$$0 \neq g \in \bigcap_{k=1}^2 (\text{Im} P_k(\alpha, \lambda))^\perp,$$

and hence condition (ii) of Theorem 5.1 is also fulfilled. Finally, Lemma 6.2 implies the fulfilment of condition (iii) of Theorem 5.1. Thus, to study the invertibility of operators in the  $C^*$ -algebra  $\mathcal{A}_{\alpha, \lambda}$  we can apply this theorem.

Consider the direct sum  $\mathfrak{M}$  of linear subspaces of  $L^2(\gamma_\alpha)$  generated by the norm one functions  $v_{\alpha, \lambda, j}$  ( $j = 1, 2$ ) given by (6.1). Since  $\dim \mathfrak{M} = 2$ , we infer from (iii) that the system  $v = \{v_{\alpha, \lambda, 1}, v_{\alpha, \lambda, 2}\}$  is an ordered basis of  $\mathfrak{M}$ . Obviously, for every  $k, j = 1, 2$ , we obtain

$$P'_k(\alpha, \lambda)v_{\alpha, \lambda, j} = \langle v_{\alpha, \lambda, j}, v_{\alpha, \lambda, k} \rangle v_{\alpha, \lambda, k}, \tag{6.7}$$

where  $P'_k(\alpha, \lambda) := P_k(\alpha, \lambda)|_{\mathfrak{M}}$ . From (6.7) it follows for every  $k = 1, 2$  that

$$[P'_k(\alpha, \lambda)v_{\alpha, \lambda, 1} \ P'_k(\alpha, \lambda)v_{\alpha, \lambda, 2}] = [v_{\alpha, \lambda, 1} \ v_{\alpha, \lambda, 2}] \text{diag}\{\delta_{k, j}\}_{j=1}^2 E_\alpha(\lambda),$$

where the  $m \times m$  matrix  $E_\alpha(\lambda)$  is given by

$$E_\alpha(\lambda) = \begin{bmatrix} \langle v_{\alpha, \lambda, 1}, v_{\alpha, \lambda, 1} \rangle & \langle v_{\alpha, \lambda, 2}, v_{\alpha, \lambda, 1} \rangle \\ \langle v_{\alpha, \lambda, 1}, v_{\alpha, \lambda, 2} \rangle & \langle v_{\alpha, \lambda, 2}, v_{\alpha, \lambda, 2} \rangle \end{bmatrix} \in \mathbb{C}^{2 \times 2}. \tag{6.8}$$

Thus, the matrix representations of the projections  $P'_k(\alpha, \lambda) \in \mathcal{B}(\mathfrak{M})$  relatively to the basis  $v$  are of the form

$$\widehat{P}_k(\alpha, \lambda) = \text{diag}\{\delta_{k, j}\}_{j=1}^2 E_\alpha(\lambda), \quad k = 1, 2. \tag{6.9}$$

By Lemma 6.2,  $\det E_\alpha(\lambda) \neq 0$  for all  $\lambda \in \mathbb{R}$ . According to (6.8), Lemma 4.2 and (4.15), we conclude that for every  $\lambda \in \mathbb{R}$  the Gram matrix  $E_\alpha(\lambda)$  is of the form

$$E_\alpha(\lambda) = \begin{bmatrix} 1 & \overline{\zeta_{\alpha, \lambda}} \\ \zeta_{\alpha, \lambda} & 1 \end{bmatrix}, \tag{6.10}$$

where  $\zeta_{\alpha, \lambda}$  is given by (4.16). By (4.18),  $\lim_{\lambda \rightarrow \pm\infty} \zeta_{\alpha, \lambda} = 0$ , and hence we conclude from (6.10) that

$$\lim_{\lambda \rightarrow \pm\infty} E_\alpha(\lambda) = I_2.$$

Thus, we obtain the following.

**LEMMA 6.3.** *The matrix function  $E_\alpha(\cdot)$  given by (6.8) admits a continuous extension to  $\overline{\mathbb{R}} = [-\infty, +\infty]$  with values  $E_\alpha(\pm\infty) = I_2$  and a non-zero determinant  $\det E_\alpha(\lambda)$  on  $\overline{\mathbb{R}}$ .*

### 7. Symbol calculus for the C\*-algebra $\mathfrak{A}_{\mathbb{K}\alpha}$

Theorem 5.1 for  $m = 2$  immediately implies the following.

THEOREM 7.1. For every  $\alpha \in (0, 2]$  and every  $\lambda \in \mathbb{R}$ , the C\*-algebra

$$\mathcal{A}_{\alpha,\lambda} := \text{alg} \{I, B_\alpha(\lambda), \tilde{B}_\alpha(\lambda)\} \subset \mathcal{B}(L^2(\gamma_\alpha))$$

is \*-isomorphic to a C\*-subalgebra of  $\mathbb{C} \oplus \mathbb{C}^{2 \times 2}$  which coincides with  $\mathbb{C} \oplus \mathbb{C}^{2 \times 2}$  if  $\alpha \in (0, 1) \cup (1, 2)$ , and this isomorphism  $\eta^0 \oplus \eta$  is given on the generators of  $\mathcal{A}_{\alpha,\lambda}$  by

$$I \mapsto 1 \oplus I_2, \quad B_\alpha(\lambda) \mapsto 0 \oplus M_\alpha(\lambda), \quad \tilde{B}_\alpha(\lambda) \mapsto 0 \oplus \tilde{M}_\alpha(\lambda), \tag{7.1}$$

where the matrices  $M_\alpha(\lambda)$  and  $\tilde{M}_\alpha(\lambda) \in \mathbb{C}^{2 \times 2}$  are defined by

$$M_\alpha(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{M}_\alpha(\lambda) = \begin{bmatrix} |\zeta_{\alpha,\lambda}|^2 & \overline{\zeta_{\alpha,\lambda}} \Delta_{\alpha,\lambda}^{1/2} \\ \zeta_{\alpha,\lambda} \Delta_{\alpha,\lambda}^{1/2} & \Delta_{\alpha,\lambda} \end{bmatrix}, \tag{7.2}$$

$\Delta_{\alpha,\lambda} := 1 - |\zeta_{\alpha,\lambda}|^2$  and  $\zeta_{\alpha,\lambda}$  is given by (4.16). For  $(\alpha, \lambda) \in \{1, 2\} \times \mathbb{R}$  the matrices  $M_\alpha(\lambda), \tilde{M}_\alpha(\lambda)$  are diagonal, and  $(\eta^0 \oplus \eta)(\mathcal{A}_{\alpha,\lambda})$  is a C\*-subalgebra of  $\mathbb{C} \oplus \mathbb{C}^{2 \times 2}$ .

*Proof.* Fix  $\alpha \in (0, 2]$  and  $\lambda \in \mathbb{R}$  and apply Theorem 5.1 to the C\*-algebra  $\mathcal{A}_{\alpha,\lambda}$ . By Lemma 6.2, for every  $\lambda \in \mathbb{R}$ , the set

$$\{g_{\alpha,\lambda}, \tilde{g}_{\alpha,\lambda}\} \subset L^2(\gamma_\alpha), \tag{7.3}$$

given by (4.8) and (4.9), is a system of linearly independent vectors in  $L^2(\gamma_\alpha)$ . Applying the Gram-Schmidt orthogonalization process to the set (7.3), we obtain the orthonormal set

$$\{e_{\alpha,\lambda}, \tilde{e}_{\alpha,\lambda}\} \subset L^2(\gamma_\alpha), \tag{7.4}$$

where

$$e_{\alpha,\lambda} = g_{\alpha,\lambda}, \quad \tilde{e}_{\alpha,\lambda} = \frac{\tilde{g}_{\alpha,\lambda} - \langle \tilde{g}_{\alpha,\lambda}, g_{\alpha,\lambda} \rangle g_{\alpha,\lambda}}{(1 - |\langle \tilde{g}_{\alpha,\lambda}, g_{\alpha,\lambda} \rangle|^2)^{1/2}}. \tag{7.5}$$

By Lemma 4.4,  $\langle \tilde{g}_{\alpha,\lambda}, g_{\alpha,\lambda} \rangle = \overline{\zeta_{\alpha,\lambda}}$ , where  $\zeta_{\alpha,\lambda}$  is given by (4.16). Then

$$\Delta_{\alpha,\lambda} = 1 - |\zeta_{\alpha,\lambda}|^2 \neq 0. \tag{7.6}$$

Let  $S_\alpha(\lambda)$  be an invertible  $2 \times 2$  matrix that transform the system (7.3) onto the orthonormal system (7.4). Then we infer from (7.5) that

$$[e_{\alpha,\lambda}, \tilde{e}_{\alpha,\lambda}] = [g_{\alpha,\lambda}, \tilde{g}_{\alpha,\lambda}] S_\alpha(\lambda) \tag{7.7}$$

where

$$S_\alpha(\lambda) = \begin{bmatrix} 1 & -\overline{\zeta_{\alpha,\lambda}} \Delta_{\alpha,\lambda}^{-1/2} \\ 0 & \Delta_{\alpha,\lambda}^{-1/2} \end{bmatrix}, \quad S_\alpha^{-1}(\lambda) = \begin{bmatrix} 1 & \overline{\zeta_{\alpha,\lambda}} \\ 0 & \Delta_{\alpha,\lambda}^{1/2} \end{bmatrix}. \tag{7.8}$$

Let us define the matrix functions

$$M_\alpha(\lambda) := \tilde{P}_k(\alpha, \lambda) = S_\alpha^{-1}(\lambda)\hat{P}_k(\alpha, \lambda)S_\alpha(\lambda) \quad (\lambda \in \mathbb{R}). \tag{7.9}$$

Then, in accordance with (5.9) and (5.10), we infer from (7.9), (6.9), (6.10) and (7.8) that

$$\begin{aligned} M_\alpha(\lambda) &= S_\alpha^{-1}(\lambda) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} E_\alpha(\lambda) S_\alpha(\lambda) \\ &= \begin{bmatrix} 1 & \overline{\zeta_{\alpha,\lambda}} \\ 0 & \Delta_{\alpha,\lambda}^{1/2} \end{bmatrix} \begin{bmatrix} 1 & \overline{\zeta_{\alpha,\lambda}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 - \overline{\zeta_{\alpha,\lambda}} \Delta_{\alpha,\lambda}^{-1/2} \\ 0 & \Delta_{\alpha,\lambda}^{-1/2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \tilde{M}_\alpha(\lambda) &= S_\alpha^{-1}(\lambda) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} E_\alpha(\lambda) S_\alpha(\lambda) \\ &= \begin{bmatrix} 1 & \overline{\zeta_{\alpha,\lambda}} \\ 0 & \Delta_{\alpha,\lambda}^{1/2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \zeta_{\alpha,\lambda} & 1 \end{bmatrix} \begin{bmatrix} 1 - \overline{\zeta_{\alpha,\lambda}} \Delta_{\alpha,\lambda}^{-1/2} \\ 0 & \Delta_{\alpha,\lambda}^{-1/2} \end{bmatrix} \\ &= \begin{bmatrix} |\zeta_{\alpha,\lambda}|^2 & \overline{\zeta_{\alpha,\lambda}} \Delta_{\alpha,\lambda}^{1/2} \\ \zeta_{\alpha,\lambda} \Delta_{\alpha,\lambda}^{1/2} & \Delta_{\alpha,\lambda} \end{bmatrix}, \end{aligned}$$

which gives (7.2). In view of (7.6) and (4.18), we deduce from (7.2) that

$$\lim_{\lambda \rightarrow \pm\infty} \tilde{M}_\alpha(\lambda) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{7.10}$$

By Theorem 5.1 and (7.9), the  $C^*$ -algebra  $\mathcal{A}_{\alpha,\lambda}$  is isomorphic to a  $C^*$ -subalgebra of  $\mathbb{C} \oplus \mathbb{C}^{2 \times 2}$ . By (7.6),  $\Delta_{\alpha,\lambda} \neq 0$  for all  $(\alpha, \lambda) \in (0, 2] \times \mathbb{R}$ . On the other hand, by (4.16),  $\zeta_{\alpha,\lambda} \neq 0$  once  $(\alpha, \lambda) \in ((0, 1) \cup (1, 2)) \times \mathbb{R}$ . This implies that in the latter case the image of the  $C^*$ -algebra  $\mathcal{A}_{\alpha,\lambda}$  under the map (7.1) coincides with  $\mathbb{C} \oplus \mathbb{C}^{2 \times 2}$ , while for  $\alpha \in \{1, 2\}$  the set  $\eta(\mathcal{A}_{\alpha,\lambda})$  in the image of  $\mathcal{A}_{\alpha,\lambda}$  under the map  $\eta^0 \oplus \eta$  given by (7.1) consists of diagonal matrices in  $\mathbb{C}^{2 \times 2}$ .  $\square$

For every  $\alpha \in (0, 2]$  and every  $A \in \mathfrak{A}_{\mathbb{K},\alpha}$ , we introduce in view of (7.1), (7.2) and (7.10) the continuous matrix functions  $\eta_{A,\alpha} : \overline{\mathbb{R}} \rightarrow \mathbb{C}^{2 \times 2}$  given by

$$\eta_{A,\alpha}(\lambda) = \eta[A_\alpha(\lambda)] \quad (\lambda \in \mathbb{R}), \quad \eta_{A,\alpha}(\pm\infty) = \lim_{\lambda \rightarrow \pm\infty} \eta_{A,\alpha}(\lambda).$$

Since the matrix functions  $\lambda \mapsto \eta_{A,\alpha}(\lambda)$  are continuous on  $\overline{\mathbb{R}}$ , we conclude from Theorem 7.1 that the invertibility criterion for an operator  $A \in \mathfrak{A}_{\mathbb{K},\alpha}$ , which is given by Theorem 3.6, is equivalent to the invertibility of the matrices  $\eta_{A,\alpha}(\lambda)$  for all  $\lambda \in \overline{\mathbb{R}}$  and the fulfillment of  $\eta^0[A_\alpha(\lambda)] \neq 0$ . Hence Theorems 3.5, 3.6 and 7.1 immediately imply the following.

THEOREM 7.2. For every  $\alpha \in (0, 2]$ , the  $C^*$ -algebra

$$\mathfrak{A}_{\mathbb{K}_\alpha} = \text{alg} \{I, B_{\mathbb{K}_\alpha}, \tilde{B}_{\mathbb{K}_\alpha}\} \subset \mathcal{B}(L^2(\mathbb{K}_\alpha))$$

is  $*$ -isomorphic to the  $C^*$ -subalgebra  $\mathbb{C} \oplus \mathfrak{S}_\alpha$  of the  $C^*$ -algebra  $\mathbb{C} \oplus C(\overline{\mathbb{R}}, \mathbb{C}^{2 \times 2})$ , and this isomorphism  $\Phi_\alpha = \Phi_\alpha^0 \oplus (\bigoplus_{\lambda \in \overline{\mathbb{R}}} \Phi_{\alpha, \lambda})$  of  $\mathfrak{A}_{\mathbb{K}_\alpha}$  onto  $\mathbb{C} \oplus \mathfrak{S}_\alpha$  is given by

$$\begin{aligned} \Phi_\alpha^0(I) &= 1, & \Phi_{\alpha, \lambda}(I) &= I_2, \\ \Phi_\alpha^0(B_{\mathbb{K}_\alpha}) &= 0, & \Phi_{\alpha, \lambda}(B_{\mathbb{K}_\alpha}) &= M_\alpha(\lambda), \\ \Phi_\alpha^0(\tilde{B}_{\mathbb{K}_\alpha}) &= 0, & \Phi_{\alpha, \lambda}(\tilde{B}_{\mathbb{K}_\alpha}) &= \tilde{M}_\alpha(\lambda), \end{aligned} \tag{7.11}$$

where the matrix functions  $M_\alpha(\cdot), \tilde{M}_\alpha(\cdot) \in C(\overline{\mathbb{R}}, \mathbb{C}^{2 \times 2})$  are defined by (7.2) for all  $\lambda \in \overline{\mathbb{R}}$ . An operator  $A \in \mathfrak{A}_{\mathbb{K}_\alpha}$  is invertible on the space  $L^2(\mathbb{K}_\alpha)$  if and only if its symbol  $\Phi_\alpha(A)$  is invertible in the  $C^*$ -algebra  $\mathbb{C} \oplus \mathfrak{S}_\alpha$ , that is, if

$$\Phi_\alpha^0(A) \neq 0 \text{ and } \det[\Phi_{\alpha, \lambda}(A)] \neq 0 \text{ for all } \lambda \in \overline{\mathbb{R}}.$$

### 8. $C^*$ -algebras $\mathfrak{B}_U$ over bounded polygonal domains $U$

Let  $U$  be a bounded polygonal domain with inner angles  $\pi\alpha_k \in (0, \pi) \cup (\pi, 2\pi]$  ( $k = 1, 2, \dots, n$ ) at corners  $z_k$ , and let  $\mathcal{S} = \{z_k : k = 1, 2, \dots, n\}$  be the set of all corners. As is well known, for a polygonal domain  $U$ ,  $\sum_{k=1}^n \pi\alpha_k = \pi(n - 2)$ .

Let us study the  $C^*$ -algebra  $\mathfrak{B}_U$  given by (1.7).

#### 8.1. Compact operators

Let  $\overline{U}$  be the closure of the polygonal domain  $U$ , and let  $\partial U$  be the boundary of  $U$ .

LEMMA 8.1. For a bounded polygonal domain  $U$  and any function  $a \in C(\overline{U})$ , the commutators  $aB_U - B_U aI$  and  $a\tilde{B}_U - \tilde{B}_U aI$  are compact on the space  $L^2(U)$ .

*Proof.* Since  $\partial U$  is a Jordan curve, then by the Riemann Mapping theorem and the Carathéodory theorem (see, e.g., [20, Sections 1.2 and 2.1]), there exists a conformal bijection of the open unit disc  $\mathbb{D}$  onto a bounded polygonal domain  $U$ , which extends to a homeomorphic map of  $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$  onto  $\overline{U} = U \cup \partial U$ . This result remains true for a bounded polygonal domain  $U$  with cuts if to distinguish two sides of cuts. Moreover, such map is given by the Schwarz-Christoffel formula (see, e.g., [16, III.9]).

Fix  $a \in C(\overline{U})$  and consider the unitary shift operator

$$W_\varphi : L^2(U) \rightarrow L^2(\mathbb{D}), \quad f \mapsto \varphi'(f \circ \varphi).$$

associated with a conformal map  $\varphi : \mathbb{D} \rightarrow U$ . Since the commutator

$$(a \circ \varphi)B_{\mathbb{D}} - B_{\mathbb{D}}(a \circ \varphi)I$$

is compact on the space  $L^2(\mathbb{D})$  (see, e.g., [29, Lemma 2.4.4]) and since  $B_U = W_\varphi^{-1} B_{\mathbb{D}} W_\varphi$ , it follows that the commutator

$$aB_U - B_U aI = W_\varphi^{-1} [(a \circ \varphi) B_{\mathbb{D}} - B_{\mathbb{D}} (a \circ \varphi) I] W_\varphi$$

is compact on the space  $L^2(U)$ . This implies the compactness of the commutator  $a\tilde{B}_U - \tilde{B}_U aI$  in view of the equality  $\tilde{B}_U = C B_U C$ , where  $Cf = \bar{f}$ .  $\square$

According to [24] (also see [1, Section 8.2]), an operator  $A \in \mathcal{B}(L^2(U))$  is called an *operator of local type* if the commutators  $cA - AcI$  are compact for every  $c \in C(\bar{U})$ . Thus, by Lemma 8.1, the operators  $B_U, \tilde{B}_U$ , and therefore all operators in the  $C^*$ -algebra  $\mathfrak{B}_U = \text{alg} \{aI, B_U, \tilde{B}_U : a \in C(\bar{U})\}$  are of local type.

Let us denote by  $\Lambda_U$  the set of all operators of local type in  $\mathcal{B} := \mathcal{B}(L^2(U))$ . It is easily seen that  $\Lambda_U$  is a  $C^*$ -subalgebra of  $\mathcal{B}$ , and  $\mathfrak{B}_U \subset \Lambda_U$ .

Repeating literally the proof of [8, Lemma 2.6], we obtain the following.

LEMMA 8.2. *For a bounded polygonal domain  $U \subset \mathbb{C}$ , the  $C^*$ -algebra  $\mathfrak{B}_U$  given by (1.7) contains all compact operators acting on the space  $L^2(U)$ .*

### 8.2. An application of the Allan-Douglas local principle

By Lemma 8.2, the  $C^*$ -algebra  $\mathfrak{B}_U$  contains the ideal  $\mathcal{K} = \mathcal{K}(L^2(U))$  of all compact operators in the  $C^*$ -algebra  $\mathcal{B} = \mathcal{B}(L^2(U))$ . Hence, the quotient  $C^*$ -algebra  $\mathfrak{B}_U^\pi := \mathfrak{B}_U / \mathcal{K}$  is well defined. To obtain a Fredholm criterion for the operators  $A \in \mathfrak{B}_U$  we need to study the invertibility of the cosets  $A^\pi := A + \mathcal{K}$  in the quotient  $C^*$ -algebra  $\mathfrak{B}_U^\pi$ . To this end we will apply the Allan-Douglas local principle to the algebra  $\mathfrak{B}_U^\pi$ .

It follows from Lemma 8.1 that  $\mathcal{Z}^\pi := \{cI + \mathcal{K} : c \in C(\bar{U})\}$  is a central subalgebra of the  $C^*$ -algebra  $\mathfrak{B}_U^\pi$ . Obviously, the commutative  $C^*$ -algebra  $\mathcal{Z}^\pi$  is (isometrically)  $*$ -isomorphic to the  $C^*$ -algebra  $C(\bar{U})$ , and therefore the maximal ideal space of  $\mathcal{Z}^\pi$  can be identified with  $\bar{U}$ . For every point  $z \in \bar{U}$ , let  $J_z^\pi$  denote the closed two-sided ideal of the quotient  $C^*$ -algebra  $\Lambda_U^\pi := \Lambda_U / \mathcal{K}$  generated by the maximal ideal

$$I_z^\pi := \{cI + \mathcal{K} : c \in C(\bar{U}), c(z) = 0\} \subset \mathcal{Z}^\pi.$$

By [1, Proposition 8.6] and [23, Proposition 2.2.5], the ideal  $J_z^\pi$  has the form

$$J_z^\pi = \{(cA)^\pi : c \in C(\bar{U}), c(z) = 0, A \in \Lambda_U\}. \tag{8.1}$$

Hence, with every  $z \in \bar{U}$  we associate the quotient  $C^*$ -algebra  $(\Lambda_U)_z^\pi := \Lambda_U^\pi / J_z^\pi$ .

The Allan-Douglas local principle (see [4, Theorem 7.47] and [2, Theorem 1.35]) implies the following invertibility criterion.

THEOREM 8.3. *An operator  $A \in \mathfrak{B}_U$  is Fredholm on the space  $L^2(U)$  if and only if for every  $z \in \bar{U}$  the coset  $A_z^\pi := A^\pi + J_z^\pi$  is invertible in the quotient  $C^*$ -algebra  $(\Lambda_U)_z^\pi$ .*

The set  $(\mathfrak{B}_U)_z^\pi := \{A_z^\pi : A \in \mathfrak{B}_U\}$  is a  $C^*$ -subalgebra of  $(\Lambda_U)_z^\pi$  (see, e.g., [2, 1.26(g)]), and hence a coset  $A_z^\pi$  associated with  $A \in \mathfrak{B}_U$  is invertible in both the  $C^*$ -algebras  $(\Lambda_U)_z^\pi$  and  $(\mathfrak{B}_U)_z^\pi$  only simultaneously.

We say that cosets  $A^\pi, B^\pi \in \mathfrak{B}_U^\pi$  are locally equivalent at a point  $z \in \overline{U}$  if  $A^\pi - B^\pi \in J_z^\pi$ , and in that case we write  $A^\pi \overset{\sim}{\sim} B^\pi$ .

Similarly to [8, Lemma 3.3], we get the following.

LEMMA 8.4. *The cosets  $B_U^\pi$  and  $\widetilde{B}_U^\pi$  are locally equivalent to zero at every point  $z \in U$ .*

### 8.3. Local study and Fredholmness for the $C^*$ -algebra $\mathfrak{B}_U$

Let  $\mathbb{C}^n$  denote the  $C^*$ -algebra of complex-valued vectors  $x = (x_1, \dots, x_n)$  with usual operations of addition and multiplication by complex scalars, with the entry-wise multiplication, the adjoint  $x^* = (\bar{x}_1, \dots, \bar{x}_n)$ , and the norm  $\|x\| = \max\{|x_1|, \dots, |x_n|\}$ .

If two  $C^*$ -algebras  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are (isometrically)  $*$ -isomorphic, we will write  $\mathcal{A}_1 \cong \mathcal{A}_2$ . For every corner  $z \in \mathcal{T}$ , we denote by  $\alpha_z \in (0, 1) \cup (1, 2]$  the value of the inner angle of  $U$  at  $z$ , which is divided by  $\pi$ .

Let us characterize the local algebras  $(\mathfrak{B}_U)_z^\pi$  for  $z \in U$ . One can see from the lemma below that there are three types of such local algebras.

LEMMA 8.5. *For the  $C^*$ -algebra  $\mathfrak{B}_U$  given by (1.7), the following holds:*

- (i) if  $z \in U$ , then  $(\mathfrak{B}_U)_z^\pi \cong \mathbb{C}$ ;
- (ii) if  $z \in \partial U \setminus \mathcal{T}$ , then  $(\mathfrak{B}_U)_z^\pi \cong \mathbb{C}^3$ ;
- (iii) if  $z \in \mathcal{T}$ , then  $(\mathfrak{B}_U)_z^\pi \cong \mathfrak{A}_{\mathbb{K}\alpha}$  with  $\alpha = \alpha_z$ .

*Proof.* (i) Let  $z \in U$ . If  $a \in C(\overline{U})$ , then  $(aI)^\pi \overset{\sim}{\sim} a(z)I^\pi$ . From Lemma 8.4 it follows that  $B_U^\pi \overset{\sim}{\sim} 0$ ,  $\widetilde{B}_U^\pi \overset{\sim}{\sim} 0$ . Hence, the generators of the  $C^*$ -algebra  $(\mathfrak{B}_U)_z^\pi$  have the form  $(aI)_z^\pi$ , where  $a \in C(\overline{U})$ , and therefore the map given by

$$(aI)_z^\pi \mapsto a(z) \quad (a \in C(\overline{U}))$$

extends to a  $C^*$ -algebra isomorphism of  $(\mathfrak{B}_U)_z^\pi$  onto  $\mathbb{C}$ .

(ii) Let now  $z \in \partial U \setminus \mathcal{T}$ . Take the Schwarz-Christoffel conformal mapping  $\beta_z : \Pi \rightarrow U$  such that  $\beta_z(0) = z$  (see, e.g., [3]). Then  $\beta'_z(0) \neq 0$ . Making use of the unitary operator

$$W_{\beta_z} : L^2(U) \rightarrow L^2(\Pi), \quad f \mapsto \beta'_z(f \circ \beta_z),$$

we deduce (see, e.g., [8, Proposition 2.2]) that

$$W_{\beta_z}(aI)W_{\beta_z}^{-1} = (a \circ \beta_z)I, \quad W_{\beta_z}B_UW_{\beta_z}^{-1} = B_\Pi, \quad W_{\beta_z}\widetilde{B}_UW_{\beta_z}^{-1} = c_z\widetilde{B}_\Pi c_z^{-1}I, \quad (8.2)$$

where  $c_z := \beta'_z/\overline{\beta'_z}$ .

Fix  $A \in \mathfrak{B}_U$ . If the coset  $A_z^\pi \in (\mathfrak{B}_U)_z^\pi$  is invertible, then in view of (8.1) there exist an operator  $B \in \mathfrak{B}_U$ , operators  $D_1, D_2 \in \Lambda_U$ , operators  $K_1, K_2 \in \mathcal{K}(L^2(U))$  and functions  $c_1, c_2 \in C(\overline{U})$  such that  $c_1(z) = c_2(z) = 0$  and

$$BA = I + c_1 D_1 + K_1, \quad AB = I + c_2 D_2 + K_2.$$

Hence, we obtain

$$\begin{aligned} (W_{\beta_z} B W_{\beta_z}^{-1})(W_{\beta_z} A W_{\beta_z}^{-1}) &= I + (c_1 \circ \beta_z) \tilde{D}_1 + \tilde{K}_1, \\ (W_{\beta_z} A W_{\beta_z}^{-1})(W_{\beta_z} B W_{\beta_z}^{-1}) &= I + (c_2 \circ \beta_z) \tilde{D}_2 + \tilde{K}_2, \end{aligned} \tag{8.3}$$

where  $\tilde{K}_1, \tilde{K}_2 \in \mathcal{K}(L^2(\Pi))$  and  $\tilde{D}_1, \tilde{D}_2 \in W_{\beta_z} \Lambda_U W_{\beta_z}^{-1}$ . For constants  $k > 0$ , we introduce the unitary dilation operators

$$U_k : L^2(\Pi) \rightarrow L^2(\Pi), \quad (U_k f)(w) = kf(kw) \text{ for all } w \in \Pi. \tag{8.4}$$

Then, in view of (1.4) and the equality  $s\text{-}\lim_{k \rightarrow 0} (U_k c_z U_k^{-1}) = (\beta'_z(0) / \overline{\beta'_z(0)}) I$ , we infer for generators (8.2) of the  $C^*$ -algebra  $W_{\beta_z} \mathfrak{B}_U W_{\beta_z}^{-1} \subset \mathcal{B}(L^2(\Pi))$  that

$$s\text{-}\lim_{k \rightarrow 0} (U_k (a \circ \beta_z) U_k^{-1}) = a(z) I, \tag{8.5}$$

$$s\text{-}\lim_{k \rightarrow 0} (U_k B_\Pi U_k^{-1}) = B_\Pi, \quad s\text{-}\lim_{k \rightarrow 0} (U_k (c_z \tilde{B}_\Pi \overline{c_z} I) U_k^{-1}) = \tilde{B}_\Pi. \tag{8.6}$$

Hence, for every  $A \in \mathfrak{B}_U$  and every  $z \in \partial U \setminus \mathcal{T}$  there exists the strong limit

$$A_z := s\text{-}\lim_{k=0} (U_k (W_{\beta_z} A W_{\beta_z}^{-1}) U_k^{-1}) \in \text{alg} \{I, B_\Pi, \tilde{B}_\Pi\}. \tag{8.7}$$

Applying now [8, Proposition 7.5], we deduce from (8.3) that  $B_z A_z = I$  and  $A_z B_z = I$ . Thus, the invertibility of the coset  $A_z^\pi \in (\mathfrak{B}_U)_z^\pi$  implies the invertibility of the operator  $A_z \in \text{alg} \{I, B_\Pi, \tilde{B}_\Pi\}$ .

On the other hand, the invertibility of the operator  $A_z \in \text{alg} \{I, B_\Pi, \tilde{B}_\Pi\}$  associated with an operator  $A \in \mathfrak{B}_U$  implies the invertibility of the operator

$$W_{\beta_z}^{-1} A_z W_{\beta_z} \in \widehat{\mathfrak{B}}_U := \text{alg} \{I, B_U, d_z \tilde{B}_U d_z^{-1} I\},$$

where  $d_z := (\beta_z^{-1})' / \overline{(\beta_z^{-1})}'$ . Since  $(d_z I)^\pi \overset{\sim}{\sim} (\overline{\beta'_z(0)} / \beta'_z(0)) I^\pi$  and then  $(d_z \tilde{B}_U d_z^{-1} I)^\pi \overset{\sim}{\sim} \tilde{B}_U^\pi$ , and since  $(aI)^\pi \overset{\sim}{\sim} a(z) I^\pi$  for all  $a \in C(\overline{U})$ , we conclude that the quotient  $C^*$ -algebras  $(\widehat{\mathfrak{B}}_U)_z^\pi$  and  $(\mathfrak{B}_U)_z^\pi$  coincide. Consequently, the invertibility of the operator  $W_{\beta_z}^{-1} A_z W_{\beta_z} \in \widehat{\mathfrak{B}}_U$  implies the invertibility of the coset  $(W_{\beta_z}^{-1} A_z W_{\beta_z})^\pi_z = A_z^\pi \in (\mathfrak{B}_U)_z^\pi$ .

Thus, the invertibility of the coset  $A_z^\pi \in (\mathfrak{B}_U)_z^\pi$  for  $A \in \mathfrak{B}_U$  is equivalent to the invertibility of the operator  $A_z \in \text{alg} \{I, B_\Pi, \tilde{B}_\Pi\}$  given by (8.7). This implies that the map  $(\mathfrak{B}_U)_z^\pi \rightarrow \text{alg} \{I, B_\Pi, \tilde{B}_\Pi\}$ , given on the generators of the  $C^*$ -algebra  $(\mathfrak{B}_U)_z^\pi$  by

$$(aI)_z^\pi \mapsto a(z) I, \quad (B_U)_z^\pi \mapsto B_\Pi, \quad (\tilde{B}_U)_z^\pi \mapsto \tilde{B}_\Pi, \tag{8.8}$$

is a  $*$ -isomorphism of the  $C^*$ -algebra  $(\mathfrak{B}_U)_z^\pi$  onto the  $C^*$ -algebra  $\text{alg}\{I, B_\Pi, \widetilde{B}_\Pi\}$ . Finally, the  $C^*$ -algebra  $\text{alg}\{I, B_\Pi, \widetilde{B}_\Pi\}$  is generated by the three pairwise orthogonal projections  $B_\Pi$ ,  $\widetilde{B}_\Pi$  and  $I - B_\Pi - \widetilde{B}_\Pi \neq I$  (see Lemma 3.2 and [27, Theorem 4.5]), which immediately implies the  $C^*$ -algebra isomorphism  $(\mathfrak{B}_U)_z^\pi \cong \mathbb{C}^3$ .

(iii) Let now  $z \in \mathcal{T}$  be a corner of opening  $\pi\alpha_z$ . Consider the conformal map  $\gamma_z : \mathbb{K}_{\alpha_z} \rightarrow U$  such that  $\gamma_z(0) = z$ , where  $\mathbb{K}_{\alpha}$  is given by (1.5). Clearly,  $\gamma_z = \beta_z \circ \varphi_{\alpha_z}^{-1}$ , where the conformal map  $\varphi_\alpha : \Pi \rightarrow \mathbb{K}_\alpha$  is given by  $\varphi_\alpha(w) = w^\alpha$  for  $\alpha \in (0, 2]$  and  $w \in \Pi$ ,  $\beta_z : \Pi \rightarrow U$  is the Schwarz-Christoffel conformal map such that  $\beta_z(0) = z$ , and  $\gamma'_z(0) \neq 0$ . Taking the unitary operator

$$W_{\gamma_z} : L^2(U) \rightarrow L^2(\mathbb{K}_{\alpha_z}), \quad f \mapsto \gamma'_z(f \circ \gamma_z),$$

we deduce from [8, Proposition 2.2] that

$$W_{\gamma_z}(aI)W_{\gamma_z}^{-1} = (a \circ \gamma_z)I, \quad W_{\gamma_z}B_UW_{\gamma_z}^{-1} = B_{\mathbb{K}_{\alpha_z}}, \quad W_{\gamma_z}\widetilde{B}_UW_{\gamma_z}^{-1} = c_z\widetilde{B}_{\mathbb{K}_{\alpha_z}}c_z^{-1}I,$$

where now  $c_z := \gamma'_z/\sqrt{\gamma'_z}$ . Applying the unitary dilation operators (8.4) considered now on the space  $L^2(\mathbb{K}_{\alpha_z})$  and [8, Proposition 2.2] again, we obtain

$$\begin{aligned} \text{s-lim}_{k \rightarrow 0}(U_k c_z U_k^{-1}) &= (\gamma'_z(0)/\overline{\gamma'_z(0)})I, & \text{s-lim}_{k \rightarrow 0}(U_k(a \circ \gamma_z)U_k^{-1}) &= a(z)I, \\ \text{s-lim}_{k \rightarrow 0}(U_k B_{\mathbb{K}_{\alpha_z}} U_k^{-1}) &= B_{\mathbb{K}_{\alpha_z}}, & \text{s-lim}_{k \rightarrow 0}(U_k(c_z \widetilde{B}_{\mathbb{K}_{\alpha_z}} \overline{c_z} I)U_k^{-1}) &= \widetilde{B}_{\mathbb{K}_{\alpha_z}}. \end{aligned}$$

Then, by analogy with part (ii) we infer that the map

$$(\mathfrak{B}_U)_z^\pi \rightarrow \mathfrak{A}_{\mathbb{K}_{\alpha_z}} = \text{alg}\{I, B_{\mathbb{K}_{\alpha_z}}, \widetilde{B}_{\mathbb{K}_{\alpha_z}}\},$$

given on the generators of the  $C^*$ -algebra  $(\mathfrak{B}_U)_z^\pi$  by

$$(aI)_z^\pi \mapsto a(z)I, \quad (B_U)_z^\pi \mapsto B_{\mathbb{K}_{\alpha_z}}, \quad (\widetilde{B}_U)_z^\pi \mapsto \widetilde{B}_{\mathbb{K}_{\alpha_z}}, \tag{8.9}$$

is a  $C^*$ -algebra isomorphism of the  $C^*$ -algebra  $(\mathfrak{B}_U)_z^\pi$  onto the  $C^*$ -algebra  $\mathfrak{A}_{\mathbb{K}_{\alpha_z}}$ , which completes the proof of part (iii).  $\square$

Combining Theorem 8.3, Lemma 8.5 and Theorem 7.2, we establish the Fredholm criterion for the  $C^*$ -algebra  $\mathfrak{B}_U$  given by (1.7), where  $U$  is a bounded polygonal domain.

**THEOREM 8.6.** *The quotient  $C^*$ -algebra*

$$\mathfrak{B}_U^\pi := \text{alg}\{aI, B_U, \widetilde{B}_U : a \in C(\overline{U})\} / \mathcal{K} \subset \mathcal{B}(L^2(U)) / \mathcal{K}$$

is  $*$ -isomorphic to the  $C^*$ -subalgebra  $\Psi(\mathfrak{B}_U^\pi)$  of the  $C^*$ -algebra

$$\left( \bigoplus_{z \in \overline{U}} \mathbb{C} \right) \oplus \left( \bigoplus_{z \in \partial U \setminus \mathcal{T}} \mathbb{C}^2 \right) \oplus \left( \bigoplus_{(z, \lambda) \in \mathcal{T} \times \overline{\mathbb{R}}} \mathbb{C}^{2 \times 2} \right), \tag{8.10}$$

and the corresponding isomorphism

$$\Psi = \left( \bigoplus_{z \in \overline{U}} \Psi_z^0 \right) \oplus \left( \bigoplus_{z \in \partial U \setminus \mathcal{I}} \Psi_z \right) \oplus \left( \bigoplus_{(z,\lambda) \in \mathcal{I} \times \overline{\mathbb{R}}} \Psi_{z,\lambda} \right) \tag{8.11}$$

is given on the generators of the  $C^*$ -algebra  $\mathfrak{B}_U^\pi$  by

$$\begin{aligned} \Psi((aI)^\pi) &:= \left( \bigoplus_{z \in \overline{U}} a(z) \right) \oplus \left( \bigoplus_{z \in \partial U \setminus \mathcal{I}} (a(z), a(z)) \right) \oplus \left( \bigoplus_{(z,\lambda) \in \mathcal{I} \times \overline{\mathbb{R}}} a(z)I_2 \right), \\ \Psi(B_U^\pi) &:= \left( \bigoplus_{z \in \overline{U}} 0 \right) \oplus \left( \bigoplus_{z \in \partial U \setminus \mathcal{I}} (1, 0) \right) \oplus \left( \bigoplus_{(z,\lambda) \in \mathcal{I} \times \overline{\mathbb{R}}} M_{\alpha_z}(\lambda) \right), \\ \Psi(\tilde{B}_U^\pi) &:= \left( \bigoplus_{z \in \overline{U}} 0 \right) \oplus \left( \bigoplus_{z \in \partial U \setminus \mathcal{I}} (0, 1) \right) \oplus \left( \bigoplus_{(z,\lambda) \in \mathcal{I} \times \overline{\mathbb{R}}} \tilde{M}_{\alpha_z}(\lambda) \right), \end{aligned} \tag{8.12}$$

where the matrices  $M_\alpha(\lambda)$ ,  $\tilde{M}_\alpha(\lambda) \in \mathbb{C}^{2 \times 2}$  are defined by (7.2) for all  $\lambda \in \overline{\mathbb{R}}$ . An operator  $A \in \mathfrak{B}_U$  is Fredholm on the space  $L^2(U)$  if and only if its symbol  $\Psi(A^\pi)$  is invertible in the  $C^*$ -algebra  $\Psi(\mathfrak{B}_U^\pi)$ , that is, if

$$\begin{aligned} \Psi_z^0(A^\pi) &\neq 0 \text{ for all } z \in \overline{U}, \\ [\Psi_z(A^\pi)]_k &\neq 0 \text{ for all } z \in \partial U \setminus \mathcal{I} \text{ and all } k = 1, 2, \\ \det[\Psi_{z,\lambda}(A^\pi)] &\neq 0 \text{ for all } z \in \mathcal{I} \text{ and all } \lambda \in \overline{\mathbb{R}}, \end{aligned}$$

where  $[\Psi_z(A^\pi)]_k$  are the  $k$ -entries of the vector  $\Psi_z(A^\pi)$ .

*Proof.* By Lemma 8.5(i), for each  $z \in U$  the map  $A_z^\pi \mapsto \Psi_z^0(A^\pi)$  is a  $*$ -isomorphism of the  $C^*$ -algebra  $(\mathfrak{B}_U^\pi)_z$  onto  $\mathbb{C}$ , while for every  $z \in \partial U \setminus \mathcal{I}$  from Lemma 8.5(ii) and (8.8) it follows that the map  $A_z^\pi \mapsto \Psi_z^0(A^\pi) \oplus \Psi_z(A^\pi)$  is a  $*$ -isomorphism of the  $C^*$ -algebra  $(\mathfrak{B}_U^\pi)_z$  onto the  $C^*$ -algebra  $\mathbb{C} \oplus \mathbb{C}^2 \cong \mathbb{C}^3$ . Further, by Lemma 8.5(iii), (8.9) and Theorem 7.2, for every  $z \in \mathcal{I}$  the  $C^*$ -algebra  $(\mathfrak{B}_U^\pi)_z$  is  $*$ -isomorphic to the  $C^*$ -subalgebra  $\mathbb{C} \oplus \mathfrak{S}_{\alpha_z}$  of  $\mathbb{C} \oplus C(\overline{\mathbb{R}}, \mathbb{C}^{2 \times 2})$ , and this isomorphism is given by  $A_z^\pi \mapsto \Psi_z^0(A^\pi) \oplus (\bigoplus_{\lambda \in \overline{\mathbb{R}}} \Psi_{z,\lambda}(A^\pi))$ , where the homomorphisms  $\Psi_z^0 : \mathfrak{B}_U^\pi \rightarrow \mathbb{C}$  for  $z \in \overline{U}$ ,  $\Psi_z : \mathfrak{B}_U^\pi \rightarrow \mathbb{C}^2$  for  $z \in \partial U \setminus \mathcal{I}$  and  $\Psi_{z,\lambda} : \mathfrak{B}_U^\pi \rightarrow \mathbb{C}^{2 \times 2}$  for  $z \in \mathcal{I}$  and  $\lambda \in \overline{\mathbb{R}}$  are given by (8.12). Hence, applying Theorem 8.3, we conclude that the  $C^*$ -algebra  $\mathfrak{B}_U^\pi$  is  $*$ -isomorphic to the  $C^*$ -subalgebra  $\tilde{\mathfrak{B}}_U$  of

$$\left( \bigoplus_{z \in U} \mathbb{C} \right) \oplus \left( \bigoplus_{z \in \partial U \setminus \mathcal{I}} (\mathbb{C} \oplus \mathbb{C}^2) \right) \oplus \left( \bigoplus_{z \in \mathcal{I}} \left( \mathbb{C} \oplus \left( \bigoplus_{\lambda \in \overline{\mathbb{R}}} \mathbb{C}^{2 \times 2} \right) \right) \right) \tag{8.13}$$

composed for all  $A^\pi \in \mathfrak{B}_U^\pi$  by the elements  $\Psi_z^0(A^\pi)$  for  $z \in U$ ,  $\Psi_z^0(A^\pi) \oplus \Psi_z(A^\pi)$  for  $z \in \partial U \setminus \mathcal{I}$  and  $\Psi_z^0(A^\pi) \oplus (\bigoplus_{\lambda \in \overline{\mathbb{R}}} \Psi_{z,\lambda}(A^\pi))$  for  $z \in \mathcal{I}$ . It is easily then seen that the  $C^*$ -subalgebra  $\tilde{\mathfrak{B}}_U$  of the  $C^*$ -algebra (8.13) is  $*$ -isomorphic to the  $C^*$ -subalgebra  $\Psi(\mathfrak{B}_U^\pi)$  of the  $C^*$ -algebra (8.10), where the isomorphism  $\Psi$  is given by (8.11) and (8.12). Thus,  $\mathfrak{B}_U^\pi \cong \Psi(\mathfrak{B}_U^\pi)$ , which implies the corresponding Fredholm criterion.  $\square$

## REFERENCES

- [1] A. BÖTTCHER AND YU. I. KARLOVICH, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators*, Progress in Mathematics **154**, Birkhäuser, Basel, 1997.
- [2] A. BÖTTCHER AND B. SILBERMANN, *Analysis of Toeplitz Operators*, 2nd edn., Springer, Berlin, 2006.
- [3] R. COURANT AND A. HURWITZ, *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen*, Springer, Berlin, 1929.
- [4] R. G. DOUGLAS, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.
- [5] A. DZHURAEV, *Methods of Singular Integral Equations*, Longman Scientific & Technical, Harlow, 1992.
- [6] E. ESPINOZA-LOYOLA, YU. I. KARLOVICH, AND O. VILCHIS-TORRES, *C\*-algebras of Bergman type operators with piecewise constant coefficients over sectors*, Integr. Equ. Oper. Theory (2015), DOI 10.1007/s00020-015-2226-5.
- [7] A. N. KARAPETYANTS, V. S. RABINOVICH, AND N. L. VASILEVSKI, *On algebras of two dimensional singular integral operators with homogeneous discontinuities in symbols*, Integr. Equ. Oper. Theory **40** (2001), 278–308.
- [8] YU. I. KARLOVICH AND L. PESSOA, *Algebras generated by Bergman and anti-Bergman projections and by multiplications by piecewise continuous coefficients*, Integr. Equ. Oper. Theory **52** (2005), 219–270.
- [9] YU. I. KARLOVICH AND L. V. PESSOA, *C\*-algebras of Bergman type operators with piecewise continuous coefficients*, Integr. Equ. Oper. Theory **57** (2007), 521–565.
- [10] YU. I. KARLOVICH AND L. V. PESSOA, *Poly-Bergman projections and orthogonal decompositions of  $L^2$ -spaces over bounded domains*, Operator Theory: Advances and Applications **181**, in: “Operator Algebras, Operator Theory and Applications” (2008), 263–282.
- [11] YU. I. KARLOVICH AND L. V. PESSOA, *C\*-algebras of Bergman type operators with piecewise continuous coefficients on bounded domains*, in: H.G.W. Begehr, F. Nicolosi (Eds.), *More Progresses in Analysis. Proceedings of the 5th International ISAAC Congress, Catania, Italy, July 25–30, 2005*. World Scientific, Singapore, 2009, 339–348.
- [12] T. KATO, *Perturbation Theory for Linear Operators*, 2nd edn., Springer, Berlin, 1984.
- [13] M. LOAIZA, *Algebras generated by the Bergman projection and operators of multiplication by piecewise continuous functions*, Integr. Equ. Oper. Theory **46** (2003), 215–234.
- [14] M. LOAIZA, *On the algebra generated by the harmonic Bergman projection and operators of multiplication by piecewise continuous functions*, Bol. Soc. Mat. Mexicana (3) **10** (2004), 179–193.
- [15] M. LOAIZA, *On an algebra of Toeplitz operators with piecewise continuous symbols*, Integr. Equ. Oper. Theory **51** (2005), 141–153.
- [16] A. I. MARKUSHEVICH, *Theory of Functions of a Complex Variable*, 2nd ed., AMS Chelsea Publishing, 1977.
- [17] S. G. MIKHLIN AND S. PRÖSSDORF, *Singular Integral Operators*, Springer, Berlin, 1986.
- [18] F. OBERHETTINGER, *Tables of Mellin Transforms*, Springer, Berlin, 1974.
- [19] B. A. PLAMENEVSKY, *Algebras of Pseudodifferential Operators*, Kluwer Academic Publishers, Dordrecht, 1989.
- [20] CH. POMMERENKE, *Boundary Behaviour of Conformal Maps*, Springer, Berlin, 1992.
- [21] V. S. RABINOVICH, S. ROCH, AND B. SILBERMANN, *Limit Operators and Their Applications in Operator Theory*, Operator Theory: Advances and Applications, vol. 150. Birkhäuser, Basel, 2004.
- [22] E. RAMÍREZ DE ARELLANO AND N. L. VASILEVSKI, *Bergman projection, three-valued functions and corresponding Toeplitz operators*, Contemporary Mathematics **212** (1998), 185–196.
- [23] S. ROCH, P. A. SANTOS, AND B. SILBERMANN, *Non-commutative Gelfand Theories. A Tool-kit for Operator Theorists and Numerical Analysts*, Springer, London, 2011.
- [24] I. B. SIMONENKO AND CHIN NGOK MIN, *Local Method in the Theory of One-Dimensional Singular Integral Equations with Piecewise Continuous Coefficients. Noetherity*, University Press, Rostov on Don, 1986 (Russian).
- [25] N. L. VASILEVSKI, *Banach algebras generated by two-dimensional integral operators with a Bergman kernel and piecewise continuous coefficients. I*, Soviet Math. (Izv. VUZ) **30**, 2 (1986), 14–24.
- [26] N. L. VASILEVSKI, *C\*-algebras generated by orthogonal projections and their applications*, Integr. Equ. Oper. Theory **31** (1998), 113–132.

- [27] N. L. VASILEVSKI, *On the structure of Bergman and poly-Bergman spaces*, Integr. Equ. Oper. Theory **33** (1999), 471–488.
- [28] N. L. VASILEVSKI, *Toeplitz operators on the Bergman spaces: Inside-the domain effects*, Contemporary Mathematics **289** (2001), 79–146.
- [29] N. L. VASILEVSKI, *Commutative Algebras of Toeplitz Operators on the Bergman Space*, Operator Theory: Advances and Applications, vol. 185. Birkhäuser, Basel, 2008.

(Received October 30, 2014)

*Yuri I. Karlovich*  
*Centro de Investigación en Ciencias*  
*Instituto de Investigación en Ciencias Básicas y Aplicadas*  
*Universidad Autónoma del Estado de Morelos*  
*Av. Universidad 1001, Col. Chamilpa*  
*C. P. 62209 Cuernavaca, Morelos, México*  
*e-mail: karlovich@uaem.mx*